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Some Spectral Properties of Positive, Irreducible Operators in R-space

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1. Introduction

The purpose of this note is to remark that the theorem obtained by F. Niiro and I. Sawashima for positive, irreducible operators in arbitrary Banach lattices can be extended to positive, irreducible operators in arbitrary R-spaces, and then, to examine the relationship between quasi-compactness and uniform ergodicity as an extension of the result by M. Lin [4].

An ordered Banach space E is called an *R-space* if its predual is a Banach lattice [1]. A positive operator in an R-space E is called to be *irreducible* if there is no closed, T-invariant ideal of E, distinct from $\{0\}$ and E; here an *ideal* I means a subspace of E satisfying the following conditions

i) if $0 \leqq y \leqq x \in I$, then $y \in I$;

ii) if $x \in I$, there are $y, z \in I$ with $y \geqq 0$, $z \geqq 0$ and $x = y - z$.

2. The spectral property of the spectral circle

As an extension of the main theorem of [6], we get the following theorem.

THEOREM 1. *Let E be an R-space and $T \in \mathfrak{L}(E)$ be a positive, irreducible operator, whose resolvent has the point $\lambda = r(T)$ as its pole. Then the spectrum of T on the spectral circle $\Gamma = \{\lambda\colon |\lambda| = r(T)\}$ coincides with the set A, where the set A consists of k-th roots of unity multiplied by $r(T)$, each of which is a simple pole of $R(\lambda, T)$, where k is a fixed positive integer determined by T.*

Since the dual operator of an irreducible operator is not necessarily irreducible, we modify the main theorem of [6] as the following lemma.

LEMMA. *Let E be a Banach lattice, $T \in \mathfrak{L}(E)$ be a positive operator whose resolvent has the point $\lambda = r(T)$ as its pole and the proper space of T for $r(T)$ be one-dimensional. Then the spectrum of T on Γ coincides with the set A mentioned at Theorem* 1.

PROOF. Since $\lambda = r(T)$ is a pole of $R(\lambda, T)$ and the proper space of T for $r(T)$ is one-dimensional, $\lambda = r(T)$ is a simple pole and therefore the proper space is PE, where P is the residual operator of $R(\lambda, T)$ at $\lambda = r(T)$. Consider the subspace $F = \{x \in E: P|x| = 0\}$. Then F is a T-invariant closed ideal in E. By using lemma 7 of [8] and the main theorem of [6], we obtain the desired result.

PROOF OF THEOREM 1. Since the dual operator $T' \in \mathfrak{L}(E')$ is a positive operator in a Banach lattice E' and the proper space of T' for $r(T')$ is one-dimensional by irreducibility of T [7], the spectrum of T' on Γ coincides with the set A by Lemma. Since the property of a pole of $R(\lambda, T)$ is the same with that of $R(\lambda, T')$, we get Theorem.

Considering the above Lemma, we see that the irreducibility of T is not so essential and obtain the following Corollary.

COROLLARY 1. *Let E be an R-space, $T \in \mathfrak{L}(E)$ be a positive operator whose resolvent has the point $\lambda = r(T)$ as its pole and the proper space of T for $r(T)$ be one-dimensional. Then the spectrum of T on Γ coincides with the set A mentioned at Theorem* 1.

PROOF. Since the proper space of T for $r(T)$ is one-dimensional, $\lambda = r(T)$ is a simple pole and the proper space of T' for $r(T')$ is one-dimensional by Th. 2 of [9]. Therefore we get Corollary by Lemma.

In the similar way as Theorem 1, we can easily obtain the following Corollary by applying Th. 2 of [5] to the dual operator.

COROLLARY 2. *Let E be an R-space, $T \in \mathfrak{L}(E)$ be a positive operator whose resolvent has the point $\lambda = r(T)$ as its pole and the proper space of T for $r(T)$ be finite dimensional. Then the spectrum of T on Γ consists of finitely many points, which are all roots of unity and poles of $R(\lambda, T)$. And moreover each proper space is finite dimensional.*

3. An application

DEFINITION. An operator $T \in \mathfrak{L}(E)$ is called to be *quasi-compact* if there exist a positive integer n and a compact operator $K \in \mathfrak{L}(E)$ such that $\|T^n - K\| < 1$.

K. Yosida examined the property of a quasi-compact* operator in a Banach space and obtained that the proper values of $T \in \mathfrak{L}(E)$ with modulus 1 are isolated proper values of finite multiplicities if T is quasi-compact [10]. By using this result and the corollary of Theorem 1, we obtain the following theorem for quasi-compact operators.

THEOREM 2. *Let E be an R-space and $T \in \mathfrak{L}(E)$ be a positive operator with $r(T)=1$. Then the following conditions are equivalent.*

i) *T is quasi-compact.*

ii) *1 is a pole of $R(\lambda, T)$ and the proper space of T for 1 is finite dimensional.*

PROOF. i)$\Rightarrow$ii) is known (cf. [2], [10]).

ii)$\Rightarrow$i): By Corollary 2 of Theorem 1, $\sigma(T) \cap \Gamma$ consists of a finite number of points $\{\lambda_1, \lambda_2, \cdots, \lambda_s\}$, each of which is a pole of $R(\lambda, T)$ and each proper space is finite dimensional. Let P_i be the residual operator of $R(\lambda, T)$ at $\lambda = \lambda_i$. Then P_i is a projection with $P_i T = T P_i$ and $P_i P_j = 0$ if $i \neq j$. So

$$E = \left(\sum_{i=1}^{s} P_i\right) E \oplus \bigcap_{i=1}^{s} (I - P_i) E .$$

Since the restriction of T to $(I - \sum_{i=1}^{s} P_i)\, E$ has no spectral points on Γ, we have $\|T^n (I - \sum_{i=1}^{s} P_i)\| \to 0$ as $n \to \infty$. So there exists a number n_0 such that

$$\|T^{n_0} (I - \sum_{i=1}^{s} P_i)\| < 1 .$$

Since $P_i E$ is finite dimensional if and only if the proper space for λ_i is finite dimensional, the subspace $(T^{n_0} \sum_{i=1}^{s} P_i) E = (\sum_{i=1}^{s} P_i T^{n_0}) E$ is finite dimensional. So $T^{n_0} \sum_{i=1}^{s} P_i$ is compact and therefore T is quasi-compact.

For a positive contraction $T \in \mathfrak{L}(E)$, equivalence between uniform ergodicity and quasi-compactness is obtained as the following Corollary, which is the extension of the result by M. Lin about a positive contraction of $C(X)$ or of $L(m)$ [4].

COROLLARY. *Let E be an R-space and $T \in \mathfrak{L}(E)$ be a positive*

* K. Yosida defined a quasi-compact operator as a quasi-completely-continuous operator.

contraction. Then the following are equivalent.

i) *T is quasi-compact.*

ii) $N^{-1}\sum_{i=1}^{N} T^i$ *converges uniformly to a finite dimensional projection.*

PROOF. Since T is a contraction, each pole on Γ is a simple pole by VIII 8.1 of [2]. For a positive operator, $N^{-1}\sum_{i=1}^{N} T^i$ converges uniformly if and only if 1 is a simple pole of $R(\lambda, T)$ by Theorems 5 and 6 of [3]. So Corollary is obtained by Th. 2.

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