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Convergence of Monotone Operators

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§1. Introduction

Let $C(X)$ be the Banach space of all continuous real-valued functions on a compact Hausdorff space X. Theorems of Korovkin type give sufficient conditions for a sequence (L_n) of positive or contractive operators on $C(X)$, or more generally, on a Banach space E to converge strongly to the identity operator whenever $(L_n g)$ converges to g for every g belonging to a certain subset C of E.

H. Bauer has considered the convergence of a net of monotone maps of an adapted space in $C(X)$ into $\boldsymbol{R}^X$ (the ordered vector space of all real-valued functions on X) for a locally compact Hausdorff space X ([2], [3]).

There arises naturally the problem to replace the identity by certain positive operators (cf. [4], [5], [6], [7]).

We consider the convergence of a net (L_i) of monotone maps of an ordered vector space E into $\boldsymbol{R}^Y$ for a set Y. More precisely, let C be a convex cone in an ordered vector space E and A a monotone sublinear map of C into R^Y. We research for sufficient conditions for $(L_i f)$ to converge pointwise to a certain $Lf \in \boldsymbol{R}^Y$ for $f \in E$ whenever $\overline{\lim}_i L_i g \leqq Ag$ for every $g \in C$, or whenever $\lim_i L_i g = Ag$ for every g belonging to a linear subspace F of E. Further, in case Y is a locally compact Hausdorff space, the analogous problems with respect to the locally uniform convergence or the order convergence instead of the pointwise convergence are considered. Moreover we apply these results to the problems with respect to regular points in potential theory, to a concept of an integral, a generalization of the Riemann-Stieltjes integral, and to the positive linear map A given in [5].

§2. The $A(C)$-boundary and $A(C)$-affine elements

Let E be an ordered vector space and C a convex cone in E. Suppose that E is C^+-bounded. Here E is called C^+-bounded when for each $f \in E$ there is $g \in C$ such that $g \geqq 0$ and $-g \leqq f \leqq g$. Let Y

be a set and A a map of C into $\boldsymbol{R}^Y$. A map A is called monotone on C if $f, g \in C$ and $f \leqq g$ imply $Af \leqq Ag$. A map A is called sublinear (resp. subadditive) on C if the following conditions (i) and (ii) (resp. (i)) are satisfied:

(i) $C \ni f, g$ implies $A(f+g) \leqq Af + Ag$,

(ii) $C \ni f, \lambda \in R^+$ implies $A(\lambda f) = \lambda Af$.

In this section we suppose that A is a monotone sublinear map of C into $\boldsymbol{R}^Y$. For $f \in E$ and $y \in Y$, put

$$\overline{Af}(g) = \inf\{Ag(y) \colon g \geqq f,\ g \in C\}\ ,$$

then $-\infty < \overline{Af}(y) < \infty$ since E is C^+-bounded and A is sublinear. The map $\overline{Af}$ of E into $\boldsymbol{R}^Y$ is monotone and sublinear. We denote, for each $y \in Y$, by $\mathfrak{M}_y$ the set of all positive linear functionals μ on E satisfying

$$\mu(g) \leqq Ag(y)$$

for each $g \in C$.

LEMMA 2.1. *For every $f \in E$ and $y \in Y$,*

$$\overline{Af}(y) = \max\{\mu(f) \colon \mu \in \mathfrak{M}_y\}\ , \tag{2.1}$$

and

$$-\overline{A(-f)} = \min\{\mu(f) \colon \mu \in \mathfrak{M}_y\}\ . \tag{2.2}$$

PROOF. The functional: $h \mapsto \overline{Ah}(y)$ is sublinear on E, there is, by the Hahn-Banach extension theorem, a linear functional μ on E satisfying

$$\mu(f) = \overline{Af}(y)$$

and

$$\mu(h) \leqq \overline{Ah}(y) \quad \text{for all} \quad h \in E$$

If $h \leqq 0$, then $\mu(h) \leqq \overline{Ah}(y) \leqq A(0)(y) = 0$. Hence μ is positive. Since $\mu(g) \leqq \overline{Ag}(y) = Ag(y)$ for every $g \in C$, we have $\mu \in \mathfrak{M}_y$. Further, if $\nu \in \mathfrak{M}_y$, then

$$\nu(f) \leqq \{\inf \nu(g) \colon g \geqq f,\ g \in C\} = \inf\{Ag(y) \colon g \geqq f,\ g \in C\} = \overline{Af}(y) = \mu(f)\ .$$

Thus we have the relation (2.1). Further, replacing f by $-f$ in (2.1), we have

$$-\overline{A(-f)}(y) = -\max\{\mu(-f) \colon \mu \in \mathfrak{M}_y\} = \min\{\mu(f) \colon \mu \in \mathfrak{M}_y\}\ .$$

An element $f \in E$ is called $A(C)$-affine if $\overline{Af} = -\overline{A(-f)}$ on Y.

PROPOSITION 2.1. *An element $f \in E$ is $A(C)$-affine if and only if the functional: $\mu \mapsto \mu(f)$ is constant on $\mathfrak{M}_y$ for each $y \in Y$.*

PROOF. From Lemma 2.1 it follows that $\overline{Af} = -\overline{A(-f)}$ on Y if and only if for each $y \in Y$ and each $\mu \in \mathfrak{M}_y$, $\mu(f) = \overline{Af}(y)$. Hence we have the conclusion.

We denote $\delta(A(C))$ the set of all $y \in Y$ for which $\mathfrak{M}_y$ consists of one element and call it the $A(C)$-boundary. Immediately we have the following proposition and the corollary by Lemma 2.1.

PROPOSITION 2.2. *A point y of Y belongs to $\delta(A(C))$ if and only if $\overline{Af}(y) = -\overline{A(-f)}(y)$ for every $f \in E$.*

COROLLARY 2.1. *$\delta(A(C)) = Y$ if and only if every element f of E is $A(C)$-affine.*

EXAMPLE 1. Let X be a compact Hausdorff space and C a convex cone in $C(X)$ containing constant functions and separating the points of X. Put $E = C(X)$ and $Y = X$. Then, it is obvious that the $I(C)$-boundary with respect to the identity map I on E is equal to the Choquet boundary of X with respect to C.

EXAMPLE 2. (cf. [5]) Let X and Y be compact Hausdorff spaces. Assume that X has at least $n+1$ points and F is a linear subspace of $C(X)$ satisfying the following assumption: given any n distincts points of X, there exists $g \in F$ such that $g(x) \geqq 0$ and $g(x) = 0$ exactly when $x = x_i$ for $i = 1, \cdots, n$. Denote by A a positive linear map of $C(X)$ into $C(Y)$ of the form

$$(Ag)(y) = \sum_{i=1}^{n} \psi_i(y) g(\varphi_i(y)) \quad (g \in C(X),\ y \in Y)\ , \tag{2.3}$$

where $\psi_i \in C^+(Y)$ and φ_i is a continuous map of Y into X for $i = 1, \cdots, n$. Then we have

PROPOSITION 2.3. $\delta(A(F)) = Y$.

PROOF. For any $y \in Y$ and any $\mu \in \mathfrak{M}_y$ we show that $\mu = \varepsilon_y \cdot A$ where ε_y is the evaluation functional at y. Put $\varphi_i(y) = x_i (i = 1, \cdots, n)$. By the assumption there exists $g \in F$ satisfying $g(x_i) = 0 (i = 1, \cdots, n)$ and $g(x) > 0$ elsewhere. Since $\mu(g) = Ag(y) = 0$, the support of the positive measure μ is contained in the set $\{x_1, \cdots, x_n\}$, and accordingly μ is represented by the form $\sum_{i=1}^{n} \alpha_i \varepsilon_{x_i}$ with some $\alpha_i \geqq 0$. For each i, there is $g_i \in F$ satisfying $g_i(x_j)=0 (j \neq i)$ and $g_i(x)>0$ elsewhere. Then $\mu(g_i)=\alpha_i g_i(x_i)$. On the other hand $\mu(g_i)=Ag_i(y)=\psi_i(y)g_i(x_i)$. Since $g_i(x_i)>0$, it follows that $\alpha_i=\psi_i(y)$. Thus we have $\mu=\sum_{i=1}^{n} \psi_i(y)\varepsilon_{\varphi_i(y)}$ and accordingly $\mu=\varepsilon_y \cdot A$. Since y is an arbitrary point, we have the conclusion.

§ 3. The locally uniform convergence

We consider the case where Y is a locally compact Hausdorff space. Let E be an ordered vector space and C a convex cone in E such that E is C^+-bounded. Suppose that A is a sublinear map of C into $C(Y)$. For a net $(L_i)_{i \in I}$ of monotone maps of E into $\boldsymbol{R}^Y$, we write

$$U-\overline{\lim}\, L_i g \leqq h \quad (\text{resp. } U-\underline{\lim}\, L_i g \geqq h) \quad \text{on} \quad S\,,$$

if for any $\varepsilon > 0$ there is an index $i_0 \in I$ such that

$$L_i g(y) < h(y)+\varepsilon \quad (\text{resp. } L_i g(y) > h(y)-\varepsilon)$$

for all $i \geqq i_0$ and all $y \in S$. We remark that a net $(L_i g)$ in $\boldsymbol{R}^Y$ converges uniformly on S to h if and only if the following two inequalities

$$U-\overline{\lim}\, L_i g \leqq h \text{ on } S \text{ and } U-\underline{\lim}\, L_i g \geqq h \text{ on } S$$

hold.

We have the following Korovkin-type theorem.

THEOREM 3.1. *Let (L_i) be a net of monotone subadditive maps of E into $\boldsymbol{R}^Y$ such that $L_i(0)=0$ and let S be a compact subset of Y. If*

$$U-\overline{\lim}\, L_i g \leqq Ag \quad on \quad S \tag{3.1}$$

for every $g \in C$, then net $(L_i f)$ converges uniformly on S to $\overline{Af}$ for each $f \in E$ satisfying $\overline{Af}=-\overline{A(-f)}$ on S.

PPOOF. Since $\overline{Af}(y)=\inf\{Ag(y): g \geqq f, g \in C\}$, there exists, for any $\varepsilon>0$, $g_j \in C (j=1, \cdots, n)$ such that $g_j \geqq f$ and

$$\overline{Af}(y)+\varepsilon > \min\{Ag_1(y), \cdots, Ag_n(y)\} \text{ for all } y \in S\,.$$

From (3.1) it follows that for any $\varepsilon>0$ there exists an index i_0 such that

$$L_i g_j(y) < Ag_j(y)+\varepsilon \quad (j=1, \cdots, n)$$

for all $y \in S$ and all index $i \geqq i_0$.

Hence

$$L_i f(y) \leqq \min\{L_i g_1(y), \cdots, L_i g_n(y)\} \leqq \min\{Ag_1(y), \cdots, Ag_n(y)\} + \varepsilon < \overline{Af}(y) + 2\varepsilon .$$

This implies

$$U - \overline{\lim}\, L_i f(y) \leqq \overline{Af}(y) \quad \text{on} \quad S . \tag{3.2}$$

Since (3.2) also holds for $-f$ and $L_i(-f) \geqq -L_i f$, we have

$$U - \underline{\lim}\, L_i f(y) \geqq -\overline{A(-f)} \text{ on } S . \tag{3.3}$$

If $\overline{Af} = -\overline{A(-f)}(y)$ on S, $(L_i f)$ converges unifomly on S to $\overline{Af}$.

COROLLARY 3.1. *If* (3.1) *holds on* Y *for every* $g \in C$, *then the net* $(L_i f)$ *converges locally uniformly on* Y *to* $\overline{Af}$ *for every* $A(C)$-*affine element.*

PROOF. The corollary follows immediately from the theorem and the definition of $A(C)$-affine elements.

COROLLARY 3.2. *Let* S *be compact subset of* $\delta(\overline{A(C)})$. *If* (3.1) *holds on* S *for every* $g \in C$, *then, for every* $f \in E$ $(L_i f)$ *converges uniformly on* S *to* $\overline{Af} = -\overline{A(-f)}$.

PROOF. Since $S \subset \delta(A(C))$, we have $\overline{Af} = -\overline{A(-f)}$ on S for all $f \in E$ by Proposition 2.2 and hence we have the conclusion.

REMARK 3.1. In particular, assume that A is a monotone sublinear map on E. Since $-\overline{A(-f)} \leqq Af \leqq \overline{Af}$, it can be concluded in Corollary 3.1 (resp. Corollary 3.2) that $(L_i f)$ converges locally uniformly on Y(resp. unifomly on S) to Af.

Next, let F be a linear subspace of E such that E is F^+-bounded and A be a monotone map of F into $\boldsymbol{R}^Y$. For $f \in E$ and $y \in Y$, put

$$\overline{Af}(y) = \inf\{Ag(y) : g \geqq f,\ g \in F\} ,$$

and

$$\underline{Af}(y) = \sup\{Ah(y) : h \leqq f,\ h \in F\} .$$

Then $-\infty < \underline{Af}(y) \leqq \overline{Af}(y) < \infty$.

Suppose that (L_i) is a net of monotone maps of E into $\boldsymbol{R}^Y$. Then we have the following theorem and corollaries.

THEOREM 3.2. *Let A be a monotone map of F into $C(Y)$ and S a compact subset of Y. If $(L_i g)$ converges uniformly on S to Ag, then $(L_i f)$ converges uniformly on S to $\overline{Af}$ for every $f \in E$ satisfying $\overline{Af} = \underline{Af}$ on S.*

PROOF. It has been shown in the proof of Theorem 3.1 that

$$\overline{Af}(y) = \inf \{Ag(y) \colon g \geqq f,\ g \in F\}$$

implies that (3.2) holds on S. Similarly,

$$\underline{Af}(g) = \sup \{Ah(y) \colon h \leqq f,\ h \in F\}$$

implies that

$$U - \underline{\lim}\, L_i f(y) \geqq \underline{Af}(y)$$

holds on S. If $\overline{Af} = \underline{Af}$ on S, then $(L_i f)$ converges uniformly on S to $\underline{Af}$.

COROLLARY 3.3. *Let A be a monotone subadditive map of F into $C(Y)$ satisfying $A(0)=0$. If $(L_i g)$ converges locally uniformly on Y to Ag, then $(L_i f)$ converges locally uniformly on Y to $\overline{Af}$ for all $A(F)$-affine element f.*

PROOF. From the asssumption follows $\underline{Af} \geqq -\overline{A(-f)}$ and hence $\overline{Af} = \underline{Af}$ on Y for an $A(F)$-affine element f. By the theorem we have the conclusion.

COROLLARY 3.4. *Let A be a positive linear map of F into $C(Y)$ and S a compact subset of $\delta(A(F))$. If $(L_i g)$ converges uniformly on S to Ag for all $g \in F$, then $(L_i f)$ converges uniformly on S to $\overline{Af}$ for all $f \in E$.*

PROOF. Since $\overline{Af} = -\overline{A(-f)} = \underline{Af}$ on S by Proposition 3.2, we have the conclusion by the theorem.

Using Proposition 2.3, Corollary 3.4 and Remark 3.1, we have immediately the following theorem which has the weaker assumption than Theorem 2 in [5].

THEOREM 3.3. *Let A be the positive linear map of E into $C(Y)$ given in Example 2 and let $(L_i g)$ be a net of monotone maps of E into $\boldsymbol{R}^Y$. If $(L_i g)$ converges uniformly on Y to Ag for all $g \in F$, then $(L_i f)$ converges uniformly on Y to Af for all $f \in E$.*

Let F be a linear subspace of E such that E is F^+-bounded and

A be a positive linear map of F into $C(Y)$. We denote by $S_u(F, A)$ the set of all $f \in E$ having the following property: for every net $(L_i)_{i \in I}$ of monotone maps of E into $\boldsymbol{R}^Y$ for which $(L_i g)$ converges locally uniformly on Y to Ag for all $g \in F$, $(L_i f)$ also converges locally uniformly to a certain $Lf \in C(Y)$ independently of (L_i).

The set $S_u(F, A)$ is characterized in the following theorem (cf. [2, Theorem 3.3]).

THEOREM 3.4. *The set $S_u(F, A)$ is equal to the set of all $A(F)$-affine elements.*

PROOF. If f is $A(F)$-affine, it follows from Corollary 3.3 that $f \in S_u(F, A)$. Conversely, suppose that $f \in S_u(, A)$. By Proposition 2.1 it suffices to prove that for every $z \in Y$, $\mu \mapsto \mu(f)$ is constant on $\mathfrak{M}_z$. Let μ be an arbitrary functional in $\mathfrak{M}_z$. Choose a base $\mathscr{B}$ of neighborhoods of z consisting of relatively compact open sets. Then for each $V \in \mathscr{B}$ there is $q_V \in C(Y)$ satifying $0 \leqq q_V \leqq 1$, $q_V(z)=1$ and $q_V(y)=0$ for all $y \in V^c$. Define

$$L_V h = \mu(h) q_V + \overline{Ah}(1-q_V)$$

for each $h \in E$. Then (L_V) is a net of monotone maps of E into $\boldsymbol{R}^Y$ and satisfies

$$|L_V g - Ag| = |Ag(z) - Ag| g_V$$

for all $g \in F$. Consequently, $(L_V g)$ converges locally uniformly on Y to Ag. Since $S_u(F, A) \ni f$, $(L_V f)$ also converges locally uniformly on Y to $Lf \in \boldsymbol{R}^Y$. In particular $(L_V f(z))$ converges to $Lf(z)$ and hence $\mu(f) = Lf(z)$. Thus it is concluded that f is $A(F)$-affine.

Let $F(G)$ be the linear subspace of E generated by a subset G of E. If $S_u(F(G), A) = E$, G is called an A-Korovkin set.

COROLLARY 3.5. *Let G a subset of Y such that E is G^+-bounded. The set G is an A-Korovkin set if and only if $Y = \delta(A(F(G)))$.*

PROOF. This is an immediate consequence of Theorem 3.4 and Corollary 2.1.

§ 4. The pointwise convergence

Let Y be a subset and C be a convex cone in an ordered vector space E such that E is C^+-bounded. Assume that A is a monotone sublinear map of C into $\boldsymbol{R}^Y$ and $(L_i)_{i \in I}$ is a net of monotone maps of E into $\boldsymbol{R}^Y$. We assign the discrete topology to Y, Y is a locally compact Hausdorff space. Since a set consisting of one point is compact, the following theorem is obtained by Theorem 3.1

THEOREM 4.1. *Let (L_i) be a net of monotone subadditive maps of E into $\boldsymbol{R}^Y$ satisfying $L_i(0)=0$. Suppose that S is a subset of Y. If*

$$\overline{\lim}\, L_i g(y) \leqq Ag(y)$$

for every $g \in C$ and every $y \in S$, then $(L_i f)$ converges pointwise on S to $\overline{Af}$ for every $f \in E$ satisfying $\overline{Af} = -\overline{A(-f)}$ on S.

EXAMPLE 3. We consider a relatively compact open set U in a strong harmonic space in the sense of H. Bauer ([1, p. 61]). Let E be the orderd vector space $C(\partial U)$ of all continuous real-valued functions on the topological boundary ∂U and C be the convex cone of all continuous functions on ∂U which are extended to be continuous on $\bar{U}$ and superharmonic in U. Consider the case where Y is the set $\{z\}$ consisting of one point $z \in \partial U$ and A is the positive linear functional on E corresponding $f \in E$ to $f(z)$. Then, we obtain the following proposition.

PROPOSITION 4.1. (cf. [1, Satz 4.4.1]) *If a point $z \in \partial U$ belongs to the $A(C)$-boundary, it is a regular point of U.*

PROOF. First we remark that we use the notations and terminologies in [1]. Let (z_n) be an arbitrary sequence of U converging to z. Put, for every $f \in E$, $L_n f = \bar{H}_f(z_n)$. Then L_n is a monotone sublinear functional on E. Since every $g \in C$ is extended to the continuons function h_g on $\bar{U}$ which is superharmonic in U, it follows that

$$\overline{\lim}\, L_n g = \overline{\lim}\, \bar{H}_g(z_n) \leqq \overline{\lim}\, h_g(z_n) = g(z) = Ag\ . \tag{4.1}$$

Since all $f \in E$ are $A(C)$-affine by the assumption and Corollary 2.1, we have, using Theorem 4.1 and (4.1),

$$\lim \bar{H}_f(z_n) = f(z) \text{ for all} f \in C(\partial U)\ .$$

Thus z is a regular point of U.

REMARK 4.1. It is obvious that a point $z \in \partial U$ belongs to the $A(C)$-boundary if and only if z is extreme-regular ([1, p. 142]).

REMARK 4.2. When F is a linear subspace of E, it is obvious that the results with respect to the pointwise convergence of the type of Theorem 3.2, Corollary 3.3 and Corollary 3.4 are also obtained.

EXAMPLE 4. Let $\mathfrak{A}$ be a field of subsets of X and μ a finitely additive positive set function defined on $\mathfrak{A}$. Define

$$F=\left\{\sum_{i=1}^{n} \alpha_i \chi_{U_i}: U_i \in \mathfrak{A},\ \alpha_i \in \boldsymbol{R},\ n \in \boldsymbol{N}\right\},$$

where χ_{U_i} is the characteristic function of U_i. If $B(X)$ is the ordered vector space of all bounded real-valued functions on X, the space F is a linear subspace of $B(X)$ containing constant functions. Consequently $B(X)$ is F^+-bounded. For every $g=\sum_{i=1}^{n} \alpha_i \chi_{U_i} \in F$, define

$$Ag=\sum_{i=1}^{n} \alpha_i \mu(U_i) .$$

Then the functional A is well-defined and positive linear on F. By Proposition 2.1 a function $f \in B(X)$ is $A(F)$-affine with respect to this map A if and only if the function: $\varphi \mapsto \varphi(f)$ is constant for any extension φ of the positive linear functional μ on F to a positive linear functional on $B(X)$. Consequently the space $I(X)$ of all $A(F)$-affine functions is a linear subspace of $B(X)$. Define

$$\mu(f)=\overline{Af}=\underline{Af} \quad \text{for each } f \in I(X) ,$$

then μ is a positive linear functional on $I(X)$.

We consider a partition Δ of X:

$$X=\bigcup_{i=1}^{n} X_i, \quad X_i \cap X_j=\phi(i \neq j), \quad X_i \in \mathfrak{A} . \tag{4.2}$$

If a partition Δ_2 is a refinement of Δ_1, we write $\Delta_1 \leqq \Delta_2$. The family of partitions of X is directed by this order relation. For a partition Δ given by (4.2) and $f \in B(X)$, define

$$M_\Delta f= \sum_{i=1}^{n} \left(\sup_{x \in X_i} f(x)\right)\mu(X_i)$$

and

$$m_\Delta f= \sum_{i=1}^{n} \left(\inf_{x \in X_i} f(x)\right)\mu(X_i) .$$

Then (M_Δ) and (m_Δ) are monotone functionals on $B(X)$. For every $g \in F$

$$\lim_{\Delta} M_\Delta g=Ag=\mu(g)$$

and

$$\lim_{\Delta} m_{\Delta} g = Ag = \mu(g) .$$

Consequently, by Remark 4.2,

$$\lim_{\Delta} M_{\Delta} f = \overline{Af} = \mu(f) \quad \text{for all} \quad f \in I(X)$$

and

$$\lim_{\Delta} m_{\Delta} f = \overline{Af} = \mu(f) \quad \text{for all} \quad f \in I(X) .$$

Conversely, suppose that $\lim M_{\Delta} f$ and $\lim_{\Delta} m_{\Delta} f$ and $\lim_{\Delta} m \cdot f$ both exist and the one is equal to the other. Put

$$\lim_{\Delta} M_{\Delta} f = \lim_{\Delta} m_{\Delta} f = k .$$

Then, given every $\varepsilon > 0$, there is a partition Δ_1 such that for all partitions $\Delta \geqq \Delta_1$

$$k - \varepsilon < m_{\Delta} f \leqq M_{\Delta} f < k + \varepsilon .$$

Assume that the partition Δ is given by (4.2) and put

$$g_1 = \sum_{i=1}^{n} \alpha_i \chi_{X_i} \quad \text{and} \quad g_2 = \sum_{i=1}^{n} \beta_i \chi_{X_i} ,$$

where $\alpha_i = \sup_{x \in X_i} f(x)$ and $\beta_i = \inf_{x \in X_i} f(x)$. Then $g_1 \geqq f$ and $f \geqq g_2$, and both g_1 and g_2 belong to F. Consequently,

$$M_{\Delta} f = Ag_1 \geqq \overline{Af} \geqq Af \geqq Ag_2 = m_{\Delta} f .$$

This implies

$$k + \varepsilon > \overline{Af} \geqq \underline{Af} > k - \varepsilon ,$$

whence $\overline{Af} = \underline{Af}$. Therefore, we have

PROPOSITION 4.2. *A function* $f \in B(X)$ *is* $A(F)$*-affine if and only if* $\lim_{\Delta} M_{\Delta} f$ *is equal to* $\lim_{\Delta} m_{\Delta} f$ *for the net* (M_{Δ}) *and* (m_{Δ}).

This integral is a generalization ot the Riemann-Stieltjes integral.

§ 5. The order convergence

Let Y be a locally compact Hausdorff space and B an adapted space in $C(Y)$. For a closed subset S of Y, the vector space $B_0(S)$ of all B^+-bounded continuous functions on S is also an adapted space and a sub-vector lattice of $\boldsymbol{R}^S$. Therefore, according to Bauer, we can consider the order topology on $B_0(S)$ ([3, §1]). This topology, by definition, is the finest locally convex topology on $B_0(S)$ for which every order interval in $B_0(S)$

$$\{f \in B_0(S): -g \leqq f \leqq g \text{ on } S\} \quad (g \in B_0(S),\ g \geqq 0)$$

is bounded.

Let $\boldsymbol{E}=\boldsymbol{E}(B^+)$ be the set of all functions ε of B^+ into $(0, +\infty]$. For every $\varepsilon \in \boldsymbol{E}$ we denote by $V_\varepsilon(S)$ the set of all B^+-bounded continuous functions f on S with following property: there exist finitely many functions $h_1, \cdots, h_r \in B^+$ and corresponding numbers $\lambda_1, \cdots, \lambda_r \in R^+$ with sum $\sum_{i=1}^{r} \lambda_i = 1$ such that

$$|f| \leqq \sum_{i=1}^{r} \lambda_i \varepsilon(h_i) h_i \quad \text{on} \quad S\,.$$

Then, Corollary 1.2 and Lemma 1.4 in [3], the family $(V_\varepsilon(S))_{\varepsilon \in \boldsymbol{E}}$ is a fundamental system of convex and symmetric neighborhoods of the constant function 0 on S with respect to the order topology on $B_0(S)$. When a net in $B_0(S)$ converges to $h \in B_0(S)$ in the order topology on $B_0(S)$, it is said to be order convergent on S to h.

The following lemma by Bauer is an extension of Dini's theorem to the locally compact case.

LEMMA 5.1. ([3, Lemma 2.1]) *Let $(g_i)_{i \in I}$ be a decreasing net in $B_0(S)$ which satisfies*

$$\inf_{i \in I} g_i(y) = 0 \quad \text{for all} \quad y \in S\,\cdot$$

Then the net $(g_i)_{i \in I}$ is order convergent on S to 0.

We remark that if a net $B_0(S)$ is order convergent on S, it converges locally uniformly on S.

Let F be a linear subspace of an ordered vector space E such that E is F^+-bounded and let Y be a locally compact Hausdorff space. Assume that the image of a positive linear map A of F into $C(Y)$ is an adapted space in $C(Y)$. Then the following Korovkin-type theorem for the order topology is a generalization of Theorem 3.1 in [3].

THEOREM 5.1. *Let S be a closed subset of Y and $(L_i)_{i \in I}$ a net of monotone maps of E into $\boldsymbol{R}^Y$. If $(L_i g)_{i \in I}$ is order convergent on S to Ag for every $g \in F$, then $(L_i f)_{i \in I}$ is order convergent on S to*

$\overline{Af}$ *for every* $f \in E$ *satisfying* $\overline{Af}=\underline{Af}$ *on* S.

PROOF. Assume that $f \in E$ satisfies $\overline{Af}=\underline{Af}$ on S. Then the restriction $\overline{Af}|S$ of $\overline{Af}$ to S is continuous on S. Put

$$\mathscr{G}_f^S=\{(\inf(Ah_1, \cdots, Ah_n)-\overline{Af})|S: n \in N, h_i \in F, h_i \geqq f\} .$$

Then $\mathscr{G}_f^S$ is a decreasing net in $B_0(S)$ and satisfies $\inf\{g(y): g \in \mathscr{G}_f^S\}=0$ for every $y \in S$. By Lemma 5.1 there exist, for any $\varepsilon \in E$, $h'_1, \cdots, h'_n \in F$ such that

$$h'_j \geqq f, \ \bar{g}=\inf(Ah'_1, \cdots, Ah'_n) \geqq \overline{Af}, \quad \text{and} \quad \bar{g}|S-\overline{Af}|S \in V_{\varepsilon/8}(S)$$

Similarly, there exist, for any $\varepsilon \in E$, $h''_1, \cdots, h''_m \in F$ such that

$$h''_j \leqq f, \ \underline{g}=\sup(Ah''_1, \cdots, Ah''_m) \leqq \underline{Af} \text{ and } \underline{Af}|S-\underline{g}|S \in V_{\varepsilon/8}(S) .$$

Since L_i is monotone,

$$L_i f-\bar{g} \leqq \inf_{1 \leqq j \leqq n} L_i h'_j - \inf_{1 \leqq j \leqq n} Ah'_j \leqq \sum_{j=1}^{n} |L_i h'_j - Ah'_j| .$$

Similarly,

$$\underline{g}-L_i f \leqq \sup_{1 \leqq j \leqq m} Ah''_j - \sup_{1 \leqq j \leqq m} L_i h''_j \leqq \sum_{j=1}^{n} |Ah''_j - L_i h''_j| .$$

Hence

$$(5.1) \qquad |L_i f-\overline{Af}| \leqq \sum_{j=1}^{m} |L_i h'_j - Ah'_j| + \sum_{j=1}^{n} |AL''_j - L_i h''_j| + |\bar{g}-\overline{Af}| + \bar{g} - \underline{g} .$$

Since for each j $(L_i h'_j)_{i \in I}$(resp. $(L_i h''_j)_{i \in L}$) is order convergent on S to Ah'_j(resp. Ah''_j), there exists an index i_0 such that for every $i \geqq i_0$

$$L_i h'_j - Ah'_j \in V_{\varepsilon/4n}(S) \quad (j=1, \cdots, n)$$

and

$$L_i h''_j - Ah''_j \in V_{\varepsilon/4m}(S) \quad (j=1, \cdots, m) .$$

Hence, from (5.1) it follows that $L_i f-\overline{Af} \in V_\varepsilon(S)$. Thus it is concluded that $(L_i f)$ is order convergent on S to $\overline{Af}$.

REMARK 5.1. It is obvious that the results with respect to the order convergence of the types of Corollary 3.3 and 3.4 are also obtained by Theorem 5.1.

Finally, we denote by $S_0(F, A)$ the set of all $f \in E$ having the following property: for every net $(L_i g)_{i \in I}$ of monotone maps of E into $\boldsymbol{R}^Y$ for which $(L_i g)$ is order convergent to Ag for all $g \in F$, $(L_i f)$

is order convergent to a certain $Lf \in B_0(Y)$ independently of (L_i). Corresponding to Theorem 3.4 we obtain the following theorem (cf. [3, Theorem 3.4]).

THEOREM 5.2. *The set $S_0(F, A)$ is equal to the set of all $A(F)$-affine elements.*

PROOF. From Theorem 5.1 it follows that the set $S_0(F, A)$ contains all $A(F)$-affine elements. Conversely, suppose that $f \in S_0(F, A)$ and $z \in Y$. It suffices to prove that $\mu \mapsto \mu(f)$ is constant on $\mathfrak{M}_z$. For each $\mu \in \mathfrak{M}_z$, we consider the net $(L_V)_{V \in \mathscr{B}}$ of monotone maps of E into $\boldsymbol{R}^Y$ used in the proof of Theorem 3.4. For given $V \in \mathscr{B}$ there is $h_V \in B^+$ satisfying $q_V \leqq h_V$ since q_V has a compact support. For any $\varepsilon \in \boldsymbol{E}$ and for any $g \in F$,

$$|L_W g - Ag| = |Ag(z) - Ag| q_W \leqq \varepsilon(h_V) h_V$$

holds for all sufficiently small $W \subset V$. Thus $(L_V g)$ is order convergent to Ag for all $g \in B$. Since $f \in S_0(F, A)$, $(L_V f)$ is also order convergent to Lf. In particular $\mu(f) = Lf(z)$. From Proposition 2.1 it follows that f is $A(F)$-affine.

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