The Balayage onto Closed Sets with Respect to Continuous Function-Kernels

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Let X be a locally compact Hausdorff space with a countable base and G be a continuous function-kernel on X such that each non-empty open set is non-negligible with respect to G.

Under the assumption that G and the adjoint kernel $\check{G}$ satisfies the continuity principle, R. Durier proved that, if G or $\check{G}$ satisfies the domination principle, G or $\check{G}$ does the balayaged principle and conversely ([2]). Further, I. Higuchi and M. Ito obtained the same conclusion without the assumption of the continuity principle ([3]).

In this paper we shall consider the balayage onto any closed non-negligible set with respect to a continuous function-kernel G satisfying the domination principle. We shall show that, if each non-empty open set is non-negligible and the convex cone of continuous potentials is adapted, then it is possible to balayage onto any closed non-negligible set. Further, we shall show that there exists a "minimum" balayaged potential uniquely up to a negligible set.

§ 1. Preliminary.

Throughout this paper we assume that X is a locally compact Hausdorff space with a countable base and G is a continuous function-kernel, i. e. an extended continuous mapping from $X \times X$ to $\boldsymbol{R}^+ \cup \{+\infty\}$ such that it is strictly positive on the diagonal set Δ and finite outside of Δ. The adjoint kernel $\check{G}$ of G is defined by $\check{G}(x, y)=G(y, x)$. Evidently G is also a continuous function-kernel.

We denote by M^+ (resp. M_k^+) the set of positive Radon measures on X (resp. the subset of M^+ of the measures of with compact support). The potential $G\mu$ of $\mu \in M^+$ is defined by

$$G\mu(x)=\int G(x, y)d\mu(y)$$

and the energy of μ is defined by $\int G\mu d\mu$. We denote by

$$\mathcal{E}:=\left\{\mu \in M_k^+ : \int G\mu d\mu < \infty\right\},$$

$$\mathcal{F} \text{ (resp. } \check{\mathcal{F}}):=\{\mu \in M_k^+ : G\mu \text{ (resp. } \check{G}\mu) \text{ is finite and continuous on } X\}$$

and for a subset F of X

$$M^+(F) := \{\mu \in M^+ : S\mu \subset F\}, \qquad \mathcal{E}(F) := \{\mu \in \mathcal{E} : S\mu \subset F\}.$$

A subset F of X is said to be negligible if $\mu(F)=0$ for all $\mu \in \mathcal{E}(F)$. Given a subset F of X, "the property holds n.e. on F" means that the property holds on F with a possible exception of a negligible set. Suppose that G satisfies the continuity principle, i.e. if $G\mu$ ($\mu \in M_k^+$) is finite and continuous as a function on $S\mu$, $G\mu$ is also finite and continuous in the whole space. If a subset F of X is non-negligible, it follows from Lusin's theorem that there exists a non-zero measure $\tau \in \mathcal{F}(F)$. Therefore, if $\tau(F)=0$ for all $\tau \in \mathcal{F}$, F is negligible.

We say that a kernel G satisfies the domination principle, if, for $\mu \in \mathcal{E}$, $\nu \in M_k^+$, $G\mu \leqq G\nu$ in X whenever $G\mu \leqq G\nu$ on $S\mu$. If G satisfies the domination principle, G satisfies the continuity principle (c.f. [3, Theorem 2]). Therefore, without the assumption of continuity principle, we have

THEOREM 1 ([3, Theorem 3]). *Assume that each non-empty open set is non-negligible with respect to G. The following statements are equivalent:*

(i) *G satisfies the domination principle,*

(ii) *$\check{G}$ satisfies the domination principle,*

(iii) *G satisfies the balayage principle,*

(iv) *$\check{G}$ satisfies the balayage principle.*

PROPOSITION 1. *Assume that G satisfies the domination principle and each non-empty open set is non-negligible. Then the inequality $G\mu \leqq G\nu$ n.e. on $S\mu$ for $\mu \in \mathcal{E}$, $\nu \in M_k^+$ implies $G\mu \leqq G\nu$ n.e. in X.*

PROOF. Let x be an arbitrary point of the complement of $S\mu$. By Theorem 1 there exists a $\tau \in M^+(S\mu)$ such that $\check{G}\tau = \check{G}\varepsilon_x$ n.e. on $S\mu$ and $\check{G}\tau \leqq \check{G}\varepsilon_x$ everywhere. Since $\check{G}\varepsilon_x$ is continuous on $S\mu$, it follows that $\tau \in \mathcal{E}$. Hence

$$G\mu(x) = \int \check{G}\varepsilon_x d\mu = \int \check{G}\tau d\mu = \int G\mu d\tau \leqq \int G\nu d\tau \leqq \int \check{G}\tau d\nu \leqq \int \check{G}\varepsilon_x d\nu = G\nu(x).$$

Therefore we have conclusion.

The following proposition will be used frequently.

PROPOSITION 2. *Assume that G satisfies the domination principle and that each non-empty open set is non-negligible. Then, for each compact set F there exists a $\tau \in \mathcal{F}$ such that $G\tau \geqq 1$ on F.*

PROOF. Since G is lower semicontinuous and $G(x, x) > 0$ for all $x \in X$, there is, for each $x \in X$, a relatively compact neighborhood U_x of x such that

$$G(z, y) > \frac{1}{2} G(x, x) \qquad \text{on } U_x \times U_x .$$

Choose finite points $x_1, x_2, \cdots, x_n$ of F satisfying $\bigcup_{i=1}^{n} U_{x_i} \supset F$. Since G satisfies the domination principle, it also satisfies continuity principle. Since U_{x_i} is non-negligible by the assumption, there exists a $\tau_i \in \mathcal{F}(U_{x_i})$ ($\tau_i(1)=1$). Then it holds that

$$G\tau_i(z)=\int G(z, y)d\tau_i(y) \geqq \frac{1}{2}G(x_i, x_i) \qquad \text{for all } z \in U_{x_i}.$$

Put $\tau=\sum_{i=1}^{n}\tau_i$. Then $G_\tau>0$ on F. Since G_τ is continuous everywhere, we can find a positive real number $b>0$ satisfying $bG\tau \geqq 1$ on F.

§ 2. Adapted spaces.

We assume that the convex cone of the potentials with compact support satisfies the following condition (R_σ) to consider the balayage onto any closed non-negligible set F.

In general, let P be a convex cone in $C(X)$ and

$$P_\sigma := \{\sum_{n=1}^{\infty} u_n : u_n \in P, \ \sum_{n=1}^{\infty} u_n \in C(X)\}.$$

(R_σ) For each $u \in P$, for each real number $\varepsilon>0$ and for each compact subset F of X, there exist a $v \in P_\sigma$ and a compact subset K of X satisfying

$$v \leqq \varepsilon \text{ on } F \quad \text{and} \quad v \geqq u \text{ on } CK.$$

When $\{K_n\}$ is an exhaustion of compact subsets of X, it is easy to see that (R_σ) is equivalent to the following condition (R'_σ):

(R'_σ) For each $u \in P$ and for each $x \in X$ $\inf\{v(x) : v \geqq u \text{ on } CK_n, v \in P_\sigma\}$

converges to zero uniformly on any compact set as $n \to \infty$.

We denote by $u \in o(v)$ for $u \in C(X)$, $v \in C^+(X)$ if for each $\varepsilon>0$ the set $\{x \in X : |u(x)| > \varepsilon v(x)\}$ is compact.

PROPOSITION 3. *If a convex cone P in $C^+(X)$ satisfies (R_σ), then, for each $u \in P$, for each $\varepsilon>0$ and for a compact set K in X, there exists a $v \in P_\sigma$ satisfying $u \in o(v)$ and $v \leqq \varepsilon$ on K.*

PROOF. Let $\{K_n\}$ be an exhaustion of compact subsets of X. We can assume that $K \subset K_1$. Let u be a function in P. By (R_σ) there exist a $v_n \in P_\sigma$ and a $m_n \in N$ such that $m_n < m_{n+1}$,

$$v_n \leqq \frac{1}{2^n}\varepsilon \quad \text{on } K_n, \qquad \text{and} \qquad v_n \geqq u \quad \text{on } CK_{m_n}.$$

Put $v := \sum_{n=1}^{\infty} v_n$. Then u is continuous everywhere and $v \in P_\sigma$. For each n and each $x \in CK_{m_n}$, it holds that

$$u_i(x) \geqq u(x) \qquad (i=1, 2, \cdots, n).$$

Hence $v(x)\geqq\sum_{i=1}^{n}v_i(x)\geqq nu(x)$ and the set $\left\{x: u(x)>\frac{1}{n}v(x)\right\}$ is compact. Therefore we have $u\in o(v)$.

PROPOSITION 4. *Let P be a convex cone in $C^+(X)$ satisfying (R_σ). Then for each $u\in P_\sigma$ there exists a $v\in P_\sigma$ with $u\in o(v)$.*

PROOF. Let $\{K_n\}$ be an exhaustion of compact subsets of X and u a function in P. Since $u=\sum_{n=1} u'_n$ $(u'_n\in P)$ converges uniformly on K_n, we can write

$$u=\sum_{n=0}^{\infty}u_n,\quad u_n\in P,\quad u_n\leqq\frac{1}{8^n}\quad\text{on } K_n\quad(n=1, 2, \cdots).$$

By Proposition 3 there exists a $v_0\in P$ with $u_0\in o(v)$. For each $n\geqq 1$ there exists a $w_n\in P$, $2^n u_n\in o(w_n)$ and $w_n<1/4^n$ on K_n. Put $v_n:=u_n+w_n$ $(n=1, 2, \cdots)$. Then $2^n u_n\in o(v_n)$, $v_n\geqq 2^n u_n$ and

$$v_n\leqq\frac{1}{4^n}+\sup_{x\in K_n}2^n u_n(x)\leqq\frac{1}{4^n}+\frac{1}{4^n}=\frac{1}{2^n}\qquad\text{on } K_n.$$

Put $v:=\sum_{n=0}^{\infty}v_n$. Then $\sum_{n=0}^{\infty}v_n$ converges uniformly on each K_n and $v\in P_\sigma$. Let ε be a positive real number. Choose r with $1/2^r\leqq\varepsilon$. Since $2^n u_n\in o(v_n)$, there exists a $m_n\in N$ such that

$$2^n u_n\leqq\varepsilon v_n\qquad\text{on } CK_{m_n}\qquad(n=0, 1, \cdots, r).$$

Put $m:=\max\{m_1, m_2, \cdots, m_r\}$. Then

$$2^n u_n\leqq\varepsilon v_n\qquad\text{on } CK_m\qquad(n=0, 1, \cdots, r).$$

Hence

$$\begin{aligned} u&=\sum_{n=0}^{\infty}u_n=\sum_{n=0}^{r}u_n+\sum_{n=r+1}^{\infty}u_n\\ &\leqq\sum_{n=0}^{\infty}\frac{\varepsilon}{2^n}v_n+\sum_{n=r+1}^{\infty}\frac{v_n}{2^n}\leqq\sum_{n=0}^{\infty}\varepsilon v_n=\varepsilon v\qquad\text{on } CK_m.\end{aligned}$$

Therefore $u\in o(v)$.

Let P be a convex cone in $C^+(X)$ satisfying (R_σ). We denote by $C(X, P_\sigma)$ the set of all continuous real-valued functions f on X such that there exists a $g\in P$ with $|f|\leqq g$. If $C(X, P_\sigma)\supset\mathcal{K}(X)$[1], $C(X, P_\sigma)$ is an adapted space in $C(X)$; it is a linear subspace H of $C(X)$ satisfying the following conditions (a_1), (a_2) and (a_3):

(a_1) each $v\in H$ is written $v=v_1-v_2$ with $v_1, v_2\geqq 0$ and $v_i\in H$ $(i=1, 2)$,

(a_2) for each $x\in X$ there is a $v\in H$ with $v\geqq 0$ and $v(x)>0$,

(a_3) for each $u\in H$ with $u\geqq 0$ there exists a $v\in H$ with $v\geqq 0$ and $u\in o(v)$.

1) We denote by $\mathcal{K}(X)$ the set of all continuous real-valued functions on X with compact support.

It is well-known that each positive linear functional φ on an adapted space H in $C(X)$ is represented by a positive measure μ on X, i. e.

$$\varphi(v)=\int v d\mu \qquad \text{for all } v\in H \qquad ([1,\ 34.6 \text{ Theorem}]).$$

Let G be a continuous function-kernel on X and put

$$\check{P}:=\{\check{G}\tau:\ \tau\in\check{\mathfrak{F}}\}.$$

Then $\check{P}$ is a convex cone in $C^{+}(X)$. If $\check{P}$ satisfies (R_σ), we obtained by Propositions 3 and 4 that any function in $\check{P}_\sigma$ is contained in $o(v)$ with some $v\in\check{P}_\sigma$. Further, if each non-empty open set is non-negligible, there exists, for each compact subset of K of X, a $f\in\check{P}$ such that $f\geqq 1$ on K. Therefore, it is easy to see that the space $C(X, \check{P}_\sigma)$ is an adapted space in $C(X)$.

PROPOSITION 5. *Assume that $\check{G}$ satisfies the domination principle, $\check{P}$ satisfies (R_σ) and that each non-empty open set is non-negligible. If $\check{G}\tau\leqq\check{G}\lambda+u$ on S_τ for $\tau\in\mathcal{E}$, $\lambda\in M_k^+$ and $u\in\check{P}_\sigma$, then the same inequality holds everywhere.*

PROOF. Let $u\in\check{P}_\sigma$. Since the convergence of $u=\sum_{n=1}^{\infty}\check{G}\mu_n$ $(\mu_n\in\mathfrak{F})$ is uniformly on $S\tau$, for each $\varepsilon>0$ there exists a $m\in N$ satisfying

$$\sum_{n=1}^{m}\check{G}\mu_n+\varepsilon>u \qquad \text{on } S\tau .$$

Choose $\tau_1\in\check{\mathfrak{F}}$ with $\check{G}\tau_1\geqq 1$ on $S\tau$. Since $\check{G}\tau\leqq\check{G}\lambda+\sum_{n=1}^{m}\check{G}\mu_n+\check{G}\tau_1$ on $S\tau$ and $\check{G}$ satisfies the domination principle, the same inequality holds everywhere. Hence $\check{G}\tau\leqq\check{G}\lambda+u+\varepsilon\check{G}\tau_1$ on X. As ε tends to zero, it follows that $\check{G}\tau\leqq\check{G}\lambda+u$ on X.

§ 3. The balayage onto closed sets.

Assume that each non-empty open set is non-negligible. Under the assumption that the convex cone $\check{P}=\{\check{G}\tau:\ \tau\in\check{\mathfrak{F}}\}$ satisfies (R_σ), we shall consider the balayage of a positive measure μ onto any closed non-negligible set F.

THEOREM 2. *Assume that G satisfies the domination principle and $\check{P}$ satisfies (R_σ). Let F be a non-negligible compact set and μ be a positive measure such that for all $u\in\check{P}_\sigma$ is μ-integrable. Then there exists a positive measure ν with $S\nu\subset F$ satisfying the following conditions.*

(i) *$G\nu\leqq G\mu$ on X,*

(ii) *$G\nu=G\mu$ n. e. on F,*

(iii) *each $u\in\check{P}_\sigma$ is ν-integrable.*

PROOF. Put

$$C(F, \check{P}_\sigma):=\{f\in C(F):\ \exists u\in\check{P}_\sigma,\ u\geqq 0,\ -u\leqq f\leqq u \text{ on } F\}.$$

Then $C(F, \check{P}_\sigma)$ is an adapted space in $C(F)$. Remark that $C(F, \check{P}_\sigma)\supset\mathcal{K}(F)$. For each $f\in C(F, \check{P}_\sigma)$, put

$$Q(f):=\inf\{\mu(\check{G}\lambda)-\mu(\check{G}\tau)+\mu(u): \lambda\in M_k^+,\ \tau\in\check{\mathfrak{F}}(F),$$
$$u\in\check{P}_\sigma,\ f\leqq\check{G}\lambda-\check{G}\tau+u \text{ on } F\}.$$

Take $v\in\check{P}_\sigma$ such that $-v\leqq f\leqq v$ on F. Then $Q(f)\leqq\mu(v)<\infty$. Further, assume that $\check{G}\lambda-\check{G}\tau+u\geqq f$ on F with $\lambda\in M_k^+$, $\tau\in\check{\mathfrak{F}}(F)$ and $u\in\check{P}_\sigma$. Since $-v\leqq f\leqq \check{G}\lambda-\check{G}\tau+u$ on F, it holds that $\check{G}\tau\leqq\check{G}\lambda+u+v$ on F. By Proposition 5, the same inequality holds everywhere. Hence $-\mu(v)\leqq\mu(\check{G}\lambda)-\mu(\check{G}\tau)+\mu(u)$. Therefore $-\infty<-\mu(v)\leqq Q(f)$. Since the mapping $f\mapsto Q(f)$ is a sublinear functional on $C(F, \check{P}_\sigma)$, there exists, by Hahn-Banach theorem, a linear functional ν on $C(F, \check{P}_\sigma)$ such that $\nu(f)\leqq Q(f)$ for all $f\in C(F, \check{P}_\sigma)$. If $f\leqq 0$, it holds that $\nu(f)\leqq Q(f)\leqq 0$. Hence ν is positive. Since ν is a positive linear functional on the adapted space $C(F, \check{P}_\sigma)$, ν is a positive measure on F such that each $f\in C(F, \check{P}_\sigma)$ is ν-integrable. Let $\lambda\in M_k^+$. Since $\check{G}\lambda$ is a positive lower semi-continuous function, it holds that

$$\nu(G\lambda)=\sup\{\nu(g): 0\leqq g\leqq G\lambda \text{ on } F,\ g\in\mathcal{K}(F)\}$$
$$\leqq\sup\{Q(g): 0\leqq g\leqq G\lambda \text{ on } F,\ g\in\mathcal{K}(F)\}\leqq\mu(G\lambda).$$

Especially, put $\lambda=\varepsilon_x$. We have

$$G\nu(x)\leqq G\mu(x) \qquad \text{for all } x\in X.$$

Let $\tau\in\check{\mathfrak{F}}(F)$. Since $\check{P}$ satisfies (R_σ), there exists, by Proposition 3, a $w\in\check{P}_\sigma$ such that $\check{G}\tau\in o(w)$. For each $\varepsilon>0$ there exists a compact set $K\subset F$ such that $\check{G}\tau\leqq\varepsilon w$ on CK. Take $g\in\mathcal{K}(F)$ such that $0\leqq g\leqq\check{G}\tau$ and $g=\check{G}\tau$ on K. Then $\check{G}\tau\leqq\varepsilon w+g$ on F. Consequently

$$\nu(-\check{G}\tau)\leqq\nu(-g)\leqq Q(-g)\leqq\varepsilon\mu(w)-\mu(\check{G}\tau).$$

Hence $-\nu(\check{G}\tau)\leqq\varepsilon\mu(w)-\mu(\check{G}\tau)$. As ε tends to zero, we have

$$-\nu(\check{G}\tau)\leqq-\mu(\check{G}\tau).$$

Therefore $\nu(\check{G}\tau)=\mu(\check{G}\tau)$ for all $\tau\in\check{\mathfrak{F}}(F)$. Hence $G\nu=G\mu$ n.e. on F.

A positive measure ν on F satisfying (i), (ii) and (iii) is called a balayaged measure of μ onto F with respect to G.

§ 4. The minimum balayaged potentials.

In this section we assume that each non-empty open set is non-negligible. In § 3 we have considered the balayage onto any non-negligible closed set. But a balayaged measure is not necessarily unique. We prepare the following dominated convergence theorem to see that the minimume balayaged potential is determined uniquely.

THEOREM 3. *Assume that G satisfies the domination principle and $\check{P}$ satisfies*

(R_σ). *Suppose that the sequence* $\{G\mu_n\}$ *of potentials of positive measures is dominated by a potential* $G\nu$ *of a positive measure* ν *such that each* $u\in\check{P}_\sigma$ *is* ν*-integrable. Then there exist a* $\mu\in M^+$ *and a subsequence* $\{\mu_{n_j}\}$ *of* $\{\mu_n\}$ *satisfying the following conditions*:

(i) $\varliminf_{j\to\infty} G\mu_{n_j}=G\mu$ *n. e. on* X,

(ii) $\lim_{j\to\infty}\int G\mu_{n_j}d\tau=\int G\mu d\tau$ *for each* $\tau\in\check{\mathfrak{F}}$,

(iii) *each* $u\in\check{P}_\sigma$ *is* μ*-integrable.*

PROOF. If $\tau\in\mathfrak{F}$, it holds that

$$\mu_n(\check{G}\tau)=\tau(G\mu_n)\leqq\tau(G\nu)=\nu(\check{G}\tau)<\infty .$$

Since each $u\in\check{P}_\sigma$ is written as $\sum_{i=1}^{\infty}\check{G}\tau_i$ where $\tau_i\in\check{\mathfrak{F}}$, it holds that $\mu_n(u)\leqq\nu(u)$. Since for each $f\in C(X, \check{P}_\sigma)$ there is a function $u\in\check{P}_\sigma$ satisfying $|f|\leqq u$, we have $|\mu_n(f)|\leqq\nu(u)<\infty$. Since the set $\{\mu_1, \mu_2, \cdots\}$ is bounded under the topology $\sigma(C(X, \check{P}_\sigma)^*, C(X, \check{P}_\sigma))$. Hence $\{\mu_1, \mu_2, \cdots\}$ is relatively compact under the topology $\sigma(C(X, \check{P}_\sigma)^*, C(X, \check{P}_\sigma))$ c. f. [1, 23.11 Theorem]). Put

$$A_n:=\overline{\{\mu_n, \mu_{n+1}, \cdots\}}$$

and take $\mu\in\bigcap_{n=1}^{\infty}A_n$. Then μ is a positive continuous linear functional on $C(X, \check{P}_\sigma)$ and hence a positive measure on X such that each $f\in C(X, \check{P}_\sigma)$ is μ-integrable. Since X has a countable base, the adapted space $C(X, \check{P}_\sigma)$ is separable (c. f. [4, Proposition 6]). We can choose a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ such that

$$\lim_{j\to\infty}\mu_{n_j}(f)=\mu(f) \qquad \text{for all } f\in C(X, \check{P}_\sigma).$$

Especially

$$\lim_{j\to\infty}\mu_{n_j}(u)=\mu(u) \qquad \text{for all } u\in\check{P}_\sigma$$

and

$$\lim_{j\to\infty}\mu_{n_j}(\check{G}\tau)=\mu(\check{G}\tau) \qquad \text{for all } \tau\in\check{\mathfrak{F}}.$$

Further, since $\lim_{j\to\infty}\mu_{n_j}(g)=\mu(g)$ for all $g\in\mathcal{K}(X)$, $\varliminf_{j\to\infty}G\mu_{n_j}\geqq G\mu$. For each $\tau\in\check{\mathfrak{F}}$ it holds that by Fatou's lemma

$$\begin{aligned}\tau(G\mu)&\leqq\tau(\varliminf_{j\to\infty}G\mu_{n_j})\leqq\varliminf_{j\to\infty}\tau(G\mu_{n_j})\\&=\varliminf_{j\to\infty}\mu_{n_j}(\check{G}\tau)=\mu(\check{G}\tau)=\tau(G\mu).\end{aligned}$$

Consequently

$$\tau(G\mu)=\tau(\varliminf_{j\to\infty}G\mu_{n_j})=\lim_{j\to\infty}\tau(G\mu_{n_j}).$$

Hence $G\mu=\varliminf G\mu_{n_j}$ n. e. on X.

Let $\{K_n\}$ be an exhaustion of compact subsets of X.

THEOREM 4. *Assume that G satisfies the domination principle and $\check{P}$ satisfies (R_σ). Let F be a non-negligible closed subset of X and μ be a positive measure on X such that each $u\in\check{P}_\sigma$ is μ-integrable. Then there exists a balayaged measure ν of μ onto F satisfying*

$$\int G\nu d\tau=\lim_{n\to\infty}\int\check{G}\lambda_n d\mu \qquad \text{for each } \tau\in\check{\mathfrak{F}}. \tag{4.1}$$

Here λ_n is a balayaged measure of τ onto $F\cap K_n$ with respect to $\check{G}$.

PROOF. Since for each $\tau\in\check{\mathfrak{F}}$ $G\mu$ is τ-integrable, the set $Q:=\{x\in X: G\mu(x)=\infty\}$ is negligible. Put

$$F_n:=F\cap K_n\cap\{x\in X:\ G\mu(x)\leqq n\}$$

and let ν_n be a balayaged measure of μ onto F_n with respect to G. Then $G\nu_n\leqq G\mu$, $G\nu_n=G\mu$, n.e. on F_n. Remark that the energy of ν_n is finite. By Theorem 3 there exists a subsequence $\{\nu_{n_j}\}$ of $\{\nu_n\}$ such that $\{\nu_{n_j}\}$ converges to vaguely a positive measure ν and $\varliminf_{j\to\infty} G\nu_{n_j}=G\nu$ n.e. on X, $\lim_{j\to\infty}\int G\nu_{n_j}d\tau=\int G\nu d\tau$ for all $\tau\in\check{\mathfrak{F}}$ and $G\nu\leqq G\mu$. Simply we use $\{\nu_i\}$ instead of $\{\nu_{n_j}\}$. Further, let $\{\lambda_m\}$ be a balayaged measure of τ onto $F\cap K_m$ with respect to $\check{G}$. Then $\check{G}\lambda_m=\check{G}\tau$ n.e. on $F\cap K_m$ and $\check{G}\lambda_m\leqq\check{G}\tau$ everywhere. Consequently the energy of λ_m is finite. Since $\check{G}\lambda_m=\check{G}\tau$ n.e. on $S\nu_i$ for all $m\leqq i$ and $\nu_i\in\check{\mathfrak{F}}$, it holds that, for each $\tau\in\check{\mathfrak{F}}$

$$\int G\nu d\tau=\lim_{i\to\infty}\int G\nu_i d\tau=\lim_{i\to\infty}\int\check{G}\tau d\nu_i=\lim_{i\to\infty}\lim_{m\to\infty}\int\check{G}\lambda_m d\nu_i$$

$$=\lim_{i\to\infty}\lim_{m\to\infty}\int G\nu_i d\lambda_m .$$

From Proposition 1, it follows that the inequality $G\nu_i=G\mu=G\nu_{i+1}$ n.e. on $S\nu_i$ implies $G\nu_i\leqq G\nu_{i+1}$ n.e. on X. Consequently

$$\int G\nu_i d\lambda_m\leqq\int G\nu_{i+1}d\lambda_m \qquad \text{for all } m\in \boldsymbol{N}.$$

Similarly,

$$\int\check{G}\lambda_m d\nu_i\leqq\int\check{G}\lambda_{m+1}d\nu_i \qquad \text{for all } i\in \boldsymbol{N}.$$

Therefore $\lim_{m\to\infty}\lim_{i\to\infty}\int G\nu_i d\lambda_m$ also exists and is equal to $\lim_{i\to\infty}\lim_{m\to\infty}\int G\nu_i d\lambda_m$. Since $\int_{CF_i}G\nu_i d\lambda_m\leqq\int_{\{G\mu>i\}}G\mu d\lambda_m$, $\int_{CF_i}G\nu_i d\lambda_m$ converges to zero as $i\to\infty$.
Hence we have

$$\lim_{i\to\infty}\lim_{m\to\infty}\int G\nu_i d\lambda_m=\lim_{m\to\infty}\lim_{i\to\infty}\int G\nu_i d\lambda_m$$

$$=\lim_{m\to\infty}\lim_{i\to\infty}\left\{\int_{F_i} G\nu_i d\lambda_m+\int_{CF_i} G\nu_i d\lambda_m\right\}$$

$$=\lim_{m\to\infty}\lim_{i\to\infty}\int_{F_i} G\nu_i d\lambda_m$$

$$=\lim_{m\to\infty}\lim_{i\to\infty}\int_{F_i} G\mu d\lambda_m=\lim_{m\to\infty}\int G\mu d\lambda_m .$$

Therefore we have the conclusion.

From (4.1) it follows that

COROLLARY 1. *The potential $G\nu$ of a balayaged measure ν satisfying* (4.1) *is uniquely determined up to a negligible set.*

COROLLARY 2. *If $G\lambda\geqq G\mu$ n.e. on F, then $G\lambda\geqq G\nu$ n.e. on X.*

PROOF. From (4.1) it follows that, for each $\tau\in\mathcal{T}$

$$\int G\nu d\tau=\lim_{m\to\infty}\int\check{G}\lambda_m d\mu=\lim_{m\to\infty}\int G\mu d\lambda_m\leqq\lim_{m\to\infty}\int G\lambda d\lambda_m$$

$$=\lim_{m\to\infty}\int\check{G}\lambda_m d\lambda\leqq\int\check{G}\tau d\lambda=\int G\lambda d\tau .$$

Hence $G\nu\leqq G\lambda$ n.e. on X.

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Added in proof

It is easy to see that, if a convex cone P in $C(X)$ satisfies (R_σ), P_σ also does (R_σ). Using this, we can proof more easily Proposition 4 by the same method used in the proof of Proposition 3.