Rotation-Invariant Cylindrical Measures

Michie Maeda

Department of Mathematics, Faculty of Science,
Ochanomizu University, Tokyo
(Received September 1, 1980)

Introduction.

In 1962, L. Gross introduced the notion "*measurable norms*" ([5]), which has played an important part in the successive researches. R. Dudly, J. Feldman and L. LeCam presented in [4] much generalization of this idea. They did not treat only Gaussian cylindrical measures. This is the point improved greatly as compared with the preceding one. Through this paper we shall call the notion introduced by L. Gross "*μ-measurable by projections*" and the later one simply "*μ-measurable*", according to [4], where μ is a cylindrical measure. If μ is the canonical Gaussian cylindrical measure on a real separable Hilbert space, then the above two notions exactly coincide with each other. However this fact is not trivial. Most techniques in the proof depend on the rotation-invariance property rather than on the characterization of the canonical Gaussian cylindrical measure. We notice this fact and consider the two measurabilities in the case of rotation-invariant cylindrical measures. It will answer partially the following question offered by A. Badrikian and S. Chevet ([2]). "*Does the measurability always imply the measurability in the sense of Gross*?"

§1. Measurable semi-norms.

First of all, we shall present two definitions of measurability which are interpreted in the introduction. We consider only the case of Hilbert spaces. L. Gross proved in [6] that a measurable norm is continuous. But here we assume the continuity of semi-norms.

Let H be a real separable Hilbert space and $p(\cdot)$ be a continuous semi-norm defined on H. Let $(a_1, \cdots, a_n)$ be a finite system of elements of H. Then by $\boldsymbol{a}$ we denote the operator from H into R_n mapping x onto the vector $((x, a_1), \cdots, (x, a_n))$, where $(\cdot, \cdot)$ is an inner product defined on H. By $\mathfrak{B}(R_n)$ we denote the Borel field on R_n. A set $Z \subset H$ is said to be a cylindrical set if there are $a_1, \cdots, a_n \in H$ and $B \in \mathfrak{B}(R_n)$ such that $Z = \boldsymbol{a}^{-1}(B)$. A map μ from the algebra of all cylindrical sets into [0, 1] is called a cylindrical measure if it satisfies the following two conditions:

(i) $\mu(H)=1$.

(ii) Restrict μ to the σ-algebra of cylindrical sets which are generated by a fixed finite system of functionals. Then each such restriction is σ-additive. Denote by $\mu_{a_1\cdots a_n}$ the restriction by the system $(a_1, \cdots, a_n)$.

For any vector space X let $FD(X)$ be the set of all finite-dimensional subspaces of X. Let P be an orthogonal projection of H with $\dim P(H)=n$ and $C=P^{-1}(D)$ for some $D\in\mathfrak{B}(P(H))$. Define γ by

$$\gamma(P^{-1}(D))=(2\pi)^{-n/2}\int_D \exp\{-(x, x)/2\}\,dx\,,$$

where dx denotes the Lebesgue measure on $P(H)$. Then γ is a cylindrical measure on H and is said the canonical Gaussian cylindrical measure. Through the paper we denote by γ the canonical Gaussian cylindrical measure on H.

DEFINITION. For any cylindrical measure μ on H, a continuous semi-norm $p(\cdot)$ will be called to be *μ-measurable by projections* if for any $\varepsilon>0$ there exists $G\in FD(H)$ such that

$$\mu((N_\varepsilon\cap F)+F^\perp)\geqq 1-\varepsilon$$

for every $F\in FD(H)$ satisfying $F\perp G$, where $N_\varepsilon=\{x\in H\,;\,p(x)<\varepsilon\}$ and $F^\perp$ is the orthogonal complementary set of F.

REMARK. This idea was introduced by L. Gross. He called it "*measurable*" but we call it "*measurable by projections*" in distinction from the next definition.

DEFINITION. For any cylindrical measure μ on H, a continuous semi-norm $p(\cdot)$ will be called to be *μ-measurable* if for any $\varepsilon>0$ there exists $G\in FD(H)$ such that

$$\mu(P_F(N_\varepsilon)+F^\perp)\geqq 1-\varepsilon$$

for every $F\in FD(H)$ satisfying $F\perp G$, where P_F is the orthogonal projection of H onto F.

The following result is trivial.

PROPOSITION 1.1. *If $p(\cdot)$ is μ-measurable by projections, then it is μ-measurable.*

Let $p(\cdot)$ be a continuous semi-norm on H. Consider the quotient space $H/p^{-1}(0)$, then $p(\cdot)$ induces a norm on this space, denote by $p^*(\cdot)$. Let us take this norm $p^*(\cdot)$ and complete $H/p^{-1}(0)$ with respect to $p^*(\cdot)$. We call it the Banach space induced by H and $p(\cdot)$.

Now we present the theorems, due to R. Dudley, J. Feldman and L. LeCam ([4]).

THEOREM 1.2. *Let $p(\cdot)$ be a continuous semi-norm on H. The following statements are equivalent.*

(i) *$p(\cdot)$ is μ-measurable.*

(ii) *Let E be the Banach space induced by H and $p(\cdot)$, and i be the canonical map of H into E, then $i(\mu)$ can be extended to a Radon measure on E (i.e., $i(\mu)$ is σ-additive).*

THEOREM 1.3. *Let $p(\cdot)$ be a continuous semi-norm on H, then the next two statements are equivalent.*

(i) *$p(\cdot)$ is γ-measurable.*

(ii) *$p(\cdot)$ is γ-measurable by projections.*

§ 2. Gross's inequalities and rotation-invariant cylindrical measures.

A cylindrical measure μ on H is called to be rotation-invariant if whenever Z is a cylindrical set and U an isometric operator from H onto H, $\mu(Z)=\mu(U(Z))$. Denote by δ_0 the Dirac's measure having the total mass 1 at the origin. Clearly, the canonical Gaussian cylindrical measure and δ_0 are rotation-invariant.

We shall start with the following lemma (cf. [2] and [8]).

LEMMA 2.1. *Suppose that H is infinite dimensional. Let μ be a rotation-invariant cylindrical measure on H. Then there exists a Borel probability measure σ_μ on $[0, \infty)$ such that*

$$(*)\qquad \mu_{e_1\cdots e_n}(A)=\int_{t>0}\gamma_n\left(\frac{A}{t}\right)d\sigma_\mu(t)+\sigma_\mu(\{0\})\delta_0(A)$$

$$=\int_A\left(\int_{t>0}\exp\left(-\frac{1}{2t^2}\|x\|^2\right)\frac{d\sigma_\mu(t)}{(\sqrt{2\pi}\,t)^n}\,dm_n(x)+\sigma_\mu(\{0\})\delta_0(A)\right.$$

for every $A\in\mathfrak{B}(R_n)$ and for every finite system $(e_1, \cdots, e_n)\subset H$ such that $(e_i, e_j)=\delta_{ij}$.

REMARK. In the above lemma, we denote by γ_n the canonical Gaussian measure on R_n, by $\|\cdot\|$ the usual norm on R_n and by m_n the Lebesgue measure on R_n.

Now we are in a position to prove the several inequalities, which proved by L. Gross in the case of γ.

LEMMA 2.2. *Under the hypothesis of Lemma 2.1, there exists a non-negative function $\Phi(r)$ defined on $[0, \infty)$ such that*

$$\mu_{e_1\cdots e_n}(A)=\int_0^\infty\Phi(r)m_n(A\cap S_r)dr+\sigma_\mu(\{0\})\delta_0(A),$$

where $S_r=\{x\,;\ x\in R_n,\ \|x\|\leqq r\}$.

RROOF. By virtue of Lemma 2.1, we have

$$\mu_{e_1\cdots e_n}(A)=\int_A\Big(\int_{t>0}\exp\Big(-\frac{1}{2t^2}\|x\|^2\Big)\frac{d\sigma_\mu(t)}{(\sqrt{2\pi}\,t)^n}\Big)dm_n(x)+\sigma_\mu(\{0\})\delta_0(A).$$

Define $\Psi(r)$ and $\varphi(t, r)$ by

$$\Psi(r)=\int_{t>0}\frac{1}{(\sqrt{2\pi}\,t)^n}\exp\Big(-\frac{1}{2t^2}r^2\Big)d\sigma_\mu(t) \qquad \text{and}$$

$$\varphi(t, r)=\frac{1}{(\sqrt{2\pi}\,t)^n}\exp\Big(-\frac{1}{2t^2}r^2\Big).$$

It is easy to see that

$$\int_a^\infty-\frac{\partial\varphi(t, r)}{\partial r}dr=\varphi(t, a) \quad \text{for every } a\in[0, \infty)$$

Then

$$\Psi(a)=\int_{t>0}\varphi(t, a)d\sigma_\mu(t)$$

$$=\int_{t>0}\Big(\int_a^\infty-\frac{\partial\varphi(t, r)}{\partial r}dr\Big)d\sigma_\mu(t)$$

$$=\int_a^\infty\Big(\int_{t>0}-\frac{\partial\varphi(t, r)}{\partial r}d\sigma_\mu(t)\Big)dr.$$

Define $\Phi(r)$ by

$$\Phi(r)=\int_{t>0}-\frac{\partial\varphi(t, r)}{\partial r}d\sigma_\mu(t),$$

that is,

$$\Phi(r)=\int_{t>0}\frac{r}{t^2}\varphi(t, r)d\sigma_\mu(t)$$

$$=\int_{t>0}\frac{r}{(\sqrt{2\pi})^n t^{n+2}}\exp\Big(-\frac{1}{2t^2}r^2\Big)d\sigma_\mu(t).$$

Since $\Psi(a)=\int_a^\infty\Phi(r)dr$, we have

$$\mu_{e_1\cdots e_n}(A)=\int_A\Psi(\|x\|)dm_n(x)+\sigma_\mu(\{0\})\delta_0(A)$$

$$=\int_A\Big(\int_0^\infty\Phi(r)1_{[\|x\|,\infty)}(r)dr\Big)dm_n(x)+\sigma_\mu(\{0\})\delta_0(A)$$

$$=\int_0^\infty\Big(\int_A\Phi(r)1_{[\|x\|,\infty)}(r)dm_n(x)\Big)dr+\sigma_\mu(\{0\})\delta_0(A)$$

$$=\int_0^\infty\Phi(r)m_n(A\cap S_r)dr+\sigma_\mu(\{0\})\delta_0(A).$$

LEMMA 2.3. *Let T be a linear symmetric invertible operator of R_n onto R_n and C be a closed convex centrally symmetric set in R_n. If $\|T^{-1}\|\leqq 1$ then* $\mu_{e_1\cdots e_n}(T(C))\geqq\mu_{e_1\cdots e_n}(C)$.

PROOF. Clearly, we can neglect the second part of the representation of $\mu_{e_1\cdots e_n}$ in the previous lemma. In [5], L. Gross proved that $m_n(T(C)\cap S_r)\geqq m_n(C\cap S_r)$ for all $r>0$. Therefore, by Lemma 2.2, we obtain easily our consequence in the lemma.

Successive lemmas are deduced from Lemma 2.3, therefore their proofs have the same processes as the case of the canonical Gaussian cylindrical measure.

LEMMA 2.4. *Let E be a n-dimensional Hilbert space and σ be a Borel probability measure on $[0, \infty)$. Let μ be a rotation-invariant probability measure on E defined as follows:*

$$\mu(A)=\int_A\left(\int_{t>0}\exp\left(-\frac{1}{2t^2}\|x\|_E^2\right)\frac{d\sigma(t)}{(\sqrt{2\pi}\, t)^n}\right)dm(x)+\sigma(\{0\})\delta_0(A)$$

for every $A\in\mathfrak{B}(E)$, where $\|\cdot\|_E$ is the norm on E and m denotes the Lebesgue measure on E. Let E_1 be a linear subspace of E and C be a closed convex centrally symmetric subset of E, then

$$\mu(C)\leqq\mu(C\cap E_1+E_1^{\perp}).$$

PROOF. We can assume that $E_1\subsetneqq E$. Let P be the orthogonal projection of E onto E_1, I be the identity operator of E. Define $P^{\perp}=I-P$ and $T_m=mP^{\perp}+P$ for every integer $m>1$, then T_m is a linear symmetric invertible operator of E. Clearly, we have $\|T_m^{-1}\|<1$. If $\{e_i\,;\,1\leqq i\leqq n\}$ is an orthonormal basis of E, then the mapping $x=\sum_{i=1}^{n}x_ie_i\mapsto(x_i)$ defines an isomorphism from E onto R_n. Therefore, Lemma 2.3 says that

$$\mu(T_m(C))\geqq\mu(C)$$

for every integer $m>1$.

By Fatou's lemma we have

$$\mu(\limsup_{m\to\infty} T_m(C))\geqq\mu(C).$$

And $P^{-1}(C)=P^{-1}(C\cap E_1)=C\cap E_1+E_1^{\perp}$. Hence, in order to complete the proof, we only have to prove that

$$P^{-1}(C)\supset\limsup_{m\to\infty} T_m(C),\ i.e.,\ E\backslash P^{-1}(C)\subset\liminf_{m\to\infty} T_m(E\backslash C).$$

For any $x\in E\backslash P^{-1}(C)$, there exists $\varepsilon>0$ such that $S_\varepsilon+Px\subset E\backslash C$, where $S_\varepsilon=\{x\in E\,;\,\|x\|<\varepsilon\}$. Choose an integer m such that $m>\|P^{\perp}x\|/\varepsilon$. Then

$$T_m^{-1}x=Px+\frac{1}{m}P^{\perp}x\in Px+S_\varepsilon\subset E\backslash C.$$

Hence we have $x\in\liminf_{m\to\infty} T_m(E\backslash C)$. Thus

$$E\backslash P^{-1}(C)\subset\liminf_{m\to\infty} T_m(E\backslash C).$$

REMARK. Given any Borel probability measure σ on $[0, \infty)$, there exists a rotation-invariant cylindrical measure μ on a Hilbert space H satisfying the relation (*) in Lemma 2.1. In this case, we call μ the rotation-invariant cylindrical measure induced by σ. In particular, if H is infinite dimensional, the above correspondence between σ and μ is a bijection.

LEMMA 2.5. *Let H_1, H_2 be real separable Hilbert spaces, σ be a Borel probability measure on $[0, \infty)$ and μ_1, μ_2 be the rotation-invariant cylindrical measures induced by σ on H_1 and on H_2 respectively. Let U be a continuous linear operator of H_1 into H_2, C_0 be a closed convex centrally symmetric set of some finite dimensional subspace of H_2 and C be a cylindrical set with the base C_0 (if H_2 is finite dimensional, then C denotes a closed convex centrally symmetric set of H_2). If $\|U\| \leqq 1$, then $\mu_1(U^{-1}(C)) \geqq \mu_2(C)$.*

PROOF. (I) Suppose that H_2 is a finite dimensional Hilbert space and U is a bijection. Then H_1 has the same dimension as H_2 has. We can decompose U as follows:

$$U = I \circ U_1,$$

where I is a linear isometric operator from H_1 onto H_2 and U_1 is a linear symmetric invertible operator of H_1 such that $\|U\| = \|U_1\|$. Since $\|U_1\| \leqq 1$, we have $\mu_1(U^{-1}(C)) = \mu_1(U_1^{-1}(I^{-1}(C))) \geqq \mu_1(I^{-1}(C)) = I(\mu_1)(C) = \mu_2(C)$. Hence the proof is complete in this case.

(II) Consider next the case that H_2 is same as (I) and U is a general form. Define $(U^{-1}(0))^{\perp} = K_1$ and $U(H_1) = K_2$. Let P_{K_1} be an orthogonal projection of H_1 onto K_1 and V be a linear bijection from K_1 onto K_2. It is easy to see that $V \circ P_{K_1}(x) = U(x)$ for all $x \in H_1$ and that $\|V\| \leqq 1$. Let μ_{K_i} be the rotation-invariant cylindrical measure on the Hilbert space K_i induced by σ, for $i = 1, 2$. We have

$$\begin{aligned} \mu_1(U^{-1}(C)) &= \mu_1(P_{K_1}^{-1}(V^{-1}(C \cap K_2))) \\ &= P_{K_1}(\mu_1)(V^{-1}(C \cap K_2)) \\ &= \mu_{K_1}(V^{-1}(C \cap K_2)). \end{aligned}$$

Since $C \cap K_2$ is a closed convex centrally symmetric subset of K_2, we can apply the consequence of (I). Then

$$\mu_{K_1}(V^{-1}(C \cap K_2)) \geqq \mu_{K_2}(C \cap K_2).$$

Therefore,

$$\mu_1(U^{-1}(C)) \geqq \mu_{K_2}(C \cap K_2) = \mu_2(C \cap K_2 + K_2^{\perp}).$$

Hence Lemma 2.4 says the conclusion.

(III) Now consider the case that H_2 is infinite dimensional. C is a cylindrical set, then there exists a finite dimensional subspace N_2 of H_2 such that $C_0 \subset N_2$ and $P_{N_2}^{-1}(C_0) = C$, where P_{N_2} is the orthogonal projection of H_2 onto N_2. It is clear that $\|P_{N_2} \circ U\| \leqq 1$ and that

$$\mu_1(U^{-1}(C))=\mu_1(U^{-1}(P_{N_2}^{-1}(C_0)))$$
$$=\mu_1((P_{N_2}\circ U)^{-1}(C_0)).$$

Apply (II), then we have

$$\mu_1((P_{N_2}\circ U)^{-1}(C_0))\geqq\mu_{N_2}(C_0),$$

where μ_{N_2} is the rotation-invariant cylindrical measure on N_2 induced by σ. Since $\mu_{N_2}(C_0)=\mu_2(C)$, we have $\mu_1(U^{-1}(C))\geqq\mu_2(C)$.

§ 3. Random functions.

Here we present another notion. We continue to assume that H is a real separable Hilbert space and μ is a cylindrical measure on H. Then there exists a pair of a probability measure space and a linear random function associated with μ (see, e.g., [2] or [7]), write (Ω, P) and Λ. Λ is a mapping from H' into $L^0(\Omega, P; \bar{R})$, where H' is a topological dual space of H, $\bar{R}$ is the extended real number field and $L^0(\Omega, P; \bar{R})$ is the space of all equivalence classes of measurable functions defined on Ω into $\bar{R}$ with respect to P. Define

$$\Lambda_A(\cdot)=\sup_{x'\in A}|\Lambda(x')(\cdot)|$$

for any subset $A\subset H'$, then clearly, we have

$$\Lambda_A(\cdot)\in L^0(\Omega, P; \bar{R}).$$

Let S be an arbitrary set. We denote by S^0 the polar of S and by *card.* S the cardinal number of S.

LEMMA 3.1. *For every real number* $t\geqq 0$,

$$P(\Lambda_A\leqq t)=\inf_S\{\mu(tS^0)\,;\ S\subset A,\ \textit{and card. } S<\infty\}.$$

PROOF. Let S be a finite subset of A. Since Λ is the associated random function with μ, we have $\mu(tS^0)=P(\Lambda_S\leqq t)$ for all $t\geqq 0$. Hence

$$P(\Lambda_A\leqq t)=\inf_S\{\mu(tS^0)\,;\ S\subset A,\ \textit{and card. } S<+\infty\}.$$

PROPOSITION 3.2. *Let μ be a rotation-invariant cylindrical measure on H, (Ω, P) and Λ be a probability measure space and a random function associated with μ.*

(I) *Let u' be a continuous linear operator of H' into H' such that $\|u'\|\leqq 1$. For any $t\geqq 0$ and for any subset $A\subset H'$, we have $P(\Lambda_A\leqq t)\leqq P(\Lambda_{u'(A)}\leqq t)$.*

(II) *Let Q_1, Q_2 be two orthogonal projections of H' such that $Q_1(H')\subset Q_2(H')$. For any $\varepsilon\geqq 0$ and for any subset $C\subset H'$, we have $P(\Lambda_{Q_1(C)}>\varepsilon)\leqq P(\Lambda_{Q_2(C)}>\varepsilon)$.*

PROOF. (I) By virtue of Lemma 3.1, we can (and do) suppose that A is finite. Let u be the adjoint of u'. Lemma 2.5 says that $\mu(tA^0)\leqq\mu(u^{-1}(tA^0))$. Therefore,

$$\begin{aligned}P(\Lambda_A \leqq t) &= \mu(tA^0)\\ &\leqq \mu(u^{-1}(tA^0))\\ &= \mu(t(u'(A))^0)\\ &= P(\Lambda_{u'(A)} \leqq t).\end{aligned}$$

(II) Since $Q_1(H') \subset Q_2(H')$, we have $Q_1 = Q_1 \circ Q_2$. In order to apply (I), we take $u' = Q_1$, $A = Q_2(C)$ and $t = \varepsilon$. Therefore,

$$P(\Lambda_{Q_2(C)} \leqq \varepsilon) \leqq P(\Lambda_{Q_1 \circ Q_2(C)} \leqq \varepsilon) = P(\Lambda_{Q_1(C)} \leqq \varepsilon).$$

Hence $P(\Lambda_{Q_1(C)} > \varepsilon) \leqq P(\Lambda_{Q_2(C)} > \varepsilon)$.

LEMMA 3.3. *Let μ be a rotation-invariant cylindrical measure on H, (Ω, P) and Λ be a probability measure space and a random function associated with μ. Let A be a directed set, $(\pi_\alpha)_{\alpha \in A}$ be a directed family of orthogonal projections of H' such that $(\pi_\alpha(x'))_{\alpha \in A}$ converges to x' for each $x' \in H'$, and C be a closed convex centrally symmetric subset of H'. Then*

$$P(\Lambda_C \leqq t) = \lim_\alpha P(\Lambda_{\pi_\alpha(C)} \leqq t)$$

for any $t > 0$.

PROOF. It follows from Proposition 3.2 that

$$P(\Lambda_C \leqq t) \leqq \liminf_\alpha P(\Lambda_{\pi_\alpha(C)} \leqq t).$$

Hence it is sufficient to show that

$$\limsup_\alpha P(\Lambda_{\pi_\alpha(C)} \leqq t) \leqq P(\Lambda_C \leqq t).$$

Now we recall the fact that every rotation-invariant cylindrical measure is of type 0, i.e., the linear random function Λ is continuous from H' into L^0 equipped with the topology of convergence in probability. Thus $(\Lambda(\pi_\alpha(x')))_{\alpha \in A}$ converges to $\Lambda(x')$ in L^0 for every $x' \in H'$. Let S be an arbitrary finite subset of C. It is also obvious that $(\Lambda_{\pi_\alpha(S)})_{\alpha \in A}$ converges to Λ_S in L^0. Then

$$\limsup_\alpha P(\Lambda_{\pi_\alpha(S)} \leqq t) \leqq P(\Lambda_S \leqq t).$$

Therefore, since $P(\Lambda_{\pi_\alpha(C)} \leqq t) \leqq P(\Lambda_{\pi_\alpha(S)} \leqq t)$, we have

$$\limsup_\alpha P(\Lambda_{\pi_\alpha(C)} \leqq t) \leqq P(\Lambda_S \leqq t).$$

Hence, by Lemma 3.1,

$$\limsup_\alpha P(\Lambda_{\pi_\alpha(C)} \leqq t) \leqq P(\Lambda_C \leqq t).$$

LEMMA 3.4. *Let μ be a rotation-invariant cylindrical measure on H, (Ω, P) and Λ be a probability measure space and a random function associated with μ. Then the following statements are equivalent.*

(i) *Given any* $\varepsilon>0$, *there exists* $K_\varepsilon\in FD(H')$ *such that* $P(\Lambda_{\pi_L(C)}>\varepsilon)<\varepsilon$ *for all* $L\in FD(H')$ *satisfying* $L\perp K_\varepsilon$, *where* π_L *is the orthogonal projection of* H' *onto* L.

(ii) *Given any* $\varepsilon>0$, *there exists* $K_\varepsilon\in FD(H')$ *such that* $P(\Lambda_{\pi_{K_\varepsilon^\perp}(C)}>\varepsilon)<\varepsilon$.

PROOF. It is clear that $\pi_{K_\varepsilon^\perp}(H')\supset\pi_L(H')$. Then it follows from Proposition 3.2 that

$$P(\Lambda_{\pi_{K_\varepsilon^\perp}(C)}>\varepsilon)\geqq P(\Lambda_{\pi_L(C)}>\varepsilon)\,.$$

Therefore we only have to show that (i) implies (ii). Let $\{M_\varepsilon^n\}_{n=1,2,\dots}$ be a chain of increasing finite dimensional subspaces of H' such that $\bigcup_n M_\varepsilon^n$ is dense in H' and $\bigcap_n M_\varepsilon^n\supset K_\varepsilon$. By Lemma 3.3, we have

$$P(\Lambda_{\pi_{K_\varepsilon^\perp}(C)}\leqq\varepsilon)=\lim_n P(\Lambda_{\pi_{M_\varepsilon^n}\circ\pi_{K_\varepsilon^\perp}(C)}\leqq\varepsilon)\,.$$

Observe that

$$\pi_{M_\varepsilon^n}\circ\pi_{K_\varepsilon^\perp}=\pi_{M_\varepsilon^n\cap K_\varepsilon^\perp}\,.$$

By (i), we have

$$P(\Lambda_{\pi_{M_\varepsilon^n\cap K_\varepsilon^\perp}(C)}>\varepsilon)<\varepsilon\,.$$

This implies that

$$P(\Lambda_{\pi_{M_\varepsilon^n\cap K_\varepsilon^\perp}(C)}\leqq\varepsilon)>1-\varepsilon\,.$$

Therefore, we have

$$P(\Lambda_{\pi_{K_\varepsilon^\perp}(C)}\leqq\varepsilon)\geqq1-\varepsilon\,,$$

and so,

$$P(\Lambda_{\pi_{K_\varepsilon^\perp}(C)}>\varepsilon)\leqq\varepsilon\,.$$

REMARK. Note that the following two conditions are equivalent.

(i) Given any $\varepsilon>0$, there exists a family $\{f_\alpha\}\subset L^0(\Omega, P;\bar{R})$ such that $P(f_\alpha>\varepsilon)\leqq\varepsilon$ for all α.

(ii) Given any $\varepsilon>0$, there exists a family $\{f_\alpha\}\subset L^0(\Omega, P;\bar{R})$ such that $P(f_\alpha>\varepsilon)<\varepsilon$ for all α.

Let C be a closed convex centrally symmetric bounded subset of H' and C^0 be the polar of C. Define a semi-norm p_{C^0} on H as follows:

$$p_{C^0}(x)=\inf\{\lambda\,|\,\lambda\geqq0,\ x\in\lambda C^0\}$$

for all $x\in H$. Clearly $p_{C^0}(x)$ is continuous.

PROPOSITION 3.5. *Let* μ *be a cylindrical measure (not only rotation-invariant) on* H *and* C *be a closed convex centrally symmetric bounded subset of* H'. *Then the following statements are equivalent.*

(i) p_{C^0} *is* μ*-measurable by projections.*

(ii) *Given any* $\varepsilon>0$, *there exists* $K_\varepsilon\in FD(H')$ *such that* $P(\Lambda_{\pi_L(C)}>\varepsilon)<\varepsilon$ *for all* $L\in FD(H')$ *satisfying* $L\perp K_\varepsilon$.

PROOF. It is easy to see that

$$P(\Lambda_{\pi_L(C)}\leqq\varepsilon)=\mu(\pi_L^{-1}(\varepsilon C^0))$$
$$=\mu(\varepsilon C^0\cap L+L^{\perp}).$$

Obviously, we have

$$\{x\in H;\ p_{C^0}(x)<\varepsilon\}\subset\varepsilon C^0\subset\{x\in H;\ p_{C^0}(x)\leqq\varepsilon\}.$$

Thus the desired conclusion follows immediately.

§4. Two measurabilities.

In this section, we prove the main theorems which show the equivalence of two measurabilities of continuous semi-norms with respect to rotation-invariant cylindrical measures. At the beginning, we introduce the following two notions. Let H be a real separable Hilbert space as ever, μ be a cylindrical measure on H, (Ω, P) and Λ be a probability measure space and a random function associated with μ. Let C be a subset of H'. It is called a μ-continuity set with respect to Λ if there exists a version of Λ, say λ, such that $\lambda(x, \omega)$ $(x\in H', \omega\in\Omega)$ is continuous relative to x on C for almost all $\omega\in\Omega$. Also, it is called a μ-bounded set with respect to Λ if there exists $g(\omega)\in L^0(\Omega, P; \bar{R})$ such that $|f(x, \omega)|\leqq g(\omega)$ for almost all $\omega\in\Omega$, for all $f\in\Lambda(x)$ and for all $x\in C$.

Now, let us start with the next lemma.

LEMMA 4.1. *Let μ be a rotation-invariant cylindrical measure on H and (Ω, P) and Λ be a probability measure space and a random function associated with μ. Let C be a compact convex centrally symmetric subset of H'.*

If C is a μ-continuity set, then p_{C^0} is μ-measurable by projections.

REMARK. Recall that p_{C^0} is a continuous semi-norm.

PROOF. Let D be a countable dense subset of C. The assumption that C is a μ-continuity set with respect to Λ implies the existence of the version $\lambda(x, \omega)$ of Λ which is continuous on C for almost all $\omega\in\Omega$. It follows from the compactness of C that

$$P\Big(\bigcup_{k=1}^{\infty}\bigcap_{\substack{(x,y)\in D\times D\\ \|x-y\|\leqq 1/k}}\{\omega;\ |\lambda(x-y, \omega)|\leqq\varepsilon\}\Big)=1$$

for all $\varepsilon>0$. This induces the following:

(**) Given any $\varepsilon>0$, there exists a $\delta>0$ such that

$$P\Big(\Big\{\omega;\ \sup_{\substack{(x,y)\in D\times D\\ \|x-y\|<\delta}}|\lambda(x-y, \omega)|>\varepsilon\Big\}\Big)<\varepsilon.$$

D is relatively compact in H'. Then, we have a subspace $F\in FD(H')$ such that $\sup_{x\in D}(\inf_{y\in F\cap D}\|x-y\|)<\delta$. Let π be an orthogonal projection onto F and $\pi^{\perp}=I-\pi$, where I is an identity operator. It follows from Proposition 3.2 that

$$P\Big(\Big\{\omega;\ \sup_{\substack{(x,y)\in D\times D\\ \|x-y\|<\delta}}|\lambda(\pi^{\perp}(x-y), \omega)|>\varepsilon\Big\}\Big)$$

$$\leqq P(\{\omega;\ \sup_{\substack{(x,y)\in D\times D\\ \|x-y\|<\delta}} |\lambda(x-y,\omega)|>\varepsilon\})\,.$$

For any $x\in D$, we have $y\in F\cap D$ satisfying $\|x-y\|<\delta$ and $\pi^{\perp}y=0$. Therefore we obtain

$$P(\Lambda_{\pi^{\perp}C}>\varepsilon)=P(\Lambda_{\pi^{\perp}D}>\varepsilon)$$
$$\leqq P(\{\omega;\ \sup_{\substack{(x,y)\in D\times D\\ \|x-y\|<\delta}} |\lambda(x-y,\omega)|>\varepsilon)\,.$$

Then (**) implies that for any $\varepsilon>0$ there exists a finite rank orthogonal projection π such that

$$P(\Lambda_{\pi^{\perp}C}>\varepsilon)<\varepsilon\,.$$

Using Lemma 3.4 and Proposition 3.5, we can complete the proof.

The following lemma has been proved by A. Badrikian more generally ([1]). But, here we offer only the part which is necessary for successive arguments.

LEMMA 4.2. *Let μ be a cylindrical measure on H, (Ω, P) and Λ be the associated pair of a probability measure space and a random function. Let $p(\cdot)$ be a continuous semi-norm on H, E be the Banach space induced by H and $p(\cdot)$, and i be the canonical map of H into E. Assume that $i(\mu)$ can be extended to a Radon measure on E. Set $A=\{x\in H;\ p(x)\leqq 1\}^{0}$, then A is a μ-continuity set with respect to Λ.*

PROOF. Let $i(\mu)=\nu$, and let ${}^{t}i$ be the transpose of i. Then $\Lambda\circ{}^{t}i$ is the associated random function of ν. By the assumption, ν can be extended to a Radon measure on E. The theorem of L. Schwartz ([7]) says that there exists a P-Lusin measurable E-valued function φ defined on Ω such that the equivalence class of $\langle\varphi(\cdot), x'\rangle$ is equivalent to $\Lambda\circ{}^{t}i(x')$ for every $x'\in E'$, where we denote by $\langle\cdot,\cdot\rangle$ the canonical bilinear functional defined on $E\times E'$. Since ${}^{t}i$ is injective, we can consider E' as the linear subspace of H' and also A as the unit ball of E'. Therefore, this implies the conclusion.

Now we are in a position to prove the main theorem.

THEOREM 4.3. *Let H be a real separable infinite dimensional Hilbert space and μ be a rotation-invariant cylindrical measure on H with $\sigma_{\mu}(\{0\})=0$, where σ_{μ} is the associated Borel probability measure on $[0, \infty)$ (cf. Lemma 2.1). Let (Ω, P) and Λ be the pair of a probability measure space and a random function associated with μ, and $p(\cdot)$ be a continuous semi-norm on H. Then the following statements are equivalent.*

(i) *$p(\cdot)$ is μ-measurable.*

(ii) *Let E be the Banach space induced by H and $p(\cdot)$, and i be the canonical map of H into E. Then $i(\mu)$ can be extended to a Randon measure on E.*

(iii) *The set $C=\{x\in H;\ p(x)\leqq 1\}^{0}$ is a compact and μ-continuity set.*

(iv) *$p(\cdot)$ is μ-measurable by projections.*

PROOF. Theorem 1.2 says the equivalence of (i) and (ii). Also Proposition 1.1 says that (iv) implies (i). Then it is sufficient to prove that (ii)$\Rightarrow$(iii)$\Rightarrow$(iv). It is easy to see that the set $\{x \in H;\ p(x) \leqq 1\}^0$ is convex centrally symmetric and $p_{C^0} = p$. Therefore, by Lemma 4.1 we have (iii)$\Rightarrow$(iv). Next we shall show that (ii) implies (iii). By Lemma 4.2 we can say that C is a μ-continuity set. Using the function φ which appeared in the proof of Lemma 4.2, we can also say that C is a μ-bounded set. A. Badrikian and S. Chevet proved that a bounded and μ-bounded set is γ-bounded ([2]). Furthermore, it is well known that a bounded and γ-bounded set is relatively compact (see, e.g., [3]). Hence C is compact.

In [2], A. Badrikian and S. Chevet investigated about the relation between the canonical Gaussian cylindrical measure γ and a rotation-invariant cylindrical measure μ with $\sigma_\mu(\{0\})=0$. They have the following:

(***) Let X be a locally convex Hausdorff space over R, and u be a weakly continuous linear operator from H into X. Then $u(\gamma)$ can be extended to a Radon measure on X if and only if $u(\mu)$ is extensible to a Radon measure on X.

It follows from (***) that the notion "γ-continuity set" is equivalent to "μ-continuity set" for any compact convex centrally symmetric sets. Therefore we have the next corollary, which is an immediate consequence of the above results.

COROLLARY 4.4. *Let $p(\cdot)$ be a continuous semi-norm on H and μ be a rotation-invariant cylindrical measure on H with $\sigma_\mu(\{0\})=0$. Then, for $p(\cdot)$, the following all measurabilities are equivalent to one another:*

(i) *γ-measurable,* (ii) *γ-measurable by projections,* (iii) *μ-measurable,* (iv) *μ-measurable by projections.*

Next, let us consider the case of $\sigma_\mu(\{0\}) \neq 0$. First, suppose that $\sigma_\mu(\{0\})=1$. For any continuous semi-norm $p(\cdot)$, we can say that $p(\cdot)$ is both μ-measurable and μ-measurable by projections. Thus we only have to consider the case of $\sigma_\mu(\{0\})=\alpha$ $(0<\alpha<1)$. By the definitions of both rotation-invariant cylindrical measures and measurabilities with respect to μ, we obtain the next lemma. The proof is trivial and so it is omitted.

LEMMA 4.5. *Let μ_1, μ_2 be two rotation-invariant cylindrical measures on H and $p(\cdot)$ be a continuous semi-norm on H. Define μ_3 by $\mu_3(Z)=a\mu_1(Z)+b\mu_2(Z)$ for every cylindrical set Z of H, where a and b are positive real numbers satisfying $a+b=1$. Then μ_3 is the rotation-invariant cylindrical measure on H.*

Furthermore, if $p(\cdot)$ is both μ_1-measurable (resp. μ_1-measurable by projections) and μ_2-measurable (resp. μ_2-measurable by projections), then $p(\cdot)$ is also μ_3-measurable (resp. μ_3-measurable by projections).

"$\sigma_\mu(\{0\})=\alpha$" implies that

$$\mu_{e_1 \cdots e_n}(A) = \int_{t>0} \gamma_n\left(\frac{A}{t}\right) d\sigma_\mu(t) + \alpha\delta_0(A),$$

where notation is same as in Lemma 2.1. Take μ_1 by

$$(\mu_1)_{e_1 \cdots e_n}(A) = \frac{1}{1-\alpha} \int_{t>0} \gamma_n\left(\frac{A}{t}\right) d\sigma_\mu(t)$$

and $\mu_2 = \delta_0$. Then we have $\mu = (1-\alpha)\mu_1 + \alpha\mu_2$. Therefore we can apply the above lemma to μ. Notice that μ_1 is the foregoing type of rotation-invariant cylindrical measure, *i. e.*, $\sigma_{\mu_1}(\{0\}) = 0$. Furthermore, if a continuous semi-norm $p(\cdot)$ is μ-measurable, then $p(\cdot)$ is also μ_1-measurable. Thus we have

THEOREM 4.6. *Let μ be a rotation-invariant cylindrical measure on H and $p(\cdot)$ be a continuous semi-norm. Then the two statements are equivalent.*

(i) *$p(\cdot)$ is μ-measurable.*

(ii) *$p(\cdot)$ is μ-measurable by projections.*

References

1) A. Badrikian: *Propriétés permanentes des trajectoires de processus*, Colloque Analyse Fonctionnelle de Bordeaux (1971), Bull. Soc. Math. France, Mémoire 31-32 (1972), 21-29.

2) A. Badrikian and S. Chevet: *Mesures Cylindriques, Espaces de Wiener et Fonctions Aléatoires Gaussiennes*, Lecture Notes in Mathematics 379, Berlin-Heidelberg-New York, Springer 1974.

3) R. M. Dudley: *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Functional Analysis 1 (1967), 290-330.

4) R. M. Dudley, J. Feldman and L. LeCam: *On seminorms and probabilities, and abstract Wiener spaces*, Ann. Math. **93** (1971), 390-408.

5) L. Gross: *Measurable functions on Hilbert space*, Trans. Amer. Math. Soc. **105** (1962), 372-390.

6) ———: *Abstract Wiener measure and infinite dimensional potential theory*, Lectures in Modern Analysis and Applications II, Lecture Notes in Mathematics 140, Berlin-Heidelberg-New York, Springer (1970), 84-116.

7) L. Schwartz: *Radon measures on arbitrary topological spaces and cylindrical measures*, Bombay, Oxford University Press 1973.

8) Y. Umemura: *Measures on infinite dimensional vector spaces*, Publ. Res. Inst. Math. Sci., Kyoto Univ. Ser. A, 1 (1965), 1-47.