On a Generalization of Denjoy Integration

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§1. Introduction. We are concerned with the *quasi-Denjoy integration* introduced by Iseki [1]. It was invented as a generalization of the Denjoy-Khintchine process of integration for functions of one real variable.

At the end of [1] there was given a sketched account of a family of functions which are GHC (see [1], §3), without being GAC (i.e. ACG; see [2], p. 223), on the unit interval [0, 1]. It thus turned out that the quasi-Denjoy integration is actually wider than that of Denjoy-Khintchine.

It is the object of the present paper to deal in detail with the formation of the above family. This will occur as follows: Fixing first a positive constant $\delta<1$, we shall attach to each closed interval I a continuous function $P(x)$ depending on δ, among others, and fulfilling certain conditions. This procedure, which is somewhat complicated, will constitute the subject matter of §2. Once $P(x)$ is obtained, it is easy to construct a continuous function $B(x)=B(x;\delta, P)$, which will be shown afterwards to be GHC, but not GAC, on [0, 1]. The construction of this function, as well as the verification, not quite simple, of its mentioned property, will be our concernment in §3. Our required family of functions will be no other than the totality of the functions $B(x;\delta, P)$ for all choices of δ and P.

The term *function* will exclusively mean a point-function defined on the whole real line $\boldsymbol{R}$ and assuming finite real values, unless another meaning is implied by the context. By *intervals*, by themselves, we shall always understand linear non-degenerate closed intervals. If f is a function and J an interval, the symbol $f(J)$ will denote the increment of f on J, while the image of J under the mapping f will be written $f[J]$, in conformity with Saks [2] (p. 99 and p. 100). The letter U will be reserved for the unit interval [0, 1]. The symbol $|J|$ will stand for the length of an interval J.

§2. Construction of the function $P(x)$. Given a positive number $\delta<1$ and an interval $I=[a, b]$, consider in I an increasing infinite sequence of points $a_1<a_2<\cdots$ tending to the point b, where we require that $a_1=a$. We shall write for brevity $I_n=[a_n, a_{n+1}]$ $(n=1, 2, \cdots)$.

LEMMA 1 (see [1], § 7). *The above sequence $a_1<a_2<\cdots$ can be so chosen as to satisfy the following condition* (i) *and, furthermore, to ensure the existence of a nonnegative continuous function $P(x)$ vanishing outside the interval $I=[a, b]$ and subject to the conditions* (ii) *to* (v) *below:*

(i) $\sum_{n=1}^{\infty}|I_{2n-1}|^\delta<\frac{1}{2}|I|^\delta$;

(ii) *$P(x)$ is a constant on each odd-numbered interval I_{2n-1} (where $n=1, 2, \cdots$)*;

(iii) *$P(x)$ is linear in x, but not a constant, on each even-numbered interval I_{2n} $(n=1, 2, \cdots)$*;

(iv) $\sum_{n=1}^{\infty}|P(I_{2n})|=+\infty$, *so that $P(x)$ is not of bounded variation on I*;

(v) $|P(J)|<|J|^\delta$ *for every interval J (which need not lie in I).*

PROOF. Writing $h=\delta^{-1}$ for brevity and choosing a number α such that $1<\alpha<h$, let us put

$$A=\frac{1}{4}|I|^\delta, \qquad M=\frac{|I|-A^h\zeta(2h)}{2\zeta(\alpha)},$$

where ζ is the Riemann zeta-function. Then $A^h\zeta(2h)=4^{-h}\cdot|I|\cdot\zeta(2h)$. But $\zeta(2h)<\zeta(2)<1+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=2<4^h$, and so $M>0$.

We now determine the required sequence $\langle a_n\rangle_{n=1,2,\cdots}$ inductively as follows $(m=1, 2, \cdots)$:

$$a_1=a,$$

$$a_{4m-2}=a_{4m-3}+\left\{\frac{A}{(2m-1)^2}\right\}^h, \qquad a_{4m-1}=a_{4m-2}+\frac{M}{m^\alpha},$$

$$a_{4m}=a_{4m-1}+\left\{\frac{A}{(2m)^2}\right\}^h, \qquad a_{4m+1}=a_{4m}+\frac{M}{m^\alpha}.$$

We then have

$$\lim_{n\to\infty}a_n=a+\sum_{n=1}^{\infty}\left(\frac{A}{n^2}\right)^h+2\sum_{m=1}^{\infty}\frac{M}{m^\alpha}$$

$$=a+A^h\zeta(2h)+|I|-A^h\zeta(2h)=b.$$

Moreover

$$\sum_{n=1}^{\infty}|I_{2n-1}|^\delta=\sum_{n=1}^{\infty}\left(\frac{A}{n^2}\right)^{h\delta}=A\cdot\zeta(2)<2A=\frac{1}{2}|I|^\delta.$$

Consequently condition (i) is satisfied.

Making use of the above sequence $\langle a_n \rangle_{n=1,2,\cdots}$, we construct a non-negative function $P(x)$ as follows (where $m=1, 2, \cdots$):

$$P(x)=\begin{cases} 0 & \text{when } x\in I_{4m-3}, \\ A_m\cdot(x-a_{4m-2}) & \text{when } x\in I_{4m-2}, \\ A_m\cdot|I_{4m-2}| & \text{when } x\in I_{4m-1}, \\ A_m\cdot(a_{4m+1}-x) & \text{when } x\in I_{4m}, \\ 0 & \text{when } x\in \boldsymbol{R}-I^\circ, \end{cases}$$

where $A_m=\frac{1}{2}\left(\frac{M}{m^\alpha}\right)^{\delta-1}$ and I° means the interior of I.

Needless to say, the function P thus defined fulfils conditions (ii) and (iii). By the relation $A_m\cdot|I_{4m-2}|=\frac{1}{2}\left(\frac{M}{m^\alpha}\right)^{\delta}\to 0$ (as $m\to+\infty$) we find further that P is continuous. We have also

$$\sum_{n=1}^{\infty}|P(I_{2n})|=2\sum_{m=1}^{\infty}|P(I_{4m})|=2\sum_{m=1}^{\infty}A_m\cdot|I_{4m}|$$

$$=\sum_{m=1}^{\infty}\left(\frac{M}{m^\alpha}\right)^{\delta}=M^\delta\sum_{m=1}^{\infty}\frac{1}{m^{\alpha\delta}}=+\infty,$$

since $1<\alpha<h=\delta^{-1}$. This establishes condition (iv).

It remains to verify condition (v) which asserts that $|P(J)|<|J|^\delta$ for every interval $J=[u, v]$. For this purpose, it is convenient to premise the following considerations:

(a) If $u, v\notin I$, then $P(u)=0=P(v)$, so that $P(J)=0$;

(b) if $u\notin I$ and $v\in I$, then necessarily $u<a$ and $P(u)=0=P(a)$, so that $P(J)=P(v)-P(a)$, where $0\leqq v-a<|J|$;

(c) similarly, if $u\in I$ and $v\notin I$, then $P(J)=P(b)-P(u)$, where we have $0\leqq b-u<|J|$;

(d) if $u\in I$ and $v=b$, then there exist in the interior of J points v' at which $P(v')=0=P(v)$.

In view of (a)~(d) above, it suffices to consider the case $J\subset[a, b)$.

Noting that $[a, b)=\bigcup_{n=1}^{\infty}I_n$, suppose first that $J\subset I_n$ for some n. Then, $P(J)$ vanishes if the number n is odd, while we find for even n that $|P(J)|=2^{-1}\cdot|I_n|^{\delta-1}\cdot|J|<|J|^\delta$. Thus condition (v) is satisfied.

In what follows, we may thus assume that $u\in I_n$ and $v\in I_m$, where $n<m$. When n is odd, then by condition (ii) the function P takes the same value at the point u and at the left-hand extremity of I_{n+1}. Hence we may restrict to even values of n. Similarly, m may also be assumed even.

For later use let us observe here that

$$P[I_l]=[0,\ 2^{-1}|I_l|^\delta] \quad \text{for} \quad l=2,\ 4,\ 6,\ \cdots.$$

The inclusions $P[I_2]\supset P[I_4]\supset\cdots$ are also worthy of note.

This being so, we proceed to treat the following five cases separately:

(1) $n=4j-2$ and $m=4j$;
(2) $n=4j-2$ and $m>4j$;
(3) $n=4j$ and $P(u)\geqq P(v)$;
(4) $n=4j$, $m=4k-2$ and $P(u)<P(v)$;
(5) $n=4j$, $m=4k$ and $P(u)<P(v)$;

where $j=1,\ 2,\ \cdots$ and $k=j+1,\ j+2,\ \cdots$.

re(1): In this case, $P(x)$ increases on I_n and decreases on I_m, and moreover $P[I_n]=[0,\ 2^{-1}|I_n|^\delta]=[0,\ 2^{-1}|I_m|^\delta]=P[I_m]$. Accordingly, either there is in I_m a point $u'<v$ at which $P(u')=P(u)$, or there is in I_n a point $v'\geqq u$ at which $P(v')=P(v)$. We are thus reduced to the case $J\subset I_l$ (l even) considered already.

re(2): Since $P[I_m]\subset P[I_{4j}]$, there is in I_{4j} a point $v'<v$ at which $P(v')=P(v)$, and the required result follows from case (1).

re(3): Noticing that $P(x)$ decreases on I_n, we can find in I_n a point $v'\geqq u$ at which $P(v')=P(v)$, and the problem reduces to the case $J\subset I_n$.

re(4): In this case, we need only choose in I_m a point $u'<v$ at which $P(u')=P(u)$.

re(5): There is in I_m a point $u'>v$ at which $P(u')=P(u)$. But we have $u'-v\leqq|I_m|\leqq|I_{4j+2}|<v-u$. Hence the result.

This completes the verification of condition (v).

§ 3. The GHC function $B(x)$ which is not GAC.

Given a positive number $\delta<1$, suppose we have attached to each interval $I=[a,\ b]$ a continuous function P which conforms to the import of Lemma 1 and is otherwise arbitrary. On account of conditions (ii) and (iii) of the same lemma, the sequence $\langle a_n\rangle$ is then uniquely associated with I. When we make mention of $\langle a_n\rangle$ and P later on, we shall write

$$a_n=a_n(I) \text{ for every } n \text{ and } P(x)=P(x\,;\,I)$$

in case definiteness of notation is required.

Generally following the indication of [1], but deviating from it in some minor points, we now go on to construct a function which is GHC, but not GAC, on the unit interval $U=[0,\ 1]$. Let us begin with the following

DEFINITION. If $f(x)$ is a continuous function and J an interval, then any maximal interval contained in J and on which f is a constant, will be called *maximal interval of constancy for f relative to J.*

EXAMPLE. For each interval I, the maximal intervals of constancy for $P(x\,;I)$ relative to I are exactly the intervals $[a_{2n-1}(I), a_{2n}(I)]$, where $n=1, 2, \cdots$.

LEMMA 2. *Given a continuous function $f(x)$ and a real constant $c\neq 0$, let $F(x)$ be the indentation* (see [1], § 7) *of $f(x)$ and suppose that the function $g(x)=f(x)+cF(x)$ is a constant on an interval I. Then $f(x)$ and $F(x)$ are likewise each a constant on I.*

PROOF. It suffices to show that $F(x)$ is a constant over I. Suppose, if possible, that this is false, so that $F[I]$ is a non-countable set by continuity of F. It follows at once, in view of the definition of the indentation F, that the function f has at least one maximal interval of constancy relative to U. As we find furthermore, the maximal open intervals contained in U and on which F is separately a non-constant linear function can be arranged in an infinite sequence $O_1, O_2, \cdots$. Plainly, the function f is a constant on each O_n. Again, the indentation F assumes at most a countable infinity of values outside the union $\bigcup_n O_n$. But $F[I]$ is non-countable as already mentioned, and so the interval I must intersect some one of the intervals O_n, say O_k. Then $f(x)$ is a constant on the interval $I\cap O_k$, whereas $F(x)$ is not. This contradicts the constancy on I of the function $g(x)=f(x)+cF(x)$ and completes the proof.

DEFINITIONS AND NOTATION. The letters n, i, j, p, q will denote positive integers in the following lines.

(1) Consider the ordered pairs of positive integers. We can arrange all of them in a distinct sequence, as follows:

$$\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 1, 3\rangle, \langle 2, 2\rangle, \langle 3, 1\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle, \cdots,$$

wherein $\langle p, q\rangle$ precedes $\langle p', q'\rangle$ if and only if either

$$\text{(a)} \quad p+q<p'+q', \quad \text{or} \quad \text{(b)} \quad p+q=p'+q' \quad \text{and} \quad p<p'.$$

When $\langle p, q\rangle$ is the i-th pair in the above sequence, we shall write $\langle p, q\rangle=\Omega(i)$ temporarily.

(2) We define the intervals K_i^n and the intervals $K_{i,j}^n$ by induction on n, as follows (the letter U always means the unit interval):

$$K_i^1=[a_{2i-1}(U), a_{2i}(U)], \quad K_{i,j}^n=[a_{2j-1}(K_i^n), a_{2j}(K_i^n)],$$

$$K_i^{n+1}=K_{p,q}^n \quad \text{where} \quad \langle p, q\rangle=\Omega(i).$$

It is clear that, when n and i are fixed, the intervals $K^n_{i,j}$ (where $j=1, 2, \cdots$) are no other than the maximal intervals of constancy for $P(x\,;\,K^n_i)$ relative to K^n_i.

(3) We shall write for brevity $K^n=\bigcup_{i=1}^{\infty} K^n_i$, so that $K^{n+1}=\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} K^n_{i,j}$. Clearly $K^1\supset K^2\supset\cdots$, that is, the sequence $\langle K^n\rangle_{n=1,2,\cdots}$ is descending.

(4) We shall also write $\mathfrak{K}^n=\{K^n_i\}_{i=1,2,\cdots}$, so that $\mathfrak{K}^n$ is a disjoint collection of intervals for every n.

(5) We define the sets L^{n+1}_i and the sets L^{n+1} by

$$L^{n+1}_i=K^n_i-\bigcup_{j=1}^{\infty} K^n_{i,j} \qquad \text{and} \qquad L^{n+1}=\bigcup_{i=1}^{\infty} L^{n+1}_i .$$

On the other hand, we set $L^1=U-K^1$.

REMARK. It should be noted that no sets L^1_i have been defined.

LEMMA 3. *Given a nonvoid disjoint collection* $\mathfrak{M}$ *of intervals contained in* U, *let us write*

$$H(x)=H(x\,;\,\mathfrak{M})=\sum_{I\in\mathfrak{M}} P(x\,;\,I) \quad \textit{for} \quad x\in\boldsymbol{R}.$$

Then $H(x)$ *is a nonnegative continuous function vanishing outside* U. *Moreover,* $H(x)<1$ *for every* x *and* $|H(J)|<|J|^\delta$ *for every interval* J.

REMARK. Evidently, the collection $\mathfrak{M}$ is at most countable.

PROOF. Let us fix any interval $I\in\mathfrak{M}$ and consider the function $f(x)=P(x\,;\,I)$ of Lemma 1. This function is nonnegative and vanishes outside the interior of I. We have further $|f(J)|<|J|^\delta$ for every interval J, by condition (v) of the same lemma. It follows from this inequality and $f(0)=0$ that $f(x)<1$ on U. We then have $f(x)<1$ for every x.

The above consideration shows at once that $H(x)$ fulfils $0\leqq H(x)<1$ for every x and vanishes outside U. Also the inequality $|H(J)|<|J|^\delta$ follows easily from the above, if we write $J=[\alpha, \beta]$ and examine the following five cases separately:

(1) One of the intervals $I_1, I_2, \cdots$ contains both α and β;
(2) both α and β are situated outside $I_1, I_2, \cdots$;
(3) $\alpha\in I_p$ for some p, but β belongs to none of $I_1, I_2, \cdots$;
(4) $\beta\in I_q$ for some q, but α belongs to none of $I_1, I_2, \cdots$;
(5) $\alpha\in I_p$ and $\beta\in I_q$ for some p and some q, where $p\neq q$.

The inequality just obtained plainly implies the continuity of $H(x)$, and the proof is complete.

DEFINITIONS. Let us define two sequences of functions $\langle H_n\rangle$ and $\langle B_n\rangle$, where $n=0, 1, \cdots$. We set first identically

$$H_0(x)=P(x\,;\,U) \qquad \text{and} \qquad B_0(x)=0\,.$$

Using the function $H(x\,;\,\mathfrak{M})$ of Lemma 3, we define further $(n=1, 2, \cdots)$

$$H_n(x)=H(x\,;\,\mathfrak{K}^n)\,, \qquad B_n(x)=\sum_{i=0}^{n-1} 2^{-i}H_i(x)\,,$$

$$B(x)=\lim_n B_n(x)=\sum_{i=0}^{\infty} 2^{-i}H_i(x)\,.$$

REMARKS. (i) Clearly $B_{n+1}(x)=B_n(x)+2^{-n}H_n(x)$ for $n=0, 1, \cdots$.

(ii) The sequence $\langle B_n\rangle_{n=0,1,\cdots}$ as defined above differs slightly from the sequence $\langle P_m\rangle_{m=1,2,\cdots}$ of [1], § 7. But this is immaterial for our purposes.

LEMMA 4. *Thus defined, $B(x)$ is a nonnegative continuous function vanishing outside U. Moreover, it is* SC(δ) *on the whole real line* (see [1], § 2).

PROOF. The first half of the assertion is obvious by Lemma 3, especially by the relation $0\leqq H(x)<1$. The second half, too, follows directly from that lemma. In fact, for every interval J,

$$|B(J)|\leqq\sum_{i=0}^{\infty} 2^{-i}|H_i(J)|\leqq\sum_{i=0}^{\infty} 2^{-i}|J|^{\delta}=2|J|^{\delta}\,.$$

LEMMA 5. *For each $n=1, 2, \cdots$, the maximal intervals of constancy for $B_n(x)$ relative to U are exactly the intervals $K_1^n, K_2^n, \cdots$. Thus, the function $H_n(x)$ is the indentation of $B_n(x)$.*

REMARK. The second half of the assertion holds good for $n=0$ also. In fact, $H_0(x)$ is the indentation of $B_0(x)$.

PROOF. Denoting the assertion by $A(n)$, we shall prove it by induction. $A(1)$ is obvious, since $B_1(x)=H_0(x)=P(x\,;\,U)$. Assuming next the truth of $A(n)$, where n is fixed, we shall deduce that of $A(n+1)$.

Given any interval J, let us denote for the nonce by $\mathfrak{M}(J)$ the collection of the maximal intervals of constancy for $B_{n+1}(x)$ relative to J. Since $B_{n+1}(x)=B_n(x)+2^{-n}H_n(x)$, we infer by the assumption $A(n)$ and Lemma 2 that each interval of the collection $\mathfrak{M}(U)$ is contained in K_i^n for some $i=1, 2, \cdots$. It therefore suffices to prove $\mathfrak{M}(K_i^n)=\{K_{i,j}^n\}_{j=1,2,\cdots}$ for each i.

In view of the above expression for $B_{n+1}(x)$ and the constancy of $B_n(x)$ on K_i^n, we find that $\mathfrak{M}(K_i^n)$ consists of the maximal intervals of constancy for $H_n(x)$ relative to K_i^n. But precisely these intervals constitute together the collection $\{K_{i,j}^n\}_{j=1,2,\cdots}$, since $H_n(x)$, by definition, coincides with $P(x\,;\,K_i^n)$ on K_i^n. This completes the proof.

REMARK. The lemma established just now shows how our functions $B_0(x)$, $B_1(x)$, $\cdots$ are connected with the lines of thought of [1], § 7.

In the rest of this paper, we shall not require the full assertion of the above lemma, but only the partial result that the function $B_n(x)$ is a constant on each of the intervals K_1^n, K_2^n, $\cdots$. This latter result can readily be proved without having any recourse to Lemma 2.

LEMMA 6. $\sum_{i=1}^{\infty} |K_i^n|^\delta < 2^{-n}$ $(n=1, 2, \cdots)$.

PROOF. Denoting this inequality by $A(n)$, we shall derive it by induction. $A(1)$ is a special case of condition (i) of Lemma 1. Thus it is the point to ascertain $A(n+1)$ under the assumption $A(n)$. Successively using the same condition and $A(n)$, we find that

$$\sum_{k=1}^{\infty} |K_k^{n+1}|^\delta = \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} |K_{i,j}^n|^\delta) \leqq \frac{1}{2} \sum_{i=1}^{\infty} |K_i^n|^\delta < 2^{-n-1}, \text{ as required.}$$

LEMMA 7. $\bigcup_{j=1}^{n} L^j = U - K^n$ $(n=1, 2, \cdots)$.

PROOF. To prove this relation inductively, let us denote it by $A(n)$. Then $A(1)$ merely restates the definition of the set L^1. Suppose next that $A(n)$ is true. The definition of L^{n+1} shows that

$$L^{n+1} = \bigcup_{i=1}^{\infty} (K_i^n - \bigcup_{j=1}^{\infty} K_{i,j}^n) = K^n - K^{n+1},$$

where $K^{n+1} \subset K^n$. Hence it follows by $A(n)$ that

$$\bigcup_{j=1}^{n+1} L^j = (\bigcup_{j=1}^{n} L^j) \cup L^{n+1} = (U - K^n) \cup L^{n+1} = U - K^{n+1},$$

which completes the proof.

LEMMA 8. *The function $B_n(x)$ is* GAC *on the set L^n for $n=1, 2, \cdots$.*

PROOF. This is obvious when $n=1$, since the set $L^1 = U - K^1$ is composed of one point and a countable infinity of open intervals on each of which the function $B_1(x) = H_0(x) = P(x; U)$ is linear.

Suppose now $n>1$ and consider any $i=1, 2, \cdots$. We have identically $B_n(x) = B_{n-1}(x) + 2^{1-n} H_{n-1}(x)$. But $H_{n-1}(x) = P(x; K_i^{n-1})$ for $x \in K_i^{n-1}$. The same argument as for the case $n=1$ then shows that $H_{n-1}(x)$ is GAC on the set $L_i^n = K_i^{n-1} - \bigcup_{j=1}^{\infty} K_{i,j}^{n-1}$. Noticing that $B_{n-1}(x)$ is a constant on K_i^{n-1} by Lemma 5, we conclude that $B_n(x)$ is GAC on L_i^n. This completes the proof, since i is arbitrary and $L^n = \bigcup_{i=1}^{\infty} L_i^n$.

LEMMA 9. *We have* $B(x)=B_n(x)$ *for* $x\in L^n$ $(n=1, 2, \cdots)$.

PROOF. Let n be fixed and consider any integer $m\geqq n$. The function $H_m(x)=H(x\,;\,\mathfrak{K}^m)$ vanishes on the set $U-K^m$, and Lemma 7 implies that $U-K^m\supset L^n$. Thus $H_m(x)=0$ on L^n for $m=n, n+1, \cdots$. It follows at once that

$$B(x)=\sum_{i=0}^{\infty}2^{-i}H_i(x)=\sum_{i=0}^{n-1}2^{-i}H_i(x)=B_n(x) \quad \text{for} \quad x\in L^n\,.$$

LEMMA 10. *The function* $B(x)$ *is* GAC *on the set* $L=\bigcup_{n=1}^{\infty}L^n$.

PROOF. This is a direct consequence of the preceding two lemmas.

NOTATION. Throughout the rest of the paper, the letter L will retain the meaning specified above and we shall write $E=U-L$.

LEMMA 11. $E=\bigcap_{n=1}^{\infty}K^n$.

PROOF. This follows immediately from Lemma 7, as follows:

$$E=U-\bigcup_{n=1}^{\infty}L^n=U-\bigcup_{n=1}^{\infty}\bigcup_{j=1}^{n}L^j=\bigcap_{n=1}^{\infty}(U-\bigcup_{j=1}^{n}L^j)=\bigcap_{n=1}^{\infty}K^n\,.$$

LEMMA 12. $\Lambda_\delta(E)=0$ (see [2], p. 53).

NOTATION. The *diameter* of a linear set X will be denoted by $\mathrm{d}(X)$.

PROOF. Let n be any positive integer. Then $E\subset K^n$ by the foregoing lemma. Hence, if we write $E_i^n=E\cap K_i^n$ for brevity, we have $E=\bigcup_{i=1}^{\infty}E_i^n$. On account of Lemma 6, this partition of E has the property $\sum_{i=1}^{\infty}[\mathrm{d}(E_i^n)]^\delta\leqq\sum_{i=1}^{\infty}|K_i^n|^\delta<2^{-n}$, and hence $\mathrm{d}(E_i^n)<2^{-\frac{n}{\delta}}$ for $i=1, 2, \cdots$.

Given an arbitrary $\varepsilon>0$, take a positive integer N so as to satisfy $2^{-\frac{N}{\delta}}<\varepsilon$. By what has already been established, we obtain $\Lambda_\delta^{(\varepsilon)}(E)<2^{-n}$ for every $n\geqq N$. This gives $\Lambda_\delta^{(\varepsilon)}(E)=0$, whence we deduce $\Lambda_\delta(E)=0$ by making $\varepsilon\to0+$.

LEMMA 13. *The closure* $\bar{E}$ *of* E *contains both the extremities of every interval belonging to the collection* $\bigcup_{n=1}^{\infty}\mathfrak{K}^n$.

PROOF. Consider any interval $K_i^n=[a, b]$ of $\mathfrak{K}^n$. We shall first prove that $a\in E$. We have $E=\bigcap_{m=1}^{\infty}K^m$ by Lemma 11, where $K^1\supset K^2\supset\cdots$. Hence it suffices to show that $a\in K^m$ for every $m\geqq n$. More precisely, for each $m \geqq n$, there is in the collection $\mathfrak{K}^m$ an interval whose left-hand extremity is a. This is obvious by induction on m.

It remains to show that $b \in \bar{E}$. Writing for short $a_j = a_j(K_i^n)$ for $j = 1, 2, \cdots$, we have $K_{i,j}^n = [a_{2j-1}, a_{2j}] \in \mathfrak{K}^{n+1}$. Hence $a_{2j-1} \in E$ for every j, by what has already been proved (where n is replaced by $n+1$). Then $b = \lim_j a_{2j-1} \in \bar{E}$, which completes the proof.

THEOREM. *The function $B(x)$ is* GHC, *without being* GAC, *on U and so the approximate derivative of B is* Q-*integrable, without being* D-*integrable, on U.*

PROOF. Lemmas 4, 10 and 12 ensure together that $B(x)$ is GHC on U.

Suppose now, if possible, that $B(x)$ is GAC on U. On account of Theorem 9.1 of [2], p. 233, the nonvoid closed set $\bar{E}$ contains a portion S (see [2], p. 41) on which $B(x)$ is AC (see [2], p. 223). Let x_0 be a point of S. There then exists, by Lemma 11, a sequence of intervals $J_1, J_2, \cdots$ such that $x_0 \in J_n \in \mathfrak{K}^n$ for $n = 1, 2, \cdots$. But we have $|J_n| < 2^{-\frac{n}{\delta}}$ for every n by Lemma 6. Hence we can choose a positive integer m such that $\bar{E} \cap J_m \subset S$. Let us fix this m in what follows.

The interval $J_m \in \mathfrak{K}^m$ coincides with one of the intervals $K_1^m, K_2^m, \cdots$, say K_i^m. We shall write $a_j = a_j(K_i^m)$ and $I_j = [a_j, a_{j+1}]$ for $j = 1, 2, \cdots$. Since $I_{2j-1} = K_{i,j}^m \in \mathfrak{K}^{m+1}$ for every j, we find by Lemma 13 that $a_j \in \bar{E}$ for every j. It follows at once that $a_j \in \bar{E} \cap K_i^m \subset S$ for every j.

Now the function $H_m(x)$, by definition, coincides on K_i^m with the function $P(x) = P(x; K_i^m)$, and we have $\sum_{j=1}^{\infty} |P(I_{2j})| = +\infty$ as condition (iv) of Lemma 1 asserts. On the other hand, Lemma 9 shows that

$$B(x) = B_{m+1}(x) = B_m(x) + 2^{-m} H_m(x) \quad \text{for} \quad x \in L_i^{m+1}.$$

And this relation holds on the closure $\bar{L}_i^{m+1}$, too, by continuity of the involved functions. But $L_i^{m+1} = K_i^m - \bigcup_{j=1}^{\infty} I_{2j-1}$, so that the points $a_2, a_3, \cdots$ belong to $\bar{L}_i^{m+1}$. Besides, $B_m(x)$ is a constant on K_i^m in virtue of Lemma 5. Collecting the above results, we derive

$$\sum_{j=1}^{\infty} |B(I_{2j})| = 2^{-m} \sum_{j=1}^{\infty} |H_m(I_{2j})| = 2^{-m} \sum_{j=1}^{\infty} |P(I_{2j})| = +\infty .$$

This contradicts the absolute continuity of $B(x)$ on the portion S and completes the proof.

References

[1] Ka. Iseki: *On Quasi-Denjoy Integration,* Proc. Japan Acad., 38 (1962), 252–257.
[2] S. Saks: *Theory of the Integral,* Warszawa-Lwów (1937).