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A Condition for a Compact Kaehlerian Space to be Locally Symmetric

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1. Let M be a Riemannian space of class C^∞ and $R=(R_{ijkl})$ be the Riemannian curvature tensor on M. If M is locally symmetric, then it satisfies

$$(*) \qquad R(X, Y).R=0$$

for any tangent vectors X and Y. Conversely, it has been studied by several authors under what additional conditions (*) implies local symmetricity. There was given by H. Takagi an example of a complete irreducible, 3-dimensional Riemannian space which satisfies (*) and is not locally symmetric.

In this note we show the following

THEOREM A. *If a compact Kählerian space of constant scalar curvature satisfies the condition* (*), *then the space is locally symmetric.*

2. Let M be a Kählerian space with complex structure tensor $\varphi=(\varphi_i^j)$. We denote by $R_1=(R_{ij})$ and R' the Ricci tensor and the scalar curvature. It is well known that the following formulas are satisfied:

$$(1) \qquad \varphi_a^r R_{rbcd}=\varphi_b^r R_{racd}\ ,$$

$$(2) \qquad \varphi_a^r R_{rb}=-\frac{1}{2}\varphi^{rs} R_{rsab}\ ,$$

It is also known that the 2-form $S=(1/2)S_{ij}dx^i\wedge dx^j$ defined by $S_{ij}=\varphi_i^r R_{rj}$ is closed. Denoting the covariant derivative and the codifferential operator by ∇_i and δ, we have

$$\begin{aligned}(\delta S)_a&=-\nabla^r(\varphi_r^s R_{sa})\\ &=\varphi_a^s\nabla^r R_{rs}=\frac{1}{2}\varphi_a^s\nabla_s R'\ ,\end{aligned}$$

and hence the 2-form S is coclosed if and only if the scalar curvature R' is constant.

A. Lichnerowicz has obtained the following integral formula which is valid in a compact orientable Riemannian space M:

$$(3) \qquad 2\int_M [(\nabla^c R^{bd} - \nabla^d R^{bc})(\nabla_c R_{bd} - \nabla_d R_{bc}) - K]\eta = \int_M (\nabla^r R^{abcd} \nabla_r R_{abcd})\eta ,$$

where η means the volume element of M and

$$K = R^{abcd} H^{\ \ \ \ r}_{bcd,ra} ,$$
$$H_{abcd,ij} = \nabla_i \nabla_j R_{abcd} - \nabla_j \nabla_i R_{abcd} .$$

The condition (*) is equivalent to $H_{abcd,ij}=0$.

3. Proof of Theorem A. Assume that the Kählerian space M is compact and the scalar curvature R' is constant. Then we have $\Delta S=0$, where $\Delta = d\delta + \delta d$ is the Laplacian operator. Hence

$$0 = (\Delta S)_{ab} = -\varphi_a^r(\nabla^s \nabla_s R_{rb} - R_r^s R_{sb} - R_b^s R_{rs}) + R_{ba}{}^{rs} \varphi_r^t R_{ts}$$

is valid, from which we have

$$\nabla^s \nabla_s R_{ab} = 2R_a^s R_{sb} - \varphi_a^u R_{ub}{}^{rs} \varphi_r^t R_{ts} .$$

Using the Bianchi's identity and (1), (2) the second term of the right hand side becomes

$$\begin{aligned} -\varphi_a^u R_{ub}{}^{rs} \varphi_r^t R_{ts} &= \varphi_a^u R_{ursb} \varphi^{rt} R_t^s + \varphi_a^u R_{usbr} \varphi^{rt} R_t^s \\ &= R_{tasb} R^{ts} + \varphi_s^u R_{uabr}(-\varphi^{st} R_t^r) \\ &= 2R_{arbs} R^{rs} . \end{aligned}$$

Hence we have

$$(4) \qquad \nabla^s \nabla_s R_{ab} = 2(R_a^s R_{sb} + R_{arbs} R^{rs}) .$$

Let us assume that the condition

$$(**) \qquad R(X, Y).R_1 = 0$$

holds good for tangent vectors X and Y. Since (**) implies that

$$R_{abi}{}^r R_{rj} + R_{abj}{}^r R_{ri} = 0 ,$$

we have

$$R_{arbs} R^{rs} = -R_{ar}{}^{rs} R_{bs} = -R_{as} R_b^s .$$

Therefore $\nabla^r\nabla_r R_{ab}=0$ is obtained from (4), and we get

$$\begin{aligned}\nabla_a R_{bc}\nabla^a R^{bc} &= \nabla_a(R_{bc}\nabla^a R^{bc})-\nabla^a\nabla_a R_{bc}R^{bc} \\ &= \nabla_a(R_{bc}\nabla^a R^{bc}) .\end{aligned}$$

Integrating this equation on M, we have

$$\int_M (\nabla_a R_{bc}\nabla^a R^{bc})\eta=0 ,$$

from which $\nabla_a R_{bc}=0$ follows. Thus we proved the following

THEOREM B. *If a compact Kählerian space of constant scalar curvature satisfies the condition* (**), *then the Ricci tensor is parallel.*

If the condition (*) is satisfied, then it is clear that (**) is valid and K in the previous section vanishes, since $H_{abcd,ij}=0$. Making use of the integral formula of A. Lichnerowicz, it is shown that the curvature tensor is parallel. This proves Theorem A.

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