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Some Remarks on Type (τ, 1)-cylindrical Measures

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Introduction.

We are concerned here with the study of cylindrical measures on Banach spaces. The relation between vector measures and cylindrical measures has been investigated by A. Goldman ([5]) and I. Kluvánek ([6]). Some kind of cylindrical measures are determined by the pair of the vector measure $\boldsymbol{m}$ and the probability P such that $\boldsymbol{m}$ is P-continuous. Such cylindrical measures are called to be of type (τ, 1). When we restrict our investigation to the field of type (τ, 1)-cylindrical measures, we can find several useful tools within the results concerning vector measures. In this paper we show two directions in these applications. Section 2 presents an improvement of the result of [8] stating about type (τ, 1)-cylindrical measures on Banach spaces with a Schauder basis. In section 3 we construct a tensor product of two cylindrical measures, however it is different from the definition of the tensor product of Gaussian measures in [2] and [3].

§ 1. Cylindrical measures and vector measures.

All the probabilities to be considered in this paper will be assumed to be Radon measures, i.e., Borel measures with the inner regularity. Let X be a real Banach space and $\mathfrak{F}(X)$ be the collection of all closed subspaces of X of finite codimension. Let $E\in\mathfrak{F}(X)$; the quotient space X/E is a finite dimensional vector space. Denote by π_E the canonical surjection $X\to X/E$. When $F\subset E$ and $E, F\in\mathfrak{F}(X)$, we can define a canonical surjection $\pi_{EF}: X/F\to X/E$ through the relation $\pi_E=\pi_{EF}\circ\pi_F$. The family $\{X/E, \pi_E; E\in\mathfrak{F}(X)\}$ forms a projective system. Suppose that on each finite-dimensional space X/E we are given a probability μ_E together with the relation $\mu_E=\pi_{EF}(\mu_F)$ if $E\supset F$. Such a projective system of probabilities $\mu=(\mu_E)_{E\in\mathfrak{F}(X)}$ will be called a cylindrical measure on X. We shall say that a cylindrical measure μ on X is a measure (or probability) if there exists a probability m on X such that μ_E is the image measure of m by π_E for every $E\in\mathfrak{F}(X)$, i.e., $\mu_E=\pi_E(m)$. We shall say that μ is the cylindrical measure associated with m and write, by a convenient abuse of language, $\mu=m$.

We shall define the image measure of a cylindrical measure with respect to a continuous linear mapping. Let X and Y be two Banach spaces and u be a continuous linear mapping of X into Y. Let μ be a cylindrical measure on X. For every $G\in\mathfrak{F}(Y)$, the subspace $E=u^{-1}(G)$ of X belongs to $\mathfrak{F}(X)$ and u induces a linear mapping $u_G: X/E\to Y/G$. The measure λ_G on Y/G, the image of μ_E by u_G, is defined by $\lambda_G=u_G(\mu_E)$ and denoted by $\lambda=u(\mu)$.

Let X' be the topological dual space of X. There is a one-to-one correspondence between continuous linear mappings $u: X\to\mathbf{R}^n$ and n-tuples $(x'_1, \cdots, x'_n)$ where each $x'_i\in X'$. Thus a cylindrical measure μ on X associates with each n-tuple $(x'_1, \cdots, x'_n)$, a probability $\mu_{x'_1\cdots x'_n}$ on $\mathbf{R}^n$.

We shall denote by $L^0(\Omega, \Sigma, P)$, where (Ω, Σ, P) is a probability space, the space of all equivalent classes of real random variables, and we shall say that f is a random function defined on X' if f is a mapping of X' into $L^0(\Omega, \Sigma, P)$.

PROPOSITION 1.1 (cf. [9]). *There exists a bijective correspondence between the cylindrical measures on X and the isonomy classes of linear random functions defined on X'.*

DEFINITION. A cylindrical measure μ is said to be of *type* 1 (resp. *type* ∞) if the associated linear random function f is continuous of X' into $L^1(\Omega, \Sigma, P)$ (resp. $L^\infty(\Omega, \Sigma, P)$). Furthermore, a cylindrical measure μ is said to be of *type* $(\tau, 1)$ if μ is of type 1 and the continuity of the function f is guaranteed even if the space X' is equipped with the Mackey topology $\tau(X', X)$.

A set function $\boldsymbol{m}$ from a σ-algebra Σ of subsets of a set Ω to a Banach space X is called a vector measure, if $\boldsymbol{m}(\bigcup_{n=1}^\infty E_n)=\sum_{n=1}^\infty \boldsymbol{m}(E_n)$ in the norm topology of X for all sequences (E_n) of pairwise disjoint members of Σ.

We denote by $(\boldsymbol{m}; \Omega, \Sigma, P)$ the couple of a vector measure and a probability space such that $\boldsymbol{m}$ is defined on Σ and $\boldsymbol{m}$ is P-continuous. Let $\boldsymbol{m}_i$ be a vector measure and $(\Omega_i, \Sigma_i, P_i)$ be a probability space for $i=1, 2$. The couple $(\boldsymbol{m}_1; \Omega_1, \Sigma_1, P_1)$ is said to be equivalent to $(\boldsymbol{m}_2; \Omega_2, \Sigma_2, P_2)$ if the family of functions $\{(dx'\circ\boldsymbol{m}_1)/dP_1; x'\in X'\}$ is isonomous to $\{(dx'\circ\boldsymbol{m}_2)/dP_2; x'\in X'\}$, where $(dx'\circ\boldsymbol{m}_i)/dP_i$ is a Radon–Nikodym derivative. By assumption we have $x'\circ\boldsymbol{m}_i\ll P_i$ for every $x'\in X'$, and therefore the above equivalency is well defined.

Now we shall show the next theorem. The result of this theorem has been suggested in [5] and [6].

THEOREM 1.2. *There exists a bijective correspondence between the type*

$(\tau, 1)$-cylindrical measures on a Banach space X and the equivalence classes of the couples $(\boldsymbol{m}; \Omega, \Sigma, P)$ of the vector measures with values in X and the probability spaces.

PROOF. Given $(\boldsymbol{m}; \Omega, \Sigma, P)$, we have $x' \circ \boldsymbol{m} \ll P$ for every $x' \in X'$. Then there exists a Radon–Nikodym derivative $(dx' \circ \boldsymbol{m})/dP \in L^1(\Omega, \Sigma, P)$ for every $x' \in X'$. It is clear that the function f of X' into $L^1(\Omega, \Sigma, P)$ defined by $f(x') = (dx' \circ \boldsymbol{m})/dP$ is the linear random function on X'. Denote by μ the associated cylindrical measure with f. Now we shall show that μ is of type $(\tau, 1)$. The vector measure $\boldsymbol{m}$ is always of bounded semivariation, then we have $\sup_{\|x'\| \leq 1} \int_\Omega |f(x')(\omega)| \, dP(\omega) < \infty$ and this means that μ is of type 1. Moreover, since the range of $\boldsymbol{m}$ is relatively weakly compact, we can show that μ is of type $(\tau, 1)$.

Conversely, let μ be a type $(\tau, 1)$-cylindrical measure on X and f be the associated linear random function with the probability space (Ω, Σ, P). The function f is continuous of X' equipped with the Mackey topology $\tau(X', X)$ into $L^1(\Omega, \Sigma, P)$. Through only elementary argument we can complete the proof. Q.E.D.

The following obvious result has been also suggested in [5].

PROPOSITION 1.3. *Let μ be a type $(\tau, 1)$-cylindrical measure on a Banach space X and $(\boldsymbol{m}; \Omega, \Sigma, P)$ be the associated couple with μ. Then the followings are equivalent:*

(i) *μ is of type ∞.*

(ii) *The set $\{\boldsymbol{m}(A)/P(A); A \in \Sigma, P(A) \neq 0\}$ is bounded in X.*

PROPOSITION 1.4. *Let X, Y be Banach spaces, u be a continuous linear mapping of X into Y and μ be a type $(\tau, 1)$-cylindrical measure on X. If the couple $(\boldsymbol{m}; \Omega, \Sigma, P)$ is associated with μ, then $(u(\boldsymbol{m}); \Omega, \Sigma, P)$ is associated with $u(\mu)$, where $u(\boldsymbol{m})$ means $u \circ \boldsymbol{m}$.*

PROOF. It is clear that $u(\boldsymbol{m})$ is a vector measure with values in Y and $u(\boldsymbol{m}) \ll P$. Let $f: X' \to L^1(\Omega, \Sigma, P)$ be the associated linear random function with μ, then $f \circ {}^t u : Y' \to L^1(\Omega, \Sigma, P)$ is associated with $u(\mu)$. The couple $(u(\boldsymbol{m}); \Omega, \Sigma, P)$ induces the linear random function h on Y'; $h: y' \mapsto (dy' \circ u(\boldsymbol{m}))/dP = (d^t u(y') \circ \boldsymbol{m})/dP$. Thus we have $h = f \circ {}^t u$. Q.E.D.

Thus we can use the results concerning vector measures for the study of type $(\tau, 1)$-cylindrical measures. These applications will be shown in the ensuing sections.

§ 2. Application I.

We shall introduce the following theorem, which is due to Z. Lipecki and K. Musiał.

Through this section we assume X to be a Banach space with a Schauder basis. We denote by $\{x_n\}$ the Schauder basis of X and by $\{x'_n\}$ the biorthogonal functionals of $\{x_n\}$.

THEOREM 2.1 ([7]). *Let (Ω, Σ, P) be a probability space and $\{f_n\}\subset L^1(\Omega, \Sigma, P)$ and a vector measure $\boldsymbol{m}: \Sigma\to X$ be such that*

$$x'_n \circ \boldsymbol{m}(E)=\int_E f_n\, dP \qquad \text{for} \quad E\in\Sigma \quad \text{and} \quad n=1, 2, \cdots .$$

Then the following arguments are equivalent:

(i) *$\boldsymbol{m}$ has a Pettis P-integrable Radon–Nikodym derivative.*

(ii) *$\sum_{n=1}^{\infty} f_n(\cdot)x_n$ converges strongly P-a.e.*

(iii) *$\sum_{n=1}^{\infty} f_n(\cdot)x_n$ converges weakly in probability P.*

Either of them implies that

$$\boldsymbol{m}(E)=(P)\int_E \sum_{n=1}^{\infty} f_n(\cdot)x_n\, dP \qquad \text{for} \quad E\in\Sigma .$$

To each type $(\tau, 1)$-cylindrical measure μ on X, there corresponds a linear random function f of X' into $L^1(\Omega, \Sigma, P)$. Let $f_n(\cdot)=f(x'_n)(\cdot)$ for $n=1, 2, \cdots$ and M' be the closed linear span of $\{x'_n\}$. Then $\sum f_n(\cdot)\langle x_n, x'\rangle$ converges in $L^1(\Omega, \Sigma, P)$ for every $x'\in M'$. Conversely, to each sequence of real random variables $\{f_n\}$ satisfying a certain condition, there corresponds a type $(\tau, 1)$-cylindrical measure μ, which induces the linear random function f such that $f(x'_n)=f_n$ (see [8]). Therefore, the sequence $\{f_n\}$ is said to be the associated sequence with μ. Using the above theorem, we can get the next result.

THEOREM 2.2. *Let μ be a type $(\tau, 1)$-cylindrical measure on X and $\{f_n\}$ be the associated sequence with μ. Then the following statements are equivalent:*

(i) *μ is a (probability) measure.*

(ii) *$\sum_{n=1}^{\infty} f_n(\cdot)x_n$ converges strongly P-a.e.*

(iii) *$\sum_{n=1}^{\infty} f_n(\cdot)x_n$ converges weakly in probability P.*

PROOF. We only have to prove that the vector measure $\boldsymbol{m}$ has a Pettis P-integrable Radon–Nikodym derivative if and only if μ is a measure, where $(\boldsymbol{m}; \Omega, \Sigma, P)$ is the associated couple with μ. Suppose that $\boldsymbol{m}$ has a Pettis P-integrable Radon–Nikodym derivative $\varphi: \Omega\to X$. It is easily seen that $\mu=\varphi(P)$ and as a consequence that μ is a measure.

Conversely, we assume that μ is a measure and denote by f the associated linear random function with μ. Since the points of X are separated by the countable subset $\{x'_n\}$ of X', there exists a weakly measurable function

$\psi: \Omega \to X$ such that for each $x' \in X'$ the function $\langle \psi(\cdot), x' \rangle$ is equal to $f(x')$ in the sense of random variables (see [9]). Therefore the proof is complete.
Q.E.D.

We have the next result as an obvious consequence.

COROLLARY 2.3. *Let μ be a type $(\tau, 1)$-cylindrical measure on X and $(\boldsymbol{m}; \Omega, \Sigma, P)$ be the associated couple with μ. If μ is of type ∞, then μ is a measure if and only if $\boldsymbol{m}$ has a Bochner integrable Radon–Nikodym derivative with respect to P.*

§ 3. Application II.

In this section we shall try to construct a product measure of two cylindrical measures. First of all, we show the following theorem, which is due to M. Duchoň and I. Kluvánek.

THEOREM 3.1 ([4]). *Let probability spaces* $(\Omega_1, \Sigma_1, P_1)$ and $(\Omega_2, \Sigma_2, P_2)$, *Banach spaces X and Y, and vector measures $\boldsymbol{m}_1: \Sigma_1 \to X$ and $\boldsymbol{m}_2: \Sigma_2 \to Y$ be given. If we denote by $\Sigma_1 \otimes \Sigma_2$ the σ-algebra generated by the sets of the form $E \times F$, $E \in \Sigma_1$, $F \in \Sigma_2$, and by $X \hat{\otimes}_\varepsilon Y$ the completed inductive tensor product of the spaces X and Y, then there exists a unique vector measure $\boldsymbol{m}: \Sigma_1 \otimes \Sigma_2 \to X \hat{\otimes}_\varepsilon Y$ such that the relation*

$$\boldsymbol{m}(E \times F) = \boldsymbol{m}_1(E) \otimes \boldsymbol{m}_2(F), \qquad E \in \Sigma_1, \quad F \in \Sigma_2$$

holds. We denote the measure $\boldsymbol{m}$ by $\boldsymbol{m}_1 \otimes_\varepsilon \boldsymbol{m}_2$.

Moreover, if $\boldsymbol{m}_1 \ll P_1$ and $\boldsymbol{m}_2 \ll P_2$, then $\boldsymbol{m}_1 \otimes_\varepsilon \boldsymbol{m}_2 \ll P_1 \otimes P_2$, where $P_1 \otimes P_2$ is the usual product measure of P_1 and P_2 defined on $\Sigma_1 \otimes \Sigma_2$.

REMARK. If both $(\Omega_1, \Sigma_1, P_1)$ and $(\Omega_2, \Sigma_2, P_2)$ are Radon probability spaces, then $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, P_1 \otimes P_2)$ is extensible to a Radon probability space (see [9]). But we do not need to use the extended probability space in the ensuing arguments, because we can remove the assumption that (Ω, Σ, P) is a Radon probability space from the definition of random functions.

Theorem 3.1 and the above remark suggest a possibility of making another type $(\tau, 1)$-cylindrical measure from two type $(\tau, 1)$-cylindrical measures.

LEMMA 3.2. *Let X_i, Y_i $(i=1, 2)$ be four Banach spaces, $(\Omega_i, \Sigma_i, P_i)$ $(i=1, 2)$ two probability spaces and $\boldsymbol{m}_i$ $(i=1, 2)$ two vector measures such that $\boldsymbol{m}_1: \Sigma_1 \to X_1$ and $\boldsymbol{m}_2: \Sigma_2 \to Y_1$. Let u (resp. v) be a continuous linear map of X_1 into X_2 (resp. of Y_1 into Y_2). If we denote by $u \hat{\otimes}_\varepsilon v$ the extension of $u \otimes v$ as a continuous linear map of $X_1 \hat{\otimes}_\varepsilon Y_1$ into $X_2 \hat{\otimes}_\varepsilon Y_2$, then we have $(u \hat{\otimes}_\varepsilon v)(\boldsymbol{m}_1 \otimes_\varepsilon \boldsymbol{m}_2) = u(\boldsymbol{m}_1) \otimes_\varepsilon v(\boldsymbol{m}_2)$.*

PROOF. If a set G is of the form $G=\bigcup_{i=1}^{k} E_i \times F_i$ where the union is disjoint and $E_i \in \Sigma_1$, $F_i \in \Sigma_2$, then we have

$$\begin{aligned}(u\hat{\otimes}_\varepsilon v)(m_1\otimes_\varepsilon m_2)(G) &= (u\hat{\otimes}_\varepsilon v)\Big(\sum_{i=1}^{k} m_1(E_i)\otimes m_2(F_i)\Big)\\ &= \sum_{i=1}^{k} u(m_1)(E_i)\otimes v(m_2)(F_i)\\ &= (u(m_1)\otimes_\varepsilon v(m_2))(G)\,.\end{aligned}$$

According to the same step used in Theorem 3.1, we can see that $(u\hat{\otimes}_\varepsilon v)(m_1\otimes_\varepsilon m_2)(G)=(u(m_1)\otimes_\varepsilon v(m_2))(G)$ is held for every $G \in \Sigma_1\otimes\Sigma_2$. Q.E.D.

THEOREM AND DEFINITION 3.3. *Let μ_1 and μ_2 be type $(\tau, 1)$-cylindrical measures on respective Banach spaces X and Y, $(m_1; \Omega_1, \Sigma_1, P_1)$ and $(m_2; \Omega_2, \Sigma_2, P_2)$ be the couples associated with μ_1 and μ_2 respectively. If we denote by $\Sigma_1\otimes\Sigma_2$ the σ-algebra generated by the sets of the form $E\times F$, $E \in \Sigma_1$, $F \in \Sigma_2$, and by $X\hat{\otimes}_\varepsilon Y$ the completed inductive tensor product of the spaces X and Y, then there exists a unique type $(\tau, 1)$-cylindrical measure μ on the space $X\hat{\otimes}_\varepsilon Y$, which induces the couple $(m_1\otimes_\varepsilon m_2; \Omega_1\times\Omega_2, \Sigma_1\otimes\Sigma_2, P_1\otimes P_2)$. We denote the measure μ by $\mu_1\otimes^\varepsilon\mu_2$. The cylindrical measure $\mu_1\otimes^\varepsilon\mu_2$ is said to be the tensor product of the cylindrical measures μ_1 and μ_2.*

PROOF. According to Theorem 3.1 and Remark, the existence of μ is clear. We only have to check the uniqueness. Denote by W the space $X\hat{\otimes}_\varepsilon Y$ and by W' the topological dual space of W. It is easily seen that $X'\otimes Y'$ is dense in the space W' equipped with the Mackey topology $\tau(W', W)$. It is therefore sufficient to see that for every $x'\otimes y' \in X'\otimes Y'$ the probability $(x'\otimes y')(\mu)$ is uniquely determined, independent of the choice of associated couples $(m_1; \Omega_1, \Sigma_1, P_1)$ and $(m_2; \Omega_2, \Sigma_2, P_2)$. For each $x'\otimes y' \in X'\otimes Y'$,

$$\begin{aligned}(x'\otimes y')\circ(m_1\otimes_\varepsilon m_2) &= (x'\circ m_1)\otimes_\varepsilon(y'\circ m_2) \qquad \text{(by Lemma 3.2)}\\ &= (x'\circ m_1)\otimes(y'\circ m_2)\,,\end{aligned}$$

where the last one means the usual product measure of two scalar-valued measures. Hence,

$$\frac{d(x'\otimes y')\circ(m_1\otimes_\varepsilon m_2)}{dP_1\otimes P_2}=\frac{d(x'\circ m_1)\otimes(y'\circ m_2)}{dP_1\otimes P_2}=\frac{dx'\circ m_1}{dP_1}\cdot\frac{dy'\circ m_2}{dP_2}\,.$$

Then, it is enough to show that $h=fg$ is isonomous to $h'=f'g'$ if f and g are respectively isonomous to f' and g', where $f \in L^1(\Omega_1, \Sigma_1, P_1)$, $f' \in L^1(\Omega_1', \Sigma_1', P_1')$, $g \in L^1(\Omega_2, \Sigma_2, P_2)$ and $g' \in L^1(\Omega_2', \Sigma_2', P_2')$. Checking the characteristic functions of the image measures $h(P_1\otimes P_2)$ and $h'(P_1'\otimes P_2')$, we can complete the proof. Q.E.D.

The following propositions will give some characterizations of the tensor product of two type $(\tau, 1)$-cylindrical measures.

PROPOSITION 3.4. *Let μ_1, μ_2 be type (τ, 1)-cylindrical measures on Banach spaces X and Y respectively. If both μ_1 and μ_2 are of type ∞, then the tensor product $\mu_1 \otimes^{\varepsilon} \mu_2$ is also of type ∞.*

PROOF. Let $(\boldsymbol{m}_i; \Omega_i, \Sigma_i, P_i)$ $(i=1, 2)$ be the associated couples with μ_i $(i=1, 2)$. For $i=1, 2$; μ_i is of type ∞, then we have a positive number M such that $\|\boldsymbol{m}_i(A)\| \leqq M P_i(A)$ for every $A \in \Sigma_i$. This result implies that there exists a constant number $M'>0$ satisfying $\|\boldsymbol{m}_1 \otimes_{\varepsilon} \boldsymbol{m}_2(G)\| \leqq M' P_1 \otimes P_2(G)$ if G is of the form $G=\bigcup_{i=1}^{k} E_i \times F_i$ where $\{E_i \times F_i\}_{i=1}^{k}$ are mutually disjoint and $E_i \in \Sigma_1$, $F_i \in \Sigma_2$. Once more we can use the same technique used in Theorem 3.1. Q.E.D.

PROPOSITION 3.5. *Let X_i, Y_i $(i=1, 2)$ be four Banach spaces and μ_i $(i=1, 2)$ be two type (τ, 1)-cylindrical measures on X_1 and on Y_1 respectively. If u (resp. v) is a continuous linear map of X_1 into X_2 (resp. of Y_1 into Y_2), then*

$$(u \hat{\otimes}_{\varepsilon} v)(\mu_1 \otimes^{\varepsilon} \mu_2) = u(\mu_1) \otimes^{\varepsilon} v(\mu_2) .$$

PROOF. According to Proposition 1.4 and Lemma 3.2, this result is obvious. Q.E.D.

PROPOSITION 3.6. *Let μ_i $(i=1, 2)$ be two type (τ, 1)-cylindrical measures, respectively on X and on Y. For any $x' \otimes y' \in X' \otimes Y'$,*

$$x'(\mu_1) \otimes^{\varepsilon} y'(\mu_2) = h(x'(\mu_1) \otimes y'(\mu_2)) ,$$

where h is the mapping of $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$, which is defined as follows:

$$h : (x, y) \longmapsto xy .$$

PROOF. It is clear that $(\mathbf{R}, \mathfrak{B}(\mathbf{R}), x'(\mu_1))$ is a probability space, where $\mathfrak{B}(\mathbf{R})$ is the Borel σ-algebra on $\mathbf{R}$. Therefore we can take the measure

$$\boldsymbol{m}_1(A) = \int_A f(t)\, dx'(\mu_1)(t) \qquad \text{for} \quad A \in \mathfrak{B}(\mathbf{R}) ,$$

where f is the identity map of $\mathbf{R}$ onto $\mathbf{R}$, and the above-mentioned probability space $(\mathbf{R}, \mathfrak{B}(\mathbf{R}), x'(\mu_1))$ as the associated couple with $x'(\mu_1)$. Since μ_1 is of type (τ, 1), it is easily checked that f belongs to $L^1(\mathbf{R}, \mathfrak{B}(\mathbf{R}), x'(\mu_1))$. By the same method we have the probability space $(\mathbf{R}, \mathfrak{B}(\mathbf{R}), y'(\mu_2))$, the identity map g of $\mathbf{R}$ onto $\mathbf{R}$ and the measure $\boldsymbol{m}_2 = \int g\, dy'(\mu_2)$. In this case, actually both measures $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are scalar-valued measures, then we have $\boldsymbol{m}_1 \otimes_{\varepsilon} \boldsymbol{m}_2 = \boldsymbol{m}_1 \otimes \boldsymbol{m}_2$ and also $(d\boldsymbol{m}_1 \otimes \boldsymbol{m}_2)/(dx'(\mu_1) \otimes y'(\mu_2)) = fg = h$. This implies $x'(\mu_1) \otimes^{\varepsilon} y'(\mu_2) = h(x'(\mu_1) \otimes y'(\mu_2))$. Q.E.D.

REMARK. As referred to earlier, the set $X' \otimes Y'$ is dense in $(X \hat{\otimes}_{\varepsilon} Y)'$ equipped with $\tau((X \hat{\otimes}_{\varepsilon} Y)', X \hat{\otimes}_{\varepsilon} Y)$ and the type (τ, 1)-cylindrical measure μ

on $X\hat{\otimes}_\varepsilon Y$ is uniquely determined by the family of the scalar-valued measures $\{(x'\otimes y')(\mu);\ x'\otimes y'\in X'\otimes Y'\}$. Moreover, Proposition 3.5 implies that $(x'\otimes y')(\mu_1\otimes^\varepsilon\mu_2)=(x'(\mu_1))\otimes^\varepsilon(y'(\mu_2))$. Therefore it is easily seen that Proposition 3.6 gave the most important characterization of the tensor product of type $(\tau, 1)$-cylindrical measures.

The next example points out the fact that the above definition of tensor product of type $(\tau, 1)$-cylindrical measures is different from the tensor product of Gaussian cylindrical measures defined by R. Carmona and S. Chevet ([2] and [3]).

EXAMPLE. Let H be a real separable Hilbert space. We identify H' and H. Let γ be the Gaussian cylindrical measure on H with variance I, where I is the identity map (see [1]). It is clear that γ is of type $(\tau, 1)$. Indeed, every Gaussian cylindrical measure on a real Banach space is of type $(\tau, 1)$. We shall show that the tensor product $\gamma\otimes^\varepsilon\gamma$ on $H\hat{\otimes}_\varepsilon H$ is not a Gaussian cylindrical measure. For $x'\otimes y'\in H'\otimes H'$, $\|x'\|=\|y'\|=1$,

$$\begin{aligned}(x'\otimes y')(\gamma\otimes^\varepsilon\gamma)&=(x'(\gamma))\otimes^\varepsilon(y'(\gamma))\\&=h(x'(\gamma)\otimes y'(\gamma)),\end{aligned}$$

where h is the same mapping used in Proposition 3.6, and also, $x'(\gamma)(A)=y'(\gamma)(A)=\frac{1}{\sqrt{2\pi}}\int_A\exp\left(-\frac{x^2}{2}\right)dx$ for every $A\in\mathfrak{B}(\mathbf{R})$. Denote by $\varphi(t)$ the characteristic function of $h(x'(\gamma)\otimes y'(\gamma))$. We have

$$\begin{aligned}\varphi(t)&=\int_{\mathbf{R}}\exp(itu)\,dh(x'(\gamma)\otimes y'(\gamma))(u)\\&=\int_{\mathbf{R}\times\mathbf{R}}\exp(ith(\omega))\,d(x'(\gamma)\otimes y'(\gamma))(\omega)\\&=\left(\frac{1}{\sqrt{2\pi}}\right)^2\int_{\mathbf{R}}\exp\left(-\frac{y^2}{2}\right)\left(\int_{\mathbf{R}}\exp\left(itxy-\frac{x^2}{2}\right)dx\right)dy\\&=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}\exp\left(-\frac{(1+t^2)y^2}{2}\right)dy\\&=\frac{1}{\sqrt{1+t^2}}.\end{aligned}$$

This implies that $\gamma\otimes^\varepsilon\gamma$ is not a Gaussian cylindrical measure.

More detailed characterization of the tensor product $\mu_1\otimes^\varepsilon\mu_2$, which is introduced in this paper, will appear elsewhere.

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