On Projective Killing Tensor

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§ 0. Introduction. Let $M^n$ be an $n$ dimensional Riemannian space with positive definite metric $g_{ab}$ with respect to local coordinate system $(x^a)_{1}$ and denote the operator of the covariant derivation by $\Gamma_e$.

A vector field $\psi^a$ is called an infinitesimal isometry or a Killing vector if it satisfies

\[ \mathcal{L}(\psi)g_{ab} = \Gamma_a \psi_b + \Gamma_b \psi_a = 0, \]

where $\mathcal{L}(\psi)$ means the operator of Lie derivation with respect to $\psi^a$, and $\psi_b = g_{bc} \psi^c$. In 1948, S. Bochner [1] introduced the notion of Killing tensor of degree $p \geq 1$ as a generalization of Killing vector. It is a skew symmetric tensor $v_{a_1...a_p}$ satisfying

\[ \Gamma_e v_{a_1...a_p} + \Gamma_{a_1} v_{ba_2...a_p} = 0. \]

Its first non-parallel example of degree $p \geq 1$ was found in 1955 by T. Fukami and S. Ishihara [2] for $p = 2$ on the 6 dimensional sphere $S^6$. As a generalization of $S^6$ the present author [7] introduced and studied in 1959 an almost Hermitian space whose fundamental form is a Killing tensor of degree 2. The Killing tensor on $S^6$ was the only example of non-parallel Killing tensor of degree $p \geq 1$ known until 1967 when Y. Ogawa [5] found them of degree odd in a Sasakian space in a chance studying C-harmonic forms. These works called our attentions to Killing tensor, and we had recent works which contain among others a new characterization of space of constant curvature, i.e.,

THEOREM 0.1. A necessary and sufficient condition for a Riemannian space to be a space of constant curvature is that for any point $m$ and any skew symmetric constants $C_{a_1...a_p}$ and $C_{a_1...a_{p+1}}$ there exists locally a Killing tensor $v_{a_1...a_p}$ satisfying $v_{a_1...a_p}(m) = C_{a_1...a_p}$ and $(\nabla_{a_1} v_{a_2...a_{p+1}})(m) = C_{a_1...a_{p+1}}$ ([9], [10]).

Killing tensor in the Euclidean space and on the sphere of con-

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1) Indices $a, b, c, ...$ run over 1 to $n$.
2) The number in brackets refers to the paper in Bibliography.
3) We shall identify a skew symmetric tensor $v_{a_1...a_p}$ with the differential form $\psi = (1/p!)(v_{a_1...a_p}dx^{a_1} \wedge ... \wedge dx^{a_p})$. 

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stant curvature have been determined completely.

The Lie derivative of Christoffel symbols \( \left\{ a_{cb} \right\} \) with respect to any vector field \( v^c \) satisfies the identity
\[
\mathcal{L}(v)\left\{ a_{cb} \right\} = \frac{1}{2} g^{ae}(\mathcal{L}(v)g_{be} + \mathcal{L}(v)g_{ce} - \mathcal{L}(v)g_{eb})
\]
\[
= \nabla_c \nabla^c v^e + R_{ecb}^e v^e,
\]
and a vector field \( v^c \) satisfying
\[
(0.3) \quad \mathcal{L}(v)\left\{ a_{cb} \right\} = 0
\]
is called an infinitesimal affine transformation or an affine Killing vector, where \( R_{ecb}^e \) is the Riemannian curvature tensor.

We remark that (0.3) is equivalent to
\[
(0.4) \quad \nabla_c \mathcal{L}(v)g_{be} + \nabla_b \mathcal{L}(v)g_{ce} - \nabla_e \mathcal{L}(v)g_{eb} = 0.
\]

A Killing vector is clearly an affine Killing vector, and the converse is true for the compact case. An affine Killing tensor is defined by an analogous equation to (0.4) taking account of the resemblance of (0.1) and (0.2).\(^4\)

Corresponding to the case of Killing vector we know

**THEOREM 0.2.** A Killing tensor is an affine Killing tensor, and the converse is true for a compact Riemannian space, ([9], [10]).

It was in 1952 that K. Yano [13] introduced first conformal Killing tensor as a generalization of conformal Killing vector. A conformal Killing vector or an infinitesimal conformal transformation is a vector field \( v^c \) which satisfies
\[
\mathcal{L}(v)g_{ab} = \nabla_a v_b + \nabla_b v_a = 2 \rho g_{ab},
\]
where \( \rho \) is a scalar function. He named a skew symmetric tensor \( v_{a_1a_p} \) a conformal Killing tensor if it satisfies
\[
(0.5) \quad \nabla_a v_{a_1a_p} + \nabla_{a_1} v_{a_2a_p} = 2 g_{ba} \rho_{a_2a_p},
\]
where \( \rho_{a_2a_p} \) is a skew symmetric tensor. Unfortunately this definition does not introduce new notion at all, because it is proved that \( \rho_{a_2a_p} \) in (0.5) vanishes identically and (0.5) reduces to (0.2). This can be seen by comparing two equations which are obtained from (0.5) by transvection with \( g^{a_1} \) and \( g^{a_2} \).

The author [11] and T. Kashiwada [3] changed its definition and they call a skew symmetric tensor \( v_{a_1a_p} \) a conformal Killing tensor if there exists a skew symmetric tensor \( \rho_{a_1a_{p-1}} \) called the associated ten-

\(^4\) S. Tachibana, [9]. S. Tachibana and T. Kashiwada, [10]. Also see §3 in this paper.
sor such that
\[ \mathcal{L}_a v_{a_1 \cdots a_p} = \mathcal{L}_a v_{b_2 \cdots a_p} \]
\[ = 2 g_{b_2 a} \rho_{a_1 \cdots a_p} - \sum_{i=2}^{p} (-1)^i (g_{b_2 a} \rho_{a_1 \cdots \hat{a}_i \cdots a_p} + g_{a_2 a} \rho_{b_2 \cdots \hat{a}_i \cdots a_p}), \]
where \( \hat{a}_i \) means that \( a_i \) is deleted. This definition seems natural because of the following theorems.

**Theorem 0.3.** For any point \( m \) of an \( n \) dimensional Riemannian space and any skew symmetric constants \( C_{a_1 \cdots a_p} \), if there exists locally a conformal Killing tensor \( v_{a_1 \cdots a_p} \) of degree \( p \) \((1 < p < n-1)\) satisfying \( v_{a_1 \cdots a_p}(m) = C_{a_1 \cdots a_p} \), then the space is conformally flat, ([3], [11]).

**Theorem 0.4.** In a space of constant curvature with \( \mathcal{R} \neq 0 \), a conformal Killing tensor \( v_{a_1 \cdots a_p} \) of degree \( p \) \((\leq n)\) is uniquely decomposed in the form
\[ v_{a_1 \cdots a_p} = w_{a_1 \cdots a_p} + q_{a_1 \cdots a_p}, \]
where \( w_{a_1 \cdots a_p} \) is a Killing tensor and \( q_{a_1 \cdots a_p} \) is a closed conformal Killing tensor. In this case, \( q_{a_1 \cdots a_p} \) is of the form
\[ q_{a_1 \cdots a_p} = -\frac{n(n-1)}{R} \mathcal{L}_a \rho_{a_2 \cdots a_p}, \]
where \( \rho_{a_2 \cdots a_p} \) is the associated tensor of \( v_{a_1 \cdots a_p} \) and \( R \) denotes the scalar curvature. Conversely, if \( w_{a_1 \cdots a_p} \) and \( \rho_{a_2 \cdots a_p} \) are Killing tensors then \( v_{a_1 \cdots a_p} \) given by (0.6) and (0.7) is a conformal Killing tensor, ([3], [11]).

The following theorem gives some meaning to our conformal Killing tensor too.

**Theorem 0.5.** Let \( M^n \) be a compact Riemannian space which is locally isometric to the direct product of \( p \) \((\leq n/2)\) dimensional Riemannian space and an \( n-p \) dimensional one. Then \( M^n \) can not admit a conformal Killing tensor of degree \( r \) satisfying \( 3(r-1) < 2p \) which is not a Killing tensor ([6]).

A vector field \( \nu^a \) is called an infinitesimal projective transformation or a projective Killing vector if there exists a scalar function \( \theta \) called the associated function satisfying
\[ \mathcal{L}(\nu)^{\{a}_{\{cb} = \mathcal{L}_c \nu^{a} + R_{ecb} \nu^e = \theta_\nu \delta^a_c + \theta_\delta \delta^a_c, \]
where \( \theta_\nu = \mathcal{L}_\nu \theta \).

The purpose of this paper is to introduce projective Killing tensor.

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5) This is a generalization of Yano-Nagano's theorem for conformal Killing vector in Einstein spaces, [15]. We remark that in a space of constant curvature the covariant derivative \( \mathcal{R}_a v_{a_1 \cdots a_p} \) of a Killing tensor \( v_{a_2 \cdots a_p} \) is a conformal Killing tensor whose associated tensor is \((-R/n(n-1))v_{a_2 \cdots a_p} \) \([3], [11]\).

6) This is a generalization of Tachibana's theorem for conformal Killing vector \([8]\). As to conformal Killing vector in the complete case, see Y. Tashiro \([12]\).
as a generalization of projective Killing vector. Though we do not
know whether it is worthy to study such tensor, it would seem partial
without it.

§1 will be devoted to notations and preliminaries. In §2 we shall
treat of a tensor field on the sphere $S^n$ in the Euclidean space which
will be taken as a model of projective Killing tensor. We shall give
in §3 an analogous formula for any skew symmetric tensor $v$ to
$\mathcal{L}(v)\{a_{cb}\}$ and define affine Killing tensor. The definition of projective
Killing tensor will be given in §4 and we shall discuss such tensor
in a space of constant curvature in §5 to get the main result corre-
A theorem to Theorem 0.4. We shall give also the corresponding theo-
rem for projective Killing tensor to the following

THEOREM 0.6. In a compact $n$ dimensional Riemannian space of
negative constant curvature, there exists no (conformal) Killing tensor of
degree $p$ ($\leq n/2$) other than the zero tensor, ([3]).

§1. Preliminaries. Consider a Riemannian space $M^n$ keeping
notations in §0. Let $v_{a_1 \cdots a_p}$ be a skew symmetric tensor identified
with the differential $p$-form $v = (1/p!)v_{a_1 \cdots a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p}$. The exterior derivative is

$$ dv = \frac{1}{(p+1)!} (dv)_{a_1 \cdots a_{p+1}} dx^{a_1} \wedge \cdots \wedge dx^{a_{p+1}}, $$

where

$$ (dv)_{e a_1 \cdots a_p} = \nabla_e v_{a_1 \cdots a_p} - \sum_{j=1}^p \nabla_{a_j} v_{e a_1 \cdots a_{j-1} a_{j+1} \cdots a_p}. $$

The index $e$ in the last term appears at the $j$-th position of index, i.e.
$v_{e a_1 \cdots a_p} = v_{a_1 \cdots a_{j-1} e a_j \cdots a_p}$ and we shall express this, if necessary, by $v_{e a_1 \cdots a_p}$.

A differential $p$-form $v$ is called to be closed if $dv = 0$, and to be
derived if there is a $(p-1)$-form $u$ such that $v = du$. The exterior co-
derivative $\delta v$ is

$$ \delta v = \frac{1}{(p-1)!} (\delta v)_{a_1 \cdots a_{p-1}} dx^{a_1} \wedge \cdots \wedge dx^{a_{p-1}} $$

with

$$ (\delta v)_{a_1 \cdots a_p} = -F^b v_{ba_1 \cdots a_p}, $$

where $F^b = g^{be} F_e$. $v$ is called to be coclosed when it satisfies $\delta v = 0$.

The Ricci identity for any tensor, say $T_{ba} \, \cdot \, f$, is

$$ F \nabla_e T_{ba} \, \cdot \, f - F \nabla_f T_{ba} = -R_{ace} T_{b} f - R_{eca} T_{b} f + R_{eca} f T_{ba}. $$

The following identity is proved easily for any skew symmetric
tensor $v_{a_1 \cdots a_p}$:

$$ F^c F^b v_{eb_1 \cdots a_p} = 0 $$

(1.1)
by taking account of Ricci identity.

The Ricci tensor and the scalar curvature are defined by

\[ R_{bc} = R_{abc}^a, \quad R = g^{ab} R_{bc}, \]

and put

\[ k = \frac{R}{n(n-1)}. \]

A Riemannian space \( M^n \) is called an Einstein space or a space of constant curvature, if

\[ R_{bc} = (n-1) g_{bc}, \quad \text{or} \]

\[ R_{abcd} = k (g_{bc} g_{ad} - g_{ac} g_{bd}) \]

is satisfied respectively. As well known, \( k \) is constant for these spaces when \( n \geq 2 \).

Let \( M^n \) be a Riemannian space immersed in a Riemannian space \( M^{n+1} \). With respect to local coordinates \( \{y^a\} \) in \( M^n \) and \( \{x^i\} \) in \( M^{n+1} \), \( M^n \) is represented locally by \( x^i = x^i(y) \). If we put

\[ B_a^i = \frac{\partial x^i}{\partial y^a}, \]

the Riemannian metric \( g_{bc} \) of \( M^n \) is related with the one \( G_{\mu\nu} \) of \( M^{n+1} \) by \( g_{bc} = G_{\mu\nu} B^\mu_b B^\nu_c \). The second fundamental tensor is defined by

\[ H_{ba}^i = V_b B_a^i = \frac{\partial B_a^i}{\partial y^b} - \left[ \frac{\lambda}{\mu\nu} \right] B_c^i - B^\mu_b \left[ \frac{\lambda}{\mu} \right] B_a^\mu, \]

where \( \left[ \frac{\lambda}{\mu\nu} \right] \) are the Christoffel's symbols formed from \( G_{\mu\nu} \) and \( V_b \) means the Van der Waerden-Bortolotti differential operator.

Let \( N^i \) be the unit normal vector field defined locally on \( M^n \), then there exists a symmetric tensor \( H_{ba} \) on \( M^n \) such that

\[ H_{ba}^i = H_{ba} N^i, \]

and we have

\[ V_b N^i = \frac{\partial N^i}{\partial y^b} + B^\nu_b \left[ \frac{\lambda}{\mu\nu} \right] N^\nu = -H_b^i B_c^i, \]

where \( H_b^i = H_{ba} g^{ba} \).

\[ \S 2. \ \textbf{A tensor field on } S^n. \] Let \( E^{n+1} \) be an \( n+1 \) dimensional Euclidean space with orthogonal coordinate \( \{x^i\} \). Consider the upper hemi-sphere \( S^n \):

\[ (x^1)^2 + \cdots + (x^{n+1})^2 = 1, \quad x^{n+1} > 0 \]

7) Indices \( \lambda, \mu, \nu, \ldots \) run over 1 to \( n+1 \).
as a Riemannian space imbedded in $E^{n+1}$. We shall denote by $N^i$ the unit (inward) normal vector field defined globally on $S^*_+$ and take $\{x^a\}$ as a local coordinate there. If we put

$$f = x^{n+1}$$

the following equations hold good:

$$N_i = -x^i, \quad \beta_a^i = \begin{cases} \delta^b_a, & \text{if } \lambda = b, \\ -\frac{x^a}{f}, & \text{if } \lambda = n+1, \end{cases}$$

$$g_{ab} = \tilde{g}_{ab} + \frac{x^ax^b}{f^2}, \quad g^{ab} = \tilde{g}^{ab} - x^ax^b,$$

$$\left[ \begin{array}{c} a \\ bc \end{array} \right] = x^a g_{bc}, \quad R_{abc}^e = g_{be} \dot{\delta}_e^a - g_{ae} \dot{\delta}_b^e,$$

$$\nabla_b B^a_i = H_{ab} N^i = g_{ab} N^i, \quad \nabla_b N^i = -B^i_b.$$  

It is known ([12], [4]) that $f$ is a special concircular scalar field, i.e., it satisfies

$$\nabla_b f = -f g_{bc}.$$  

Now denote by $E^n$ the $n$-plane defined by $x^{n+1} = 1$. If we consider the projection from the origin of $E^{n+1}$, it is a diffeomorphism between $S^*_+$ and $E^n$ and induces a projective transformation on $S^*_+$ from an affine transformation on $E^n$. Especially a projective Killing vector field $v^a$ on $S^*_+$ is obtained by the projection from a constant vector field on $E^n$ which is extended naturally to a constant one $u^i$ ($= u_i$) over $E^{n+1}$. They are related by

$$v_a = f u_a B^a_i.$$  

Generalizing this geometrical fact, we shall adopt

$$v_{a_1 \ldots a_p} = f u_{a_1 \ldots a_p} B_{a_1}^{i_1} \ldots B_{a_p}^{i_p}$$

as the model of projective Killing tensor whose exact definition will be given in §4, where $u_{a_1 \ldots a_p}$ is a constant skew symmetric tensor over $E^{n+1}$.

Define a skew symmetric tensor $\theta_{a_1 \ldots a_p}$ for this $v_{a_1 \ldots a_p}$ by

$$\theta_{a_1 \ldots a_p} = f u_{\mu_{a_1 \ldots a_p}} N^\mu B_{a_1}^{i_1} \ldots B_{a_p}^{i_p},$$

then we have

$$\nabla_b \theta_{a_1 \ldots a_p} = (\nabla_b f) u_{\mu_{a_1 \ldots a_p}} N^\mu B_{a_1}^{i_1} \ldots B_{a_p}^{i_p} - v_{\delta_b a_1 \ldots a_p}.$$  

Simple computations show the validity of the following equations:

$$\nabla_b v_{a_1 \ldots a_p} = (\nabla_b f) u_{a_1 \ldots a_p} B_{a_1}^{i_1} \ldots B_{a_p}^{i_p} + \sum_{\ell=1}^p (-1)^{i-\ell} g_{ba_\ell} \theta_{a_1 \ldots \hat{a}_\ell \ldots a_p},$$
\[
(2.1) \quad \nabla_\alpha \nabla_\beta v_{\alpha_1 \cdots \alpha_p} = -g_{\beta \alpha} v_{\alpha_1 \cdots \alpha_p} + \sum_{i=1}^{p} g_{\alpha i} v_{\alpha_1 \cdots \beta \cdots \alpha_p} + \sum_{i=1}^{p} (-1)^{i-1} (g_{\beta \alpha} \nabla_\epsilon \theta_{\alpha_1 \cdots \hat{\alpha_i} \cdots \alpha_p} + g_{\alpha i} \nabla_\epsilon \theta_{\alpha_1 \cdots \hat{\alpha_i} \cdots \alpha_p}).
\]

§ 3. Affine Killing tensor. Let \(v_{\alpha_1 \cdots \alpha_p}\) be a skew symmetric tensor field in a Riemannian space \(M^n\). We shall put

\[
(3.1) \quad A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p} = \nabla_\alpha v_{\alpha_1 \cdots \alpha_p} + \nabla_{\alpha_1} v_{\beta \alpha_2 \cdots \alpha_p}
\]

which corresponds to (0.1). Next, corresponding to (0.4), we shall consider

\[
(3.2) \quad T_{\epsilon \beta, \alpha_1 \cdots \alpha_p} = (1/2) (\nabla_\epsilon A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p} + \nabla_\beta A_{\epsilon \alpha_1, \alpha_2 \cdots \alpha_p} - \nabla_{\alpha_1} A_{\epsilon \beta, \alpha_2 \cdots \alpha_p}).
\]

By virtue of Ricci identity we can get the following equation

\[
(3.2) \quad T_{\epsilon \beta, \alpha_1 \cdots \alpha_p} = \nabla_\epsilon \nabla_\beta v_{\alpha_1 \cdots \alpha_p} + (1/2) \sum_{i=1}^{p} R_{\epsilon \beta \alpha i} v_{\alpha_1 \cdots \epsilon \cdots \alpha_p} - (1/2) (R_{\epsilon \beta \alpha i} + R_{\epsilon \alpha i \beta}) v_{\alpha_1 \cdots \alpha_p} - (1/2) \sum_{i=2}^{p} (R_{\epsilon \beta \alpha i} v_{\alpha_1 \cdots \epsilon \cdots \alpha_p} + R_{\epsilon \alpha i \beta} v_{\alpha_1 \cdots \alpha_p}).
\]

We shall call a skew symmetric tensor \(v_{\alpha_1 \cdots \alpha_p}\) an affine Killing tensor if its \(T_{\epsilon \beta, \alpha_1 \cdots \alpha_p}\) vanishes identically. Though this definition differs from ones in [9] and [10], Theorem 0.2 is still true. The proof is as follows.

For a Killing tensor \(v_{\alpha_1 \cdots \alpha_p}\), its \(A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p}\) vanishes and hence \(T_{\epsilon \beta, \alpha_1 \cdots \alpha_p}\) = 0 which shows that \(v_{\alpha_1 \cdots \alpha_p}\) is an affine Killing tensor. Conversely, let \(v_{\alpha_1 \cdots \alpha_p}\) be a skew symmetric tensor and consider \(A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p}\) given by (3.1). Taking account of Ricci identity we can get

\[
\nabla_\beta A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p} = \nabla_\alpha v_{\alpha_1 \cdots \alpha_p} + R_{\alpha_1} v_{\alpha_2 \cdots \alpha_p} - \sum_{i=2}^{p} R_{\beta \alpha i} v_{\alpha_1 \cdots \epsilon \cdots \alpha_p} + \nabla_{\alpha_1} \nabla_\beta v_{\alpha_2 \cdots \alpha_p},
\]

\[
\nabla_\beta (A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p} v_{\alpha_1 \cdots \alpha_p}) = \nabla_\beta A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p} v_{\alpha_1 \cdots \alpha_p} + (1/2) A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p} A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p}.
\]

Integrating the last equation we have

**Theorem 3.1.** In a compact orientable Riemannian space \(M^n\), the integral formula

\[
\int_M \left( \nabla_\beta v_{\alpha_1 \cdots \alpha_p} + R_{\alpha_1} v_{\alpha_2 \cdots \alpha_p} - \sum_{i=2}^{p} R_{\beta \alpha i} v_{\alpha_1 \cdots \epsilon \cdots \alpha_p} \right) v_{\alpha_1 \cdots \alpha_p} + (1/2) A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p} A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p} \right) d\alpha = 0
\]

holds good for any skew symmetric tensor field \(v_{\alpha_1 \cdots \alpha_p}\), where \(d\alpha\) means the volume element of \(M^n\) and \(A_{\beta \alpha_1, \alpha_2 \cdots \alpha_p}\) is given by (3.1).
Assume $M^n$ to be compact orientable and let $v_{a_1 \cdots a_p}$ be an affine Killing tensor. Then, taking account of (3.3), we have

$$
(3.4) \quad \nabla^b \nabla^a v_{a_1 \cdots a_p} + {R}^{a_1}_b v_{e_2 \cdots e_p} = \sum_{s=1}^p R^e_{a_1 a_1} v_{s e_2 \cdots e_p} = 0,
$$

$$
(3.5) \quad \nabla^e \nabla^a v_{e a_2 \cdots a_p} = 0.
$$

Thus $A_{a_1 a_2 \cdots a_p} = 0$ follows from Theorem 3.1.

We remark that in a space of constant curvature the equation (3.3) becomes the following simple form:

$$
(3.6) \quad T_{e_1 a_1 \cdots a_p} = \nabla_e \nabla^a v_{a_1 \cdots a_p} + \frac{p}{e} g_{e a_1} v_{a_1 \cdots a_p} + \sum_{s=1}^p g_{e a_1} v_{a_1 \cdots a_p}.
$$

§ 4. Projective Killing tensor. We shall define projective Killing tensor in this section. A projective Killing vector is defined by (0.8) whose left hand side corresponds to $T_{e_1 a_1 \cdots a_p}$. Thus the problem is how to determine $T_{e_1 a_1 \cdots a_p}$ the form corresponding to the right hand side of (0.8). Now let take $v_{a_1 \cdots a_p}$ on $S^n$ in § 2 and calculate (3.6) for it. Substituting (2.1) and $k = 1$ in (3.6) we can get

$$
T_{e_1 a_1 \cdots a_p} = \sum_{i=1}^p (-1)^{i-1} (g_{e a_1} \nabla^a \theta_{a_1 \cdots a_i} + g_{e a_1} \nabla^a \theta_{a_1 \cdots a_i} a_p).
$$

Thus it is natural that we would admit the following definition. A skew symmetric tensor $v_{a_1 \cdots a_p}$ will be called a projective Killing tensor of degree $p$ if there exists a skew symmetric tensor $\theta_{a_1 \cdots a_{p-1}}$ such that

$$
(4.1) \quad \nabla^e \nabla^a v_{a_1 \cdots a_p} + (1/2) \sum_{i=1}^p R^{e a_1}_b v_{a_1 \cdots a_p} = (1/2)(R_{c a_1 b} + R_{c a_1 b}) v_{a_1 \cdots a_p}
$$

$$
= \sum_{i=1}^p \frac{1}{i} (-1)^{i-1} (g_{e a_1} \nabla^a \theta_{a_1 \cdots a_i} a_p + g_{e a_1} \nabla^a \theta_{a_1 \cdots a_i} a_p).
$$

$\theta_{a_1 \cdots a_{p-1}}$ is called the associated tensor and $\theta = (1/(p-1)! \theta_{a_2 \cdots a_p} dx^2 \wedge \cdots \wedge dx^p$ the associated form.

Especially, (4.1) becomes (0.8) when $p = 1$. Thus our definition of projective Killing tensor is a generalization of projective Killing vector.

By transvection (4.1) with $g^{eb}$ it follows

$$
(4.2) \quad \nabla^e \nabla^a v_{a_1 \cdots a_p} + R_{a_1 e} v_{a_2 \cdots a_p} = \sum_{i=1}^p R^b_{a_1 a_1} v_{b a_2 \cdots a_p} = 2 \sum_{i=1}^p (-1)^{i-1} \nabla \theta_{a_1 \cdots a_i a_p} = 2(d \theta)_{a_1 \cdots a_p}.
$$
On the other hand, transvecting (4.1) with \( g_{\alpha \beta} \), we have

\[
(4.3) \quad \nabla_c \nabla^c \theta_{\alpha \beta} = (n - p + 2) \nabla_c \theta_{\alpha \beta} + \sum_{i=2}^{n} (-1)^{i-1} g_{\alpha \beta} \nabla^c \theta_{\alpha_i \beta \cdots \alpha_p}
\]

i.e.

\[
(4.4) \quad -\nabla_c (\partial \nu)_{\alpha_2 \cdots \alpha_p} = (n - p + 2) \nabla_c \theta_{\alpha_2 \cdots \alpha_p} + \sum_{i=2}^{p} (-1)^{i} g_{\alpha \beta} (\partial \theta)_{\alpha_2 \cdots \alpha_i}.
\]

Some simple computations show the following equation to be valid.

\[
(4.5) \quad -(d \nu)_{\alpha_2 \cdots \alpha_p} = (n - p + 2) (d \theta)_{\alpha_2 \cdots \alpha_p}.
\]

We proceed to give some applications of these equations.

Let \( \nu_{\alpha_1 \cdots \alpha_p} \) be a projective Killing tensor whose associated form \( \theta \) is closed. Then we have (3.4) from (4.2), and have

\[
(\nabla_c \nabla^c \nu_{\alpha_2 \cdots \alpha_p}) g^{\alpha_2 \cdots \alpha_p} = 0
\]

from (4.3). Thus \( A_{\alpha_1 \alpha_2 \cdots \alpha_p} = 0 \) follows from Theorem 3.1, which means that \( \nu_{\alpha_1 \cdots \alpha_p} \) is a Killing tensor. Thus we have

**Theorem 4.1.** In a compact Riemannian space, a projective Killing tensor is a Killing tensor if the associated form is closed.

Let \( \nu \) be a projective Killing tensor which is coclosed. Then the associated form is closed by (4.5). Hence

**Theorem 4.2.** In a compact Riemannian space, a coclosed projective Killing tensor is a Killing tensor.

When \( p = 1 \) these theorems reduce to well-known theorems on projective Killing vectors.

As is well known, a projective Killing vector which is a conformal Killing vector at a time is affine. We shall generalize this to the case of tensor. Let \( \nu_{\alpha_1 \cdots \alpha_p} \) be a conformal Killing tensor and \( \rho = (1/(p-1)! \rho_{\alpha_2 \cdots \alpha_p} dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_p} \) its associated form. By definition, \( \nu_{\alpha_1 \cdots \alpha_p} \) is a skew symmetric tensor satisfying

\[
\nabla_c \nu_{\alpha_1 \cdots \alpha_p} + \nabla_{\alpha_1} \nu_{\alpha_2 \cdots \alpha_p} = 2g_{\alpha \beta} \rho_{\alpha_2 \cdots \alpha_p} - \sum_{i=2}^{p} (-1)^{i-1} (g_{\alpha \beta} \rho_{\alpha_1 \cdots \alpha_i \cdots \alpha_p} + g_{\alpha \beta_1} \rho_{\beta_2 \cdots \alpha_1 \cdots \alpha_p}).
\]

It is known that the following equations are valid: 8)

\[
(4.6) \quad \nabla^c \nu_{\alpha_2 \cdots \alpha_p} = (n - p + 1) \rho_{\alpha_2 \cdots \alpha_p}, \quad -\partial \nu = (n - p + 1) \rho,
\]

\[
(4.7) \quad \nabla_c \nabla^c \nu_{\alpha_1 \cdots \alpha_p} + R_{\alpha_1 \cdots \alpha_p} \nu_{\alpha_2 \cdots \alpha_p} - \sum_{i=2}^{p} R_{\alpha_1 \cdots \alpha_i} \nu_{\alpha_2 \cdots \alpha_i \cdots \alpha_p} = -(n - p) \rho_{\alpha_1 \cdots \alpha_p} + (d \rho)_{\alpha_1 \cdots \alpha_p}.
\]

Now suppose that \( \nu_{\alpha_1 \cdots \alpha_p} \) is a projective Killing tensor at a time. Then, comparing (4.7) with (4.2), we have

8) T. Kashiwada, [3], S. Sato, [6].
\[ 2(d\theta)_{a_1 a_p} = -(n-2)\nabla_{a_1 a_2 \cdots a_p} + (d\rho)_{a_1 a_p}, \]

from which it follows that \( \nabla_{a_1 a_2 \cdots a_p} \) is skew symmetric. Substituting \( (d\rho)_{a_1 a_p} = p \nabla_{a_1 a_2 \cdots a_p} \) into the last equation we get

\[ 2pd\theta = (2p-n)d\rho, \]

On the other hand, \( -d\vartheta = (n-p+2)d\theta \) follows from (4.5) and taking account of (4.6) we have

\[ (n-p+2)d\theta = (n-p+1)d\rho. \]

Eliminating \( d\rho \) from (4.8) and (4.9) we get \((n+2)(n-p)d\theta = 0 \) and hence \( \theta \) is closed for \( p < n \). Consequently by virtue of Theorem 4.1 we obtain

**Theorem 4.3.** In a compact \( n \) dimensional Riemannian space, if a projective Killing tensor of degree \( p \) \((< n)\) is a conformal Killing tensor at a time, then it is a Killing tensor.

§ 5. **Projective Killing tensor in a space of constant curvature.**

In this section we shall restrict our attention to a space \( M^n \) of constant curvature and assume \( n > 2 \).

A projective Killing tensor \( v_{a_1 \cdots a_p} \) is a skew symmetric tensor satisfying

\[ \nabla^b \nabla^b v_{a_1 \cdots a_p} = k (-g_{eb} v_{a_1 \cdots a_p} + \sum_{i=1}^{p} g_{ca_1} v_{a_i b \cdots a_p}) \]

\[ + \sum_{i=2}^{p} (-1)^{i-1} (g_{ca_1} \nabla^b \theta_{a_2 \cdots a_i a_p} + g_{ba_1} \nabla^b \theta_{a_2 \cdots a_i a_p}), \]

where \( \theta_{a_1 \cdots a_p} \) is skew symmetric.

Operating \( \nabla^b \) to (5.1) and changing \( a_i \) to \( e \), we have

\[ \nabla^e \nabla^b v_{a_2 \cdots a_p} = k (-g_{eb} v_{a_2 \cdots a_p} + \sum_{i=2}^{p} g_{ca_1} \nabla^e v_{a_i b \cdots a_p}) \]

\[ + \sum_{i=2}^{p} (-1)^{i-1} (g_{ca_1} \nabla^b \theta_{a_2 \cdots a_i a_p} + g_{ba_1} \nabla^b \theta_{a_2 \cdots a_i a_p}) \]

\[ + \nabla^e \nabla^b \theta_{a_2 \cdots a_p} + \nabla^e \theta_{a_2 \cdots a_p}. \]

First we shall calculate the left hand side of (5.2) which is the sum of the following \( a_i \) to \( a_4 \):

\[ a_1 = \nabla^e \nabla^b v_{a_2 \cdots a_p}, \quad a_2 = -R_{eb} v_{a_2 \cdots a_p}, \]

\[ a_3 = R_{eb} v_{a_2 \cdots a_p}, \quad a_4 = -\sum_{i=2}^{p} R_{ca_1} \nabla^e v_{a_i b \cdots a_p}. \]

Taking account of (1.2), (1.3) and (4.3) we have
\[ a_1 = V_e^c V^e_\alpha \gamma'_{a_2-\alpha_2} a_p + R_{e \alpha}^c \gamma'_{a_2-\alpha_2} a_p - \sum_{i=2}^p R_{e \alpha i}^c \gamma'_{a_2-\alpha_2} a_p \]
\[ = (n-p+2) V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p + \sum_{i=2}^p \left( (-1)^{i-1} g_{b a i} V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p \right) \]
\[ +(n-p) k V_e^c V_{b} a_2-\alpha_p, \]
\[ a_2 = k (-g_e^c V^c_\alpha \gamma'_{a_2-\alpha_2} a_p + V_e^c \gamma_{b} a_2-\alpha_p), \]
\[ a_3 = (n-1) k V_e^c \gamma_{a_2-\alpha_p}, \]
\[ a_4 = -(p-1) k V_e^c \gamma_{a_2-\alpha_p}. \]

Substituting these values to the left hand side of (5.2) we have
\[ (n-p) k (V_e^c V_{b} a_2-\alpha_p + \gamma_{b} a_2-\alpha_p) = k \sum_{i=2}^p g_{ca i} V^c_\alpha \gamma'_{a_2-\alpha_2} a_p \]
\[ - (n-p+1) V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p + V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p \]
\[ + \sum_{i=2}^p (-1)^{i-1} g_{b a i} (V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p - V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p) \]
\[ + \sum_{i=2}^p (-1)^{i-1} g_{c a i} V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p. \]

On the other hand, it follows by Ricci identity that
\[ V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p = V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p - k \sum_{i=2}^p (g_{a_2 \alpha i} \theta_{a_2-\alpha_2} a_p - g_{b a i} \theta_{a_2-\alpha_2} a_p), \]
\[ V^c_\alpha \theta_{a_2-\alpha_2} a_p = V^c_\alpha \theta_{a_2-\alpha_2} a_p + (n-p+1) k \theta_{a_2-\alpha_2} a_p. \]

Substituting these two equations into (5.3) we can get
\[ (n-p) k (V_e^c \gamma_{b} a_2-\alpha_p + \gamma_{b} a_2-\alpha_p) = -(n-p) V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p \]
\[ + \sum_{i=2}^p (-1)^{i-1} (n-p) k g_{b a i} \theta_{a_2-\alpha_2} a_p \]
\[ + \sum_{i=2}^p g_{c a i} (k V_e^c \gamma_{b} a_2-\alpha_p + (-1)^{i-1} V^c_\alpha \theta_{a_2-\alpha_2} a_p \]
\[ + (-1)^{i-1} (n-p+2) k \theta_{a_2-\alpha_2} a_p). \]

Transvecting (5.4) with \( g^{b_2} \) we have
\[ k V_e^c \gamma_{b_2} a_2-\alpha_p = V_e^c V^c_\alpha \theta_{b_2-\alpha_2} a_p + (n-p+2) k \theta_{b_2-\alpha_2} a_p, \]
and substituting this equation into (5.4) we have for \( n > p \)
\[ k (V_e^c \gamma_{b_2} a_2-\alpha_p + \gamma_{b} a_2-\alpha_p) = -V_e^c V^c_\alpha \theta_{a_2-\alpha_2} a_p - k \sum_{i=2}^p g_{a_2 \alpha i} \theta_{a_2-\alpha_2} a_p. \]

When \( p = 1 \), (5.5) reduces to
\[ k (V_e^c \gamma_{b} a_2-\alpha_p + \gamma_{b} a_2-\alpha_p) = -V_e^c V^c_\alpha \theta = (-1/2) (V_e^c V^c_\alpha \theta + V^c_\alpha V^c_\alpha \theta), \]
where \( \theta \) is a scalar function. If \( k \neq 0 \), the vector \( \omega^b \) defined by
\[ w_b = v_b - (1/2k) \nabla_b \theta \]
is a Killing vector. Thus a projective Killing vector \( v^b \) is represented in the form
\[ v_b = w_b - (1/2k) \nabla_b \theta, \]
where \( w^b \) is a Killing vector and \( \nabla_b \theta \) is a derived projective Killing vector. This fact has been got by K. Yano and T. Nagano [15] for an Einstein space. We shall generalize this to the case of tensor.

From (5.5) we have
\[ \nabla_c \nabla_b \theta_{a_1 .. a_p} = B_{c b, a_1 .. a_p} - k \sum_{j=2}^{p} g_{a_j} \theta_{a_2 .. a_p} \]
for a projective Killing tensor \( v_{a_1 .. a_p} \), where
\[ B_{cb,a_1 .. a_p} = - k (\nabla_c \nabla_b \theta_{a_1 .. a_p} + \nabla_c \nabla_{a_2 .. a_p}). \]
Now we put \( u_{a_1 .. a_p} = (d\theta)_{a_1 .. a_p} \) which is by definition
\[ u_{a_2 .. a_p} = \nabla_b \theta_{a_2 .. a_p} - \sum_{j=2}^{p} \nabla_j \theta_{a_2 .. a_p}. \]
Thus it holds that
\[ \nabla_c u_{a_2 .. a_p} = \nabla_c \nabla_b \theta_{a_2 .. a_p} - \sum_{j=2}^{p} \nabla_c \nabla_j \theta_{a_2 .. a_p}. \]
On the other hand we have from (5.6)
\[ \nabla_c \nabla_j \theta_{a_2 .. a_2 .. a_p} = B_{ca, a_2 .. a_p} - \sum_{j=2}^{p} g_{a j} \theta_{a_2 .. a_p} - k g_{a j} \theta_{a_2 .. a_p} \]
for \( 2 \leq j \leq p \). Substituting the last equations and (5.6) into (5.7), we get
\[ \nabla_c u_{a_2 .. a_p} = B_{cb, a_2 .. a_p} - \sum_{j=2}^{p} B_{ca, a_2 .. a_p} - \sum_{j=2}^{p} g_{a j} \theta_{a_2 .. a_p} - k \sum_{j=2}^{p} g_{a j} \theta_{a_2 .. a_p} \]
from which it follows
\[ \nabla_c u_{a_2 .. a_p} + \nabla_c u_{a_2 .. a_p} = 2B_{cb, a_2 .. a_p} - \sum_{j=2}^{p} (B_{ca, a_2 .. a_p} + B_{ca, a_2 .. a_p}) \]
\[ = -(p+1)k \sum_{j=2}^{p} (\nabla_c \nabla_a .. a_p + \nabla_c \nabla_{a_2 .. a_p}). \]
Thus we have
\[ \nabla_c (\kappa p+1) \nabla \theta_{a_2 .. a_p} + \nabla_c \nabla_{a_2 .. a_p} = 0. \]
If we assume \( k \neq 0 \) and put
\[ \omega_{a_2 .. a_p} = v_{a_2 .. a_p} + \frac{1}{(p+1)k} u_{a_2 .. a_p}, \]
it is a Killing tensor. Hence we have

**Theorem 5.1.** In a space of constant curvature with non-vanishing \( k = R/n(n-1) \), any projective Killing tensor \( v_{a_1\cdots a_p} \) \((p < n)\), is decomposed uniquely in the form

\[
v_{a_1\cdots a_p} = w_{a_1\cdots a_p} + q_{a_1\cdots a_p},
\]

where \( w_{a_1\cdots a_p} \) is a Killing tensor and \( q_{a_1\cdots a_p} \) is a closed projective Killing tensor. \( q_{a_1\cdots a_p} \) is given by exactly

\[
q_{a_1\cdots a_p} = -\frac{1}{(p+1)k} (d\theta)_{a_1\cdots a_p}
\]

in terms of the associated tensor \( \theta_{a_1\cdots a_p} \).

The uniqueness follows from the following lemma.

**Lemma.** In a space of constant curvature \((k \neq 0)\), if a Killing tensor of degree \( p \) \((p < n)\) is closed, then it is the zero tensor, ([3], [11]).

**Corollary 5.2.** In a space of constant curvature \((k \neq 0)\), if \( \theta_{a_1\cdots a_p} \) \((p < n)\) is the associated tensor of a projective Killing tensor, then \((d\theta)_{a_1\cdots a_p} \)
is a projective Killing tensor whose associated tensor is \(-(p+1)k\theta_{a_1\cdots a_p} \).

We shall give an application which corresponds to Theorem 5.6.

Under the assumption in Corollary 5.2, we have

\[
d\delta d\theta = (p+1)(n-p+2)kd\theta
\]

from (4.5), because of \( \delta w = 0 \). For the compact case, making use of integral and taking account of what \( d \) and \( \delta \) are dual each other, we can get

**Theorem 5.3.** In a compact space of negative constant curvature, any projective Killing tensor is a Killing tensor.

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**Bibliography**

(1968), 257–264.


