Projective Normality and the Defining Equations of Ample Invertible Sheaves on Elliptic Ruled Surfaces with $e \geqq 0$

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Let $\pi: X \to C$ be an elliptic ruled surface defined over an algebraically closed field k. We define an invariant e of X by $-e = \mathrm{Min}\{C'^2 | C' \text{ is a section of } \pi\}$. Then a section C_0 with $C_0^2 = -e$ is unique up to linear equivalence, and any divisor D on X can be written uniquely $nC_0 + \pi^*\mathfrak{b}$ where $\mathfrak{b}$ is a divisor on an elliptic curve C. According to Hartshorne [1], we denote $\pi^*\mathfrak{b}$ by $\mathfrak{b}f$. A necessary and sufficient condition of ampleness of divisors on X is known in [1], that is:
a divisor $D \sim nC_0 + \mathfrak{b}f$ on an elliptic ruled surface X with invariant e is ample $\Leftrightarrow$ $n>0$ and $\deg \mathfrak{b} > ne$, if $e \geqq 0$; $n>0$ and $\deg \mathfrak{b} > \frac{1}{2}ne$, if $e=-1$.

Our purpose is to give a necessary and sufficient condition for divisors to be normally generated, where a divisor D is said to be normally generated if D is ample and $\Gamma(jD) \otimes \Gamma(D) \to \Gamma((j+1)D)$ is surjective, for every $j \geqq 1$. Our result is as follows.

THEOREM 3.3. *Let X be an elliptic ruled surface with $e \geqq 0$, and let $D \sim nC_0 + \mathfrak{b}f$ be a divisor on X. Then*

$$D \text{ is very ample} \Leftrightarrow n \geqq 1,\ \deg \mathfrak{b} \geqq ne+3$$
$$\Leftrightarrow D \text{ is normally generated.}$$

In this case, $I(D) = \mathrm{Ker}[S\Gamma(D) \to \bigoplus_{j=0}^{\infty} \Gamma(jD)]$ *is generated by its elements of degree* 2 *and* 3.

It is to be regreted that we cannot obtain a similar result for X with $e=-1$*.

Our main tool for the proof is cohomology of invertible sheaves. If X is an elliptic ruled surface corresponding to a decomposable locally free sheaf $\mathcal{E}$ of rank 2 on C, then we can compute the dimension of cohomology of any invertible sheaf on X. The result of this computation is listed in §2. But in case X is one corresponding to an indecomposable locally free sheaf, the author cannot complete the table. It is one of the reasons why we do not discuss normal generation of ample invertible sheaves on X with $e=-1$.

* Added in proof; Recently the author obtained a similar result to Theorem 3.3 also for X with $e=-1$, which will be discussed in a separate paper.

We state our main results in § 3 and prove them in the following sections. In § 7 we refer to rational ruled surfaces. We will have the same kind of result as Theorem 3.3.

NOTATIONS. Throughout this paper, a *variety* is a projective variety over an algebraically closed field k. A *surface* will mean a non-singular projective surface over k, and a *point* will mean a closed point. For a divisor D on a non-singular projective variety V, we denote by $\mathcal{O}_V(D)$ the invertible sheaf associated to D. By abuse of notation, we sometimes use D itself instead of $\mathcal{O}_V(D)$. We denote by $h^i(D)$ the dimension over k of the i-th cohomology $H^i(V, \mathcal{O}_V(D))$ which is sometimes denoted by $H^i(\mathcal{O}_V(D))$ or $H^i(D)$ briefly.

§ 1. Previous results.

We begin by stating the definition of a ruled surface. Some of its general properties are described in [1].

DEFINITION. A *ruled surface* is a surface X together with a surjective morphism $\pi: X \to C$ to a non-singular projective curve C, such that the fibre X_y is isomorphic to $\boldsymbol{P}^1$ for every point $y \in C$, and that π admits a section (i.e., a morphism $\sigma: C \to X$ such that $\pi \circ \sigma = 1_C$). An *elliptic ruled surface* is a ruled surface over an elliptic curve.

If $\pi: X \to C$ is a ruled surface, then it is possible to write $X \cong \boldsymbol{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank 2 on C with the property that $H^0(\mathcal{E}) \neq 0$ and for all invertible sheaves $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$, we have $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$. In this case we say $\mathcal{E}$ is normalized. This does not determine $\mathcal{E}$ uniquely, but it does determine an invariant $e = -\deg \wedge^2 \mathcal{E}$. Note that this definition of e is equivalent to the previous one. We denote by $\mathfrak{e}$ the divisor on C corresponding to the invertible sheaf $\wedge^2 \mathcal{E}$.

PROPOSITION 1.1 ([1, V, 2.6]). *Let $\pi: X \to C$ and E be as above. Then there is a one-to-one correspondence between sections $\sigma: C \to X$ and surjections $\mathcal{E} \to \mathcal{L} \to 0$, where $\mathcal{L}$ is an invertible sheaf on C, given by $\mathcal{L} = \sigma^* \mathcal{O}_X(C_0)$. Under this correspondence, $\mathcal{N} = \mathrm{Ker}(\mathcal{E} \to \mathcal{L})$ is an invertible sheaf on C, and $\mathcal{N} \cong \pi_*(\mathcal{O}_X(C_0) \otimes \mathcal{O}_X(-D))$, where $D = \sigma(C)$, and $\pi^* \mathcal{N} \cong \mathcal{O}_X(C_0) \otimes \mathcal{O}_X(-D)$.*

From now on, let $\pi: X \to C$ be an elliptic ruled surface, determined by a normalized locally free sheaf $\mathcal{E}$.

THEOREM 1.2 ([1, V, 2.12, 2.15]).

(a) *If $\mathcal{E}$ is decomposable, then $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$ for some $\mathcal{L}$ with $\deg \mathcal{L} \leqq 0$. Therefore $e \geqq 0$. All values of $e \geqq 0$ are possible.*

(b) *If $\mathcal{E}$ is indecomposable, then $e = 0$ or -1, and there is exactly one such ruled surface over C for each of two values of e.*

REMARK 1.3.

(a) If $\mathcal{E}$ is an indecomposable normalized locally free sheaf of rank 2 of degree 0, then $\mathcal{E}$ is unique up to isomorphism and it is realized as a non-trivial extension of $\mathcal{O}_C$ by itself.

(b) If $\mathcal{E}$ is an indecomposable locally free sheaf of rank 2 of degree 1, then it is normalized. So in case $e=-1$, we may assume that $\mathcal{E}$ is a non-trivial extension of $\mathcal{O}_C(x)$ by $\mathcal{O}_C$, where x is a point of C.

§ 2. Cohomology of invertible sheaves on elliptic ruled surfaces.

Let $\pi: X \to C$ be an elliptic ruled surface, corresponding to a normalized $\mathcal{E}$, and let $D \sim nC_0+\mathfrak{b}f$ be a divisor on X, where $\mathfrak{b}$ is a divisor of degree m on C. If $n \geqq 0$, then $\pi_* \mathcal{O}_X(D) \cong S^n(\mathcal{E}) \otimes \mathcal{O}_C(\mathfrak{b})$, where $S^n(\mathcal{E})$ is the n-th symmetric power of $\mathcal{E}$, and $H^i(X, \mathcal{O}_X(D)) \cong H^i(C, \pi_* \mathcal{O}_X(D))$. Immediately we see that for any D with $n \geqq 0$, $H^2(D)=0$.

First suppose that $\mathcal{E}$ is decomposable. If $n \geqq 0$, then $S^n(\mathcal{E}) \otimes \mathcal{O}_C(\mathfrak{b}) \cong \bigoplus_{j=0}^{n} (j\mathfrak{e}+\mathfrak{b})$, so we can compute the dimension of its cohomology groups. By Serre's duality, we can compute $h^i(D)$ for $D \sim nC_0+\mathfrak{b}f$ with $n \leqq -2$. Before dealing with the case when $n=-1$, we note the exact sequences:

(1) $0 \longrightarrow \mathcal{O}_X(-C_0) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C_0} \cong \mathcal{O}_C \longrightarrow 0$;

(2) $0 \longrightarrow \mathcal{O}_X(-X_y) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_y} \cong \mathcal{O}_{P^1} \longrightarrow 0$, where $y \in C$;

tensored with $\mathcal{O}_X(D)$ respectively, and the resulting cohomology sequences:

$$\begin{aligned}(1)' \quad 0 &\longrightarrow H^0(D-C_0) \longrightarrow H^0(D) \longrightarrow H^0(C, n\mathfrak{e}+\mathfrak{b}) \\ &\longrightarrow H^1(D-C_0) \longrightarrow H^1(D) \longrightarrow H^1(C, n\mathfrak{e}+\mathfrak{b}) \\ &\longrightarrow H^2(D-C_0) \longrightarrow H^2(D) \longrightarrow 0;\end{aligned}$$

$$\begin{aligned}(2)' \quad 0 &\longrightarrow H^0(D-yf) \longrightarrow H^0(D) \longrightarrow H^0(\boldsymbol{P}^1, \mathcal{O}_P(n)) \\ &\longrightarrow H^1(D-yf) \longrightarrow H^1(D) \longrightarrow H^1(\boldsymbol{P}^1, \mathcal{O}_P(n)) \\ &\longrightarrow H^2(D-yf) \longrightarrow H^2(D) \longrightarrow 0.\end{aligned}$$

Now put $n=-1$ in (2)$'$, then we have $H^0(-C_0+(\mathfrak{b}-y)f) \cong H^0(-C_0+\mathfrak{b}f)$. After the repetition, we see that for any divisor $\mathfrak{b}'$ on C $H^0(-C_0+\mathfrak{b}f) \cong H^0(-C_0+\mathfrak{b}'f)$. If $\deg \mathfrak{b}' < -e$, then $H^0(-C_0+\mathfrak{b}'f) \cong H^0(-2C_0+\mathfrak{b}'f)$ by (1)$'$. Since $H^0(-2C_0+\mathfrak{b}'f)=0$, $H^0(-C_0+\mathfrak{b}f)=0$. By the same way, we have $H^2(-C_0+\mathfrak{b}f)=0$. Then $h^1(-C_0+\mathfrak{b}f)$ is given by Riemann-Roch theorem. Now we have finished the study in the case when $\mathcal{E}$ is decomposable, and get the following table.

Table I
Dimension of cohomology of $D \sim nC_0 + \mathfrak{b}f$ on X with $e > 0$.

<table>
<tr><th>n</th><th colspan="2">$m = \deg \mathfrak{b}$</th><th>$h^0(D)$</th><th>$h^1(D)$</th><th>$h^2(D)$</th></tr>
<tr><td rowspan="3">$n \geqq 0$</td><td colspan="2">$m \geqq ne+1$</td><td>$m(n+1) - \frac{e}{2}n(n+1)$</td><td>0</td><td rowspan="3">0</td></tr>
<tr><td rowspan="2">$0 \leqq m \leqq ne$</td><td>for $0 \leqq {}^{\forall} j \leqq n$ $j\mathfrak{e} + \mathfrak{b} \not\sim 0$</td><td>$m(N+1) - \frac{e}{2}N(N+1)$</td><td>$m(n+N+2) - \frac{e}{2}(n^2+n+N^2+N)$</td></tr>
<tr><td>$0 \leqq {}^{\exists} M \leqq n$ s. t. $M\mathfrak{e} + \mathfrak{b} \sim 0$</td><td>$\frac{1}{2}m(M+1)+1$</td><td>$\frac{m}{2}(2n+M+3) - \frac{e}{2}n(n+1)+1$</td></tr>
<tr><td>$n = -1$</td><td colspan="2"></td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>$n \leqq -2$</td><td colspan="2"></td><td colspan="3">easy by Serre's duality</td></tr>
</table>

where $N = \left[\frac{m}{e}\right]$.

Dimension of cohomology of $D \sim nC_0 + \mathfrak{b}f$ on X with $e = 0$.

<table>
<tr><th>n</th><th>$m = \deg \mathfrak{b}$</th><th>$h^0(D)$</th><th>$h^1(D)$</th><th>$h^2(D)$</th></tr>
<tr><td rowspan="3">$n \geqq 0$</td><td>$m > 0$</td><td>$m(n+1)$</td><td>0</td><td rowspan="3">0</td></tr>
<tr><td>$m = 0$</td><td>N'</td><td>N'</td></tr>
<tr><td>$m < 0$</td><td>0</td><td>$-m(n+1)$</td></tr>
<tr><td>$n = -1$</td><td></td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>$n \leqq -2$</td><td></td><td colspan="3">easy by Serre's duality</td></tr>
</table>

where $N' = \#\{j \mid 0 \leqq j \leqq n,\ j\mathfrak{e} + \mathfrak{b} \sim 0\}$.

Next we deal with the case when $\mathcal{E}$ is indecomposable. Using the exact sequences (1)′ and (2)′ as above, we get the following tables.

Table II ($e=0$).

<table>
<tr><th>n</th><th colspan="3">$m=\deg\mathfrak{b}$</th><th>$h^0(D)$</th><th>$h^1(D)$</th><th>$h^2(D)$</th></tr>
<tr><td rowspan="5">$n\geqq 0$</td><td colspan="3">$m>0$</td><td>$m(n+1)$</td><td>0</td><td rowspan="5">0</td></tr>
<tr><td rowspan="3">$m=0$</td><td colspan="2">$\mathfrak{b}\nsim 0$</td><td>0</td><td>0</td></tr>
<tr><td rowspan="2">$\mathfrak{b}\sim 0$</td><td>$n\geqq 2$</td><td>?</td><td>?</td></tr>
<tr><td>$n=0, 1$</td><td>1</td><td>1</td></tr>
<tr><td colspan="3">$m<0$</td><td>0</td><td>$-m(n+1)$</td></tr>
<tr><td>$n=-1$</td><td colspan="3"></td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>$n\leqq -2$</td><td colspan="3"></td><td colspan="3">easy by Serre's duality</td></tr>
</table>

Table III ($e=-1$).

<table>
<tr><th>n</th><th>$m=\deg\mathfrak{b}$</th><th>$h^0(D)$</th><th>$h^1(D)$</th><th>$h^2(D)$</th></tr>
<tr><td rowspan="3">$n\geqq 0$</td><td>$m\geqq 0$</td><td>$\frac{1}{2}(n+1)(2m+n)$</td><td>0</td><td rowspan="3">0</td></tr>
<tr><td>$-n\leqq m<0$</td><td>?</td><td>?</td></tr>
<tr><td>$m<-n$</td><td>0</td><td>$-\frac{1}{2}(n+1)(2m+n)$</td></tr>
<tr><td>$n=-1$</td><td></td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>$n\leqq -2$</td><td></td><td colspan="3">easy by Serre's duality</td></tr>
</table>

§ 3. Main results.

PROPOSITION 3.1. *Let X be an elliptic ruled surface corresponding to $\mathcal{E}$ and $D\sim C_0+\mathfrak{b}f$ an effective divisor other than C_0 on X. Then*:

(a) *when $\mathcal{E}\cong\mathcal{O}_C\oplus\mathfrak{e}$ with $\mathfrak{e}\ncong\mathcal{O}_C$, D is a section if and only if either $\mathfrak{b}\sim-\mathfrak{e}$ or* $\deg\mathfrak{b}\geqq e+1$;

(b) *when $\mathcal{E}\cong\mathcal{O}_C\oplus\mathcal{O}_C$, D is a section if and only if* $\deg\mathfrak{b}\geqq 2$;

(c) *when $\mathcal{E}$ is indecomposable, D is a section if and only if* $\deg\mathfrak{b}\geqq e+1$.

PROFOSITION 3.2. *Let X be an elliptic ruled surface with invariant e and $D\sim C_0+\mathfrak{b}f$ a divisor on X. Then*:

(a) *D is a base point free section* $\Leftrightarrow$ $\deg\mathfrak{b}\geqq e+2$; *or* $e\geqq 2$, $\mathfrak{b}\sim-\mathfrak{e}$ *or* $\mathcal{E}\cong\mathcal{O}_C\oplus\mathcal{O}_C$, $\mathfrak{b}\sim 0$;

(b) *D is very ample* $\Leftrightarrow$ $\deg\mathfrak{b}\geqq e+3$ $\Leftrightarrow$ *D is normally generated. In this case, $I(D)=\mathrm{Ker}[S\Gamma(D)\to\bigoplus_{j=0}^{\infty}\Gamma(jD)]$ is generated by its elements of degree 2 and 3, where $S\Gamma(D)$ is the symmetric algebra of $\Gamma(D)$ over k.*

THEOREM 3.3. *Let X be an elliptic ruled surface with $e\geqq 0$ and $D\sim nC_0+\mathfrak{b}f$ a divisor on X. Then*

$$D \text{ is very ample} \Leftrightarrow n\geqq 1,\ \deg\mathfrak{b}\geqq ne+3$$
$$\Leftrightarrow D \text{ is normally generated.}$$

In this case, $I(D)=\mathrm{Ker}[S\Gamma(D)\to\bigoplus_{j=0}^{\infty}\Gamma(jD)]$ is generated by its elements of degree 2 and 3.

§4. Proof of Proposition 3.1.

We can prove (a) and (b) easily. We demonstrate only (c). First we discuss the case when $e=-1$. Suppose that $C_0+\mathfrak{b}f$ is a section, by Proposition 1.1 we have the exact sequence

$$0\longrightarrow-\mathfrak{b}\longrightarrow\mathcal{E}\longrightarrow\mathfrak{b}+x\longrightarrow 0\,.$$

This extension of $\mathfrak{b}+x$ by $-\mathfrak{b}$ is non-trivial, since $\mathcal{E}$ is indecomposable. Hence $\mathrm{Ext}^1_C(\mathcal{O}_C(\mathfrak{b}+x),\ \mathcal{O}_C(-\mathfrak{b}))=H^1(\mathcal{O}_C(-2\mathfrak{b}-x))$ must have a non-trivial element, and hence we have $\deg\mathfrak{b}\geqq 0$. Conversely suppose that $\deg\mathfrak{b}\geqq 0$, then we can take a non-zero element $\xi\in\mathrm{Ext}^1_C(\mathcal{O}_C(\mathfrak{b}+x),\ \mathcal{O}_C(-\mathfrak{b}))$, and we get a non-trivial extension $\mathcal{E}'$ of $\mathfrak{b}+x$ by $-\mathfrak{b}$ corresponding to ξ. By Remark 1.3, $\mathcal{E}'$ is normalized, therefore $\Gamma(\mathcal{E}')$ has a non-zero element, which determines an injective map $\mathcal{O}_C\to\mathcal{E}$. Then the quotient $\mathcal{E}'/\mathcal{O}_C$ is an invertible sheaf by the normality of $\mathcal{E}'$ and $\mathcal{E}'/\mathcal{O}_C\cong\wedge^2\mathcal{E}'\cong\mathcal{O}_C(x)$. So we see that $\mathcal{E}'$ is a non-trivial extension of $\mathcal{O}_C(x)$ by $\mathcal{O}_C$ and $\mathcal{E}'\cong\mathcal{E}$. The surjection $\mathcal{E}\to\mathfrak{b}+x\to 0$ gives rise to the section D.

Next we discuss the case when $e=0$. Suppose that $C_0+\mathfrak{b}f$ is a section. By the same way as above, it is seen $\mathrm{Ext}^1_C(\mathfrak{b},\ -\mathfrak{b})\neq 0$. So we have $\deg\mathfrak{b}\geqq 1$. Conversely assume that $\deg\mathfrak{b}\geqq 1$, then there exists a non-zero element $t\in H^0(\mathfrak{b})$, and it determines an injective map $\alpha:\mathcal{O}_C\xrightarrow{\otimes t}\mathfrak{b}$. Let β be the map $\mathfrak{b}\to 2\mathfrak{b}$ induced by α. We can choose an element $\xi_0\neq 0$ of $\mathrm{Hom}_k(H^0(2\mathfrak{b}),\ k)\cong H^1(-2\mathfrak{b})$ such that $\mathrm{Image}[H^0(\beta):\ H^0(\mathfrak{b})\to H^0(2\mathfrak{b})]\subset\mathrm{Ker}\,\xi_0$. Let the following exact sequence be the non-trivial extension of $\mathfrak{b}$ by $-\mathfrak{b}$ corresponding to $\xi_0\in\mathrm{Ext}^1_C(\mathfrak{b},\ -\mathfrak{b})\cong H^1(-2\mathfrak{b})$.

$$(3)\qquad 0\longrightarrow-\mathfrak{b}\longrightarrow\mathcal{E}'\longrightarrow\mathfrak{b}\longrightarrow 0.$$

We consider the commutative diagram induced by (3)

$$\begin{array}{ccc} H^0(\mathcal{O}_C) & \xrightarrow{\ \delta\ } & H^1(-2\mathfrak{b}) \\ {\scriptstyle H^0(\alpha)}\downarrow & & \downarrow{\scriptstyle H^0(\beta)'} \\ H^0(\mathfrak{b}) & \xrightarrow{\ \gamma\ } & H^1(-\mathfrak{b}), \end{array}$$

where $H^0(\beta)'$ is the dual of $H^0(\beta)$. Since $\delta(1)=\xi_0$, we have $H^0(\beta)'(\xi_0)=\gamma(t)=0$. This implies that γ cannot be injective, so $\operatorname{Ker}\gamma=H^0(C, \mathcal{E})\neq 0$. On the other hand, there exists an invertible sheaf $\mathcal{L}$ of degree 0 such that $\mathcal{E}'\otimes\mathcal{L}\cong\mathcal{E}$. Hence for any invertible sheaf $\mathcal{M}$ of degree <0 $H^0(\mathcal{E}'\otimes\mathcal{M})=H^0(\mathcal{E}\otimes\mathcal{M}\otimes\mathcal{L}^{-1})=0$ by the normality of $\mathcal{E}$. So we see that $\mathcal{E}'$ is also normalized and $\mathcal{E}'\cong\mathcal{E}$ by Remark 1.3. The surjection $\mathcal{E}\to\mathfrak{b}\to 0$ gives rise to the section D.

§ 5. Proof of Proposition 3.2.

Before starting the proof, we make some remarks on divisors on C and state a generalized lemma of Castelnuovo.

REMARK 5.1. *Let C be an elliptic curve over k, and $\mathfrak{b}$ a divisor on C. Then:*
(a) *$\mathfrak{b}$ is free from base points* $\Leftrightarrow$ $\deg\mathfrak{b}\geqq 2$;
(b) *$\mathfrak{b}$ is very ample* $\Leftrightarrow$ $\deg\mathfrak{b}\geqq 3$ $\Leftrightarrow$ *$\mathfrak{b}$ is normally generated.*

LEMMA 5.2 (*Generalized lemma of Castelnuovo* [2]). *Suppose $\mathcal{L}$ is an invertible sheaf on a variety V such that $\Gamma(\mathcal{L})$ has no base points. Let $\mathcal{F}$ be a coherent sheaf on V such that $H^i(\mathcal{F}\otimes(-i\mathcal{L}))=0$, for every $i\geqq 1$. Then the map $\Gamma(\mathcal{F}\otimes(i-1)\mathcal{L})\otimes\Gamma(\mathcal{L})\to\Gamma(\mathcal{F}\otimes i\mathcal{L})$, for every $i\geqq 1$ is surjective.*

PROOF OF PROPOSITION 3.2 (a). Assume that D is free from base points, then $D|_{C_0}=\mathfrak{e}+\mathfrak{b}$ is either trivial or base point free on C_0. In the latter case, we have $\deg D|_{C_0}=-e+m\geqq 2$ by Remark 5.1, where $m=\deg\mathfrak{b}$. In the former case, that is $\mathfrak{b}\sim-\mathfrak{e}$, we restrict D to the section D itself. Since $D|_D\sim 2\mathfrak{b}+\mathfrak{e}\sim-\mathfrak{e}$, we get either $\deg(-\mathfrak{e})=e\geqq 2$ or $-\mathfrak{e}\sim 0$. Conversely if D is a divisor with $\deg\mathfrak{b}\geqq e+2$, then Proposition 3.1 shows that D is a section. We have only to prove that for any point $P\in X$, there exists a section s of $\Gamma(D)$ such that $(s)_0$ does not contain P. Let $y=\pi(P)$. Using the exact sequence (2)$'$ in § 2, we get the surjection

$$H^0(X, D)\longrightarrow H^0(\boldsymbol{P}^1, O_{\boldsymbol{P}^1}(1))\longrightarrow 0,$$

because $H^1(D-yf)=0$ in the previous tables. There is a section $\bar{s}\in H^0(O_{\boldsymbol{P}}(1))$ such that $(\bar{s})_0\not\ni y$. By the above surjection, there exists a section $s\in\Gamma(D)$ such that $s|_{X_y}=\bar{s}$. Then s is what we want. Next we assume that $D\sim C_0+(-\mathfrak{e})f$ and $e\geqq 2$. Because D is a section by Proposition 3.1 and $D\cdot C_0=0$, there are no base points of $\Gamma(D)$ on C_0. When $P\notin C_0$, we choose an effective divisor $\mathfrak{b}\in|-\mathfrak{e}|$ so that it does not contain $\pi(P)$, then $C_0+\mathfrak{b}f\in|D|$ does not contain P. It follows that D has no base points. Finally in case $X\cong\boldsymbol{P}^1\times C$, it is clear that

C_0 is free from base points.

PROOF OF PROPOSITION 3.2 (b). If D is very ample, then by Remark 5.1 $\deg\mathfrak{b}\geqq e+3$. We have only to show that if $\deg\mathfrak{b}\geqq e+3$, then D is normally generated. First suppose $X\not\cong \boldsymbol{P}^1\times C$, then by Proposition 3.1 there exists a section $C'=C_0+\mathfrak{b}'f$ with $\deg\mathfrak{b}'=e+1$. We consider the exact sequence

$$(4)\quad 0\longrightarrow \mathcal{O}_X(-C')\longrightarrow \mathcal{O}_X\longrightarrow \mathcal{O}_{C'}\longrightarrow 0$$

tensored with D, and resulting cohomology sequence

$$(5)\quad 0\longrightarrow \Gamma((\mathfrak{b}-\mathfrak{b}')f)\longrightarrow \Gamma(D)\longrightarrow \Gamma(C,\ \mathfrak{b}'+\mathfrak{e}+\mathfrak{b})\longrightarrow 0\,.$$

Tensoring $\Gamma(D)$ to (5) and $2D$ to (4) respectively, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma((\mathfrak{b}-\mathfrak{b}')f)\otimes\Gamma(D) & \longrightarrow & \Gamma(D)^{\otimes 2} & \longrightarrow & \Gamma(\mathfrak{b}'+\mathfrak{e}+\mathfrak{b})\otimes\Gamma(D) & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & \Gamma(C_0+(2\mathfrak{b}-\mathfrak{b}')f) & \longrightarrow & \Gamma(2D) & \longrightarrow & \Gamma(C,\ 2(\mathfrak{b}'+\mathfrak{e}+\mathfrak{b})) & \longrightarrow & 0\,. \end{array}$$

The map γ is surjective because $\mathfrak{b}'+\mathfrak{e}+\mathfrak{b}$ is ample with normal generation by Remark 5.1. We can apply Lemma 5.2 to α for $(\mathfrak{b}-\mathfrak{b}')f$ is free from base points and $H^i(D-i(\mathfrak{b}-\mathfrak{b}')f)=0$, $i=1,\ 2$. So we see that α is surjective, and therefore β is surjective. We also obtain the surjective map

$$\Gamma(tD)\otimes\Gamma(D)\longrightarrow\Gamma(X,\ (t+1)D),\quad \text{for every } t\geqq 2,$$

by Lemma 5.2. Since D is ample this shows the projective normality of D.

Next in case $X\cong\boldsymbol{P}^1\times C$, we must use another technique, because there is no sections of type C'. Let m be the degree of $\mathfrak{b}$. Since D is free from base points by Proposition 3.2 (a), it induces the morphism $\varphi_D\colon X\to\boldsymbol{P}^N$, where $N=h^0(D)-1=2m-1$. φ_D is factored by the following two morphisms;

$$id_{\boldsymbol{P}^1}\times\varphi_{\mathfrak{b}}\colon \boldsymbol{P}^1\times C\hookrightarrow \boldsymbol{P}^1\times\boldsymbol{P}^{m-1},$$

where $\varphi_{\mathfrak{b}}$ is the closed immersion corresponding to a very ample divisor $\mathfrak{b}$ on C, and

$$s\colon \boldsymbol{P}^1\times\boldsymbol{P}^{m-1}\hookrightarrow\boldsymbol{P}^N,$$

where s is the Segre embedding. The projective normality of D is shown by the next lemma.

LEMMA 5.3. *Let V be a variety in $\boldsymbol{P}^N$ via f and W a variety in $\boldsymbol{P}^M$ via g. Let φ be the commposition $f\times g\colon V\times W\hookrightarrow\boldsymbol{P}^N\times\boldsymbol{P}^M$, and s the Segre embedding $\boldsymbol{P}^N\times\boldsymbol{P}^M\hookrightarrow\boldsymbol{P}^L$, where $L=(N+1)(M+1)-1$.*

Then for every integer t, the restriction $\Gamma(\mathcal{O}_{\boldsymbol{P}^L}(t))\to\Gamma(\mathcal{O}_{\boldsymbol{P}^L}(t)|_{V\times W})$ is surjective if and only if both $\Gamma(\mathcal{O}_{\boldsymbol{P}^N}(t))\to\Gamma(\mathcal{O}_{\boldsymbol{P}^N}(t)|_V)$ and $\Gamma(\mathcal{O}_{\boldsymbol{P}^M}(t))\to\Gamma(\mathcal{O}_{\boldsymbol{P}^M}(t)|_W)$ are surjective.

Now to complete the proof, we need some notations and propositions, which are due to [3]. Let V be a variety and $\mathcal{M}$ an invertible sheaf on V. Let U be a subspace of $\Gamma(X, \mathcal{M})$. We define $n(U, \mathcal{M})$ by $n(U, \mathcal{M})=\mathrm{Min}\{m \geqq 1 \mid$ for every $j \geqq m$, $U^{\otimes j} \to \Gamma(j\mathcal{M})$ is surjective$\}$. We denote $\mathrm{Ker}[S^j(U) \to \Gamma(j\mathcal{M})]$ by $I_j(U, \mathcal{M})$ and $\bigoplus_{j=0}^{\infty} I_j(U, \mathcal{M})$ by $I(U, \mathcal{M})$.

PROPOSITION 5.4. *Let Y be a non-singular projective curve over k. Let $\mathcal{L}$ be an invertible sheaf on Y and U a subspace of $\Gamma(\mathcal{L})$. Assume that U is very ample. We define $i(\mathcal{L})$ and μ as follows:*

$$i(\mathcal{L})=\mathrm{Min}\{m \geqq 1 \mid h^1(j\mathcal{L})=0, \textit{ for all } j \geqq m\} ;$$

$$\mu=\mathrm{Max}\{i(\mathcal{L})+2,\ n(U, \mathcal{L})+1\} .$$

Then $I(U, \mathcal{L})$ is generated by $I_2(U, \mathcal{L}), \cdots, I_\mu(U, \mathcal{L})$.

PROPOSITION 5.5. *Let V be a variety and $\mathcal{M}$ a very ample invertible sheaf on V. Let Y be a non-singular member of $|\mathcal{M}|$ and $\mathcal{L}$ the restriction of $\mathcal{M}$ to Y. We define U the image of the restriction map $\Gamma(V, \mathcal{M}) \to \Gamma(Y, \mathcal{L})$. Assume that $I(U, \mathcal{L})$ is generated by its elements of degree $2, \cdots, \mu$. Then $I(\mathcal{M})= I(\Gamma(\mathcal{M}), \mathcal{M})$ is generated by $I_2(\mathcal{M}), \cdots, I_\nu(\mathcal{M})$, where $\nu=\mathrm{Max}\{\mu, n(\Gamma(\mathcal{M}), \mathcal{M})+1\}$.*

Now we return to the proof of the proposition. Since D is very ample, we can choose a non-singular irreducible curve $Y \in |D|$. We consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow D \longrightarrow D|_Y \longrightarrow 0 .$$

Let U be the image of the restriction map $H^0(D) \to H^0(Y, D|_Y)$.

First we claim $n(U, D|_Y)=2$. We use the commutative diagram

$$\begin{array}{ccccc} \Gamma(D)^{\otimes 2} & \longrightarrow & U \otimes \Gamma(D) & \longrightarrow & 0 \\ \beta \downarrow & & \downarrow \gamma & & \\ \Gamma(2D) & \longrightarrow & \Gamma(2D|_Y) & \longrightarrow & H^1(D)=0 . \end{array}$$

Because D is normally generated, β is surjective. So γ is also surjective and we obtain the surjection $U^{\otimes 2} \to \Gamma(2D|_Y)$. To show that $\varphi_t : U^{\otimes t} \to \Gamma(tD|_Y)$ is surjective for every $t \geqq 3$, we use the induction on t and the following commutative diagrams:

$$\begin{array}{ccccc} \Gamma((t-1)D)\otimes\Gamma(D) & \longrightarrow & \Gamma((t-1)D|_Y)\otimes\Gamma(D) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\gamma'} & & \\ \Gamma(tD) & \longrightarrow & \Gamma(tD|_Y) & \longrightarrow & 0; \end{array}$$

$$\begin{array}{ccc} \Gamma((t-1)D)\otimes U & \xrightarrow{\ \psi\ } & \Gamma(tD|_Y) \\ \uparrow{\scriptstyle\varphi_{t-1}\otimes 1_U} & \nearrow{\scriptstyle\varphi_t} & \\ U^{\otimes t} & & \end{array}$$

Since β' is surjective, γ' is also surjective and therefore ψ is surjective. For $\varphi_{t-1}\otimes 1_U$ is surjective by the induction hypothesis, φ_t is surjective. Next we claim $i(D|_Y)=1$. In the exact sequence

$$H^1(tD)\longrightarrow H^1(Y,\ tD|_Y)\longrightarrow H^2((t-1)D), \quad t\geqq 1;$$

the two outside cohomology are vanishing, hence the middle one is vanishing also. So we get that $I(U,\ D|_Y)$ is generated by its elements of degree 2 and 3, by Proposition 5.4. Consequently Proposition 5.5, we conclude that $I(D)$ is generated by its elements of degree 2 and 3.

§ 6. Proof of Theorem 3.3, examples and problems.

PROOF OF THEOREM 3.3. Assume that D is very ample, then it is clear that $n\geqq 1$ and $D\cdot C_0\geqq 3$ by Remark 5.1. Conversely we assume that $D\sim nC_0+\mathfrak{b}f$ with $n\geqq 2$ and $\deg\mathfrak{b}\geqq ne+3$. The proof of the projective normality of D is similar to the previous proof in case $n=1$. Consider the following two commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(D-C_0)\otimes\Gamma(D) & \longrightarrow & \Gamma(D)^{\otimes 2} & \longrightarrow & \Gamma(C,\ n\mathfrak{e}+\mathfrak{b})\otimes\Gamma(D) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \Gamma(2D-C_0) & \longrightarrow & \Gamma(2D) & \longrightarrow & \Gamma(C,\ 2(n\mathfrak{e}+\mathfrak{b})) & \longrightarrow & 0; \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(D-C_0)^{\otimes 2} & \longrightarrow & \Gamma(D)\otimes\Gamma(D-C_0) & \longrightarrow & \Gamma(C,\ n\mathfrak{e}+\mathfrak{b})\otimes\Gamma(D-C_0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma'} & & \\ 0 & \longrightarrow & \Gamma(2D-2C_0) & \longrightarrow & \Gamma(2D-C_0) & \longrightarrow & \Gamma(C,\ (2n-1)\mathfrak{e}+2\mathfrak{b}) & \longrightarrow & 0. \end{array}$$

Since $\Gamma(C,\ n\mathfrak{e}+\mathfrak{b})\otimes\Gamma(C,\ (n-1)\mathfrak{e}+\mathfrak{b})\rightarrow\Gamma((2n-1)\mathfrak{e}+2\mathfrak{b})$ is surjective by Lemma 5.2, γ' is surjective. By the induction hypothesis β' is surjective and by the projective normality of $n\mathfrak{e}+\mathfrak{b}$, γ is surjective. Therefore both α and β are surjective. The surjectivity of the map $\Gamma(tD)\otimes\Gamma(D)\rightarrow\Gamma((t+1)D)$, for every $t\geqq 2$, is given by Lemma 5.2. Since D is ample, we conclude that D is normally generated.

Mimicking the proof of proposition 3.2, we can prove the rest of the theorem easily.

EXAMPLE 6.1. Let $D\sim C_0+\mathfrak{b}f$ be a divisor on $X\cong \boldsymbol{P}^1\times C$. When $\deg \mathfrak{b}=3$, D is normally generated and and $I(D)$ is generated by its elements of degree 2 and 3 but not only by those of degree 2. If $\deg \mathfrak{b}\geqq 4$, then $I(D)$ is generated by its elements of degree 2.

EXAMPLE 6.2. Let C be an elliptic curve in $\boldsymbol{P}^2$ defined by $y_3y_2^2=y_1^3-y_1y_3^2$. We put $-\mathfrak{e}\sim\mathcal{O}_P(1)|_C$. Let Y be the cone over C with vertex $P_0=(1:0:0:0)$. If we blow up the point P_0, we obtain an elliptic ruled surface $X\subset \boldsymbol{P}^2\times \boldsymbol{P}^3$, which is isomorphic to $\boldsymbol{P}(\mathcal{E})$ where $\mathcal{E}=\mathcal{O}_C\oplus\mathfrak{e}$. Let s be the Segre embedding from $\boldsymbol{P}^2\times \boldsymbol{P}^3$ to $\boldsymbol{P}^{11}$. We may choose coordinates $X_{01}, X_{02}, X_{03}, \cdots X_{33}$ of $\boldsymbol{P}^{11}$ such that $s^*X_{ij}=x_iy_j$, where x_i's are the coordinates of $\boldsymbol{P}^3$. Since $s^*\mathcal{O}_{P^{11}}(1)|_X \sim C_0+2(-\mathfrak{e})f$ is normally generated by Proposition 3.2, we get the following commutative diagram

$$
\begin{array}{ccccc}
\boldsymbol{P}^2\times\boldsymbol{P}^3 & \overset{s}{\hookrightarrow} & \boldsymbol{P}^{11} & & (X_{01}:,\cdots,:X_{13}:X_{21}:X_{22}:X_{23}:X_{31}:X_{32}:X_{33}) \\
\uparrow & & \downarrow p & & \downarrow \\
X & \overset{\varphi_D}{\hookrightarrow} & \boldsymbol{P}^8 & & (X_{01}:,\cdots,:X_{13}:X_{22}:X_{23}:X_{33})
\end{array}
$$

where $D\sim C_0+2(-\mathfrak{e})f$ and p is the projection. Then $I(D)$ is generated by the following eighteen conics and one cubic.

$$
\begin{array}{lll}
X_{01}X_{12}=X_{02}X_{11}, & X_{01}X_{13}=X_{03}X_{11}, & X_{01}X_{22}=X_{02}X_{12}, \\
X_{01}X_{23}=X_{03}X_{12}, & X_{01}X_{33}=X_{03}X_{13}, & X_{02}X_{13}=X_{03}X_{12}, \\
X_{02}X_{23}=X_{03}X_{22}, & X_{02}X_{33}=X_{03}X_{23}, & X_{11}X_{22}=X_{12}^2, \\
X_{11}X_{23}=X_{13}X_{12}, & X_{11}X_{33}=X_{13}^2, & X_{12}X_{23}=X_{13}X_{22}, \\
X_{12}X_{33}=X_{13}X_{23}, & X_{22}X_{33}=X_{23}^2, &
\end{array}
$$

$$
\begin{array}{ll}
X_{03}X_{22}=X_{01}X_{11}-X_{01}X_{33}, & X_{13}X_{22}=X_{11}^2-X_{11}X_{33}, \\
X_{23}X_{22}=X_{12}X_{11}-X_{12}X_{33}, & X_{33}X_{22}=X_{13}X_{11}-X_{13}X_{33}, \\
X_{03}X_{02}^2=X_{01}^3-X_{01}X_{03}^2. &
\end{array}
$$

PROBLEM 6.3. Let X be an elliptic ruled surface with $e\geqq 0$ and D a very ample divisor on X. Find a necessary and sufficient condition so that $I(D)$ is generated by its elements of degree 2.

PROBLEM 6.4. Let X be an elliptic ruled surface with $e=-1$ and D a divisor on X. Find a necessary and sufficient condition for D to be normally generated.

§7. Projective normality of ample invertible sheaf on rational ruled surfaces.(*)

By a rational ruled surface, we understand a ruled surface $\pi: X \to \boldsymbol{P}^1$. Since any locally free sheaf of rank 2 on $\boldsymbol{P}^1$ is decomposable, for each $e \geqq 0$ there is exactly one rational ruled surface X_e with invariant e, given by $\mathcal{E} \cong \mathcal{O}_P \oplus \mathcal{O}_P(-e)$. Then we can compute $h^i(D)$ for any divisor D nC_0+mf on X, where mf is the divisor corresponding to the invertible sheaf $\pi^*\mathcal{O}_P(m)$, and we obtain that if $m \geqq ne-1$, then $h^1(D)=h^2(D)=0$. Using this, we have the following.

THEOREM 3.1. *Let X_e be a rational ruled surface and $D \sim nC_0+mf$ a divisor on X. Then the following four conditions are equivalent to each other*:

(i) $n>0$ *and* $m>ne$;

(ii) D *is ample*;

(iii) D *is very ample*;

(iv) D *is normally generated.*

In this case, $I(D)=\mathrm{Ker}[S\Gamma(D) \to \bigoplus_{j=0}^{\infty} \Gamma(jD)]$ *is generated by its elements of degree* 2.

PROOF. The equivalence of (i), (ii) and (iii) is in [1]. It is sufficient to show that D is normally generated when D satisfies the above three conditions. We can choose a non-singular irreducible curve $Y \in |D|$, since D is very ample. Let g be the genus of Y and $\mathcal{L}$ be the restriction of D to Y. Then we have $\deg \mathcal{L} \geqq 2g+2$ by Adjunction formula and the condition (i). So $\mathcal{L}$ is ample with normal generation and $I(\mathcal{L})$ is generated by its elements of degree 2 by [4]. We consider the exact sequence,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow D \longrightarrow D|_Y = \mathcal{L} \longrightarrow 0$$

tensored with $(t-1)D$, for $t \geqq 1$, and the resulting commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma((t-1)D)\otimes\Gamma(D) & \longrightarrow & \Gamma(tD)\otimes\Gamma(D) & \longrightarrow & \Gamma(t\mathcal{L})\otimes\Gamma(D) & \longrightarrow & 0. \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & \Gamma(tD) & \longrightarrow & \Gamma((t+1)D) & \longrightarrow & \Gamma((t+1)\mathcal{L}) & \longrightarrow & 0. \end{array}$$

γ is surjective by the projective normality of $\mathcal{L}$. When $t=1$, α is surjective, so is β. When $t \geqq 2$, α is surjective by the induction hypothesis, so β is also surjective. It follows that D is normally generated. Finally by Proposition 5.5, we conclude that $I(D)$ is generated by its elements of degree 2.

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