Natural Science Report, Ochanomizu University, Vol. 18, No. 2, 1967
お茶の水女子大学 自然科学報告 第18巻 第2号

On Co-torsion Groups and Algebraically Compact Groups

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(*Received September 10, 1967*)

All groups considered in this paper are abelian groups. Homological terminology and notations follow [5] and [2]; throughout this paper Q and Z mean the additive groups of rationals and rational integers respectively. A group A is called *algebraically compact* if $\mathrm{Pext}(G, A)=0$ for any group G, see [1] on page 83. A reduced group A is called *co-torsion* if $\mathrm{Ext}(G, A)=0$ for any torsion-free group G; it is known that a reduced group A is co-torsion if and only if $\mathrm{Ext}(Q, A)=0$. Of course, a reduced algebraically compact group is co-torsion.

Our main purpose is to examine the relationship between co-torsion groups and algebraically compact groups. Moreover we have given simple, homological proofs for some well-known theorems on abelian groups. We have only assumed the following basic knowledge on abelian groups, see [1] on page 98 and on page 112.

Let G be a reduced p-group. Then G has a basic subgroup $\sum_{n=1}^{\infty} B_n$ where B_n is a direct sum of cyclic groups of order p^n, such that

(I) $$0 \to \sum B_n \to G \to D \to 0 \qquad \text{(pure exact)},$$

(II) $$0 \to G/G^1 \to (\prod B_n)_t \to E \to 0 \qquad \text{(pure exact)}$$

where D and E are divisible, $G^1=\bigcap_{n=1}^{\infty} p^n G$ and $(\prod B_n)_t$ is the torsion part of $\prod B_n$.

The author wishes to express her sincere gratitude to Professor Mieo Nishi who kindly read this paper and made many valuable suggestions.

First, we review a proof of the fact that $\mathrm{Pext}(C, A)=(\mathrm{Ext}(C, A))^1$ which is very useful in this paper. Here, $(\mathrm{Ext}(C, A))^1=\bigcap_n n\cdot\mathrm{Ext}(C, A)$.

Let E be a pure exact sequence belonging to $\mathrm{Pext}(C, A)$ such that

$$E:\quad 0 \to A \xrightarrow{\kappa} B \xrightarrow{\sigma} C \to 0 .$$

Set $C[n]=\sum \langle c_\alpha\rangle$ where $C[n]=\{x\in C : nx=0\}$ and $\langle c_\alpha\rangle$ is the cyclic group generated by $c_\alpha \in C$. Since E is a pure exact sequence, for each c_α we can choose an element b_α from the inverse image of c_α by σ such that the order of b_α is equal to the order of c_α. Set $\sum\langle b_\alpha\rangle=B'$. Then $\kappa A\cap B'=0$, since $\{c_\alpha\}$ is an independent set and if $mc_\alpha=0$ for some integer m, then $mb_\alpha=0$. Hence $\operatorname{Ker} n\sigma=\kappa A\oplus B'$. We get a commutative diagram

$$\begin{array}{lccccccccc} E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0 \\ & & & \| & & \downarrow & \overset{n\sigma}{\searrow} & \downarrow & & \\ & 0 & \to & A & \to & \dfrac{B}{B'} & \to & nC & \to & 0 . \end{array}$$

On the other hand, there exists a short exact sequence $E'\in\operatorname{Ext}(C, A)$ such that

$$\begin{array}{lccccccccc} & & & & & & & 0 & & \\ & & & & & & & \downarrow & & \\ & 0 & \to & A & \to & \dfrac{B}{B'} & \to & nC & \to & 0 \\ & & & \| & & \downarrow & & \downarrow & & \\ E': & 0 & \to & A & \to & X & \to & C & \to & 0 , \end{array}$$

since $0\to nC\to C$ gives rise to $\operatorname{Ext}(C, A)\to\operatorname{Ext}(nC, A)\to 0$. Hence $E=nE'$, i. e. $\operatorname{Pext}(C, A)\subset(\operatorname{Ext}(C, A))^1$.

Suppose $E\in(\operatorname{Ext}(C, A))^1$. For each integer n there exists $E'\in\operatorname{Ext}(C, A)$ such that

$$\begin{array}{lccccccccc} E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0 \\ & & & \| & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle n} & & \\ E': & 0 & \to & A & \xrightarrow[\kappa^*]{} & X & \xrightarrow[\sigma^*]{} & C & \to & 0 . \end{array}$$

We shall show $\kappa A\cap nB\subset n\kappa A$. Set $B'=\{b'\in B : nb'\in\kappa A\}$. Then $\sigma^*\nu B'=n\sigma B'=\sigma nB'=0$. Hence $\nu B'\subset\kappa^*A=\nu\kappa A$. Since $nB'\subset\kappa A$ and $\nu:\kappa(A)\cong\kappa^*(A)$, $\nu(nB')\subset\nu(n\kappa A)$ implies $\kappa A\cap nB=nB'\subset n\kappa A$. Now we have completed the proof of $\operatorname{Pext}(C, A)=(\operatorname{Ext}(C, A))^1$.

Theorem 1 appears in [4], see [2] on page 371. However we shall give another proof based on homological methods. Also we shall give a simple proof for Lemma 1 which appears in [3].

LEMMA 1. *Let B be a group of bounded order and G be an arbitrary group. Then* $\operatorname{Pext}(G, B)=0$.

PROOF. Suppose $n\cdot B=0$. Then $n\cdot\operatorname{Ext}(G, B)=0$. Hence $\operatorname{Pext}(G, B)=(\operatorname{Ext}(G, B))^1=0$.

THEOREM 1. *A group is both co-torsion and torsion if and only if it is of bounded order. Hence a torsion, co-torsion group is algebraically compact by Lemma 1.*

PROOF. The "if part" follows immediately from the fact that $\mathrm{Ext}(G,\ )=\mathrm{Pext}(G,\ )$ for a torsion-free group G and Lemma 1. Conversely, suppose G is both co-torsion and torsion. Since the exact sequence: $0\to Z\to Q\to Q/Z\to 0$ gives rise to $\mathrm{Hom}(Q, G)=0\to \mathrm{Hom}(Z, G)\cong G\to \mathrm{Ext}(Q/Z, G)\to \mathrm{Ext}(Q, G)=0$, we see that $G\cong \mathrm{Ext}(Q/Z, G)$. Let $G=\sum_p G_p$ be the decomposition into the p-primary parts of G; then similarly $G_p\cong \mathrm{Ext}(Q/Z, G_p)$. Hence

$$G\cong \mathrm{Ext}(Q/Z, G)\cong \prod_p \mathrm{Ext}(Z(p^\infty), G)\cong \prod_p \mathrm{Ext}(Z(p^\infty), G_p)$$
$$\cong \prod_p \mathrm{Ext}(Q/Z, G_p)\cong \prod_p G_p .$$

This relation implies the number of non-zero components G_p's is finite. The rest we should do is to show each G_p is of bounded order.

Suppose G is a co-torsion p-group and $\sum B_n$ is a basic subgroup of G. Consider the exact suquence (II) at the beginning of this paper. Since $(\prod B_n)_t$ is reduced and $\mathrm{Ext}(Q, (\prod B_n)_t)=0$, $(\prod B_n)_t$ is a co-torsion group. The exact sequence:

$$0\to (\prod B_n)_t\to \prod B_n\to \prod B_n/(\prod B_n)_t\to 0$$

must split, since $\prod B_n/(\prod B_n)_t$ is of torsion-free. $(\prod B_n)_t$ cannot be a direct summand of $\prod B_n$ unless $\prod B_n=(\prod B_n)_t$. In fact, if there exists a torsion-free element $(x_1, x_2, \cdots)$ in $\prod B_n$, we can find positive integers $n_1<n_2<\cdots<n_i<\cdots$ and $x_{n_i}\in B_{n_i}$ such that $p^i x_{n_i}\neq 0$ for $i=1, 2, \cdots$. The element in $\prod B_n$ whose n_i-th coordinate is $p^i x_{n_i}$ and otherwise is 0 has infinite height in the quotient group $\prod B_n/(\prod B_n)_t$. This is a contradiction. Hence $\sum B_n$ is of bounded order and $G=\sum B_n$ by Lemma 1 and the sequence (I).

Following lemma is called Sasiada's Theorem in [1] on page 83. We shall give a simple proof for this.

LEMMA 2. *A group A is algebraically compact if and only if* $\mathrm{Ext}(Q, A)=\mathrm{Pext}(Q/Z, A)=0$.

PROOF. Suppose $\mathrm{Ext}(Q, A)=\mathrm{Pext}(Q/Z, A)=0$. Let G be an arbitrary group and G_t be its torsion part. There exists a pure exact sequence: $0\to B\to G_t\to D\to 0$ where B is a direct sum of finite cyclic groups and D is a divisible torsion group. $\mathrm{Pext}(D, A)=0$ since $\mathrm{Pext}(Q/Z, A)=0$, and $\mathrm{Pext}(B, A)=0$ since B is a direct sum of finite cyclic groups. $\mathrm{Pext}(D, A)\to \mathrm{Pext}(G_t, A)\to \mathrm{Pext}(B, A)$ implies $\mathrm{Pext}(G_t, A)=0$. Considering the injective envelope of G/G_t (see [5]), $\mathrm{Ext}(Q, A)=0$ implies $\mathrm{Ext}(G/G_t, A)=0$. Now, $\mathrm{Pext}(G, A)=0$ since the pure exact sequence: $0\to G_t\to G\to G/G_t\to 0$ gives rise to an exact sequence: $\mathrm{Pext}(G/G_t, A)\to \mathrm{Pext}(G, A)\to \mathrm{Pext}(G_t, A)$.

THEOREM 2. *If A is of torsion-free, then* $\mathrm{Pext}(Q/Z, A)=0$. *Hence a torsion-free, co-torsion group is algebraically compact by Lemma 2.*

PROOF. Let $\sum Q$ be the injective envelope of A. The short exact sequence : $0\to A\to\sum Q\to D\to 0$ gives rise to $\mathrm{Hom}(Q/Z, \sum Q)=0\to\mathrm{Hom}(Q/Z, D)\to\mathrm{Ext}(Q/Z, A)\to\mathrm{Ext}(Q/Z, \sum Q)=0$. Hence $\mathrm{Ext}(Q/Z, A)\cong\mathrm{Hom}(Q/Z, D)$. $\mathrm{Hom}(Q/Z, D)$ is a direct summand of a direct product of p-adic integers (see [2] on pp. 372–373). Hence $\mathrm{Pext}(Q/Z, A)\cong(\mathrm{Ext}(Q/Z, A))^1\cong(\mathrm{Hom}(Q/Z, D))^1=0$.

COROLLARY. $\mathrm{Pext}(Q/Z, A)\cong\mathrm{Pext}(Q/Z, A_t)$.

PROOF. From the pure exact sequence: $0\to A_t\to A\to A/A_t\to 0$, we get an exact sequence :

$$\mathrm{Hom}(Q/Z, A/A_t)\to\mathrm{Pext}(Q/Z, A_t)\to\mathrm{Pext}(Q/Z, A)\to\mathrm{Pext}(Q/Z, A/A_t)\,,$$

where $\mathrm{Hom}(Q/Z, A/A_t)=0$ since A/A_t is of torsion-free, and $\mathrm{Pext}(Q/Z, A/A_t)=0$ by Theorem 2. Therefore $\mathrm{Pext}(Q/Z, A)\cong\mathrm{Pext}(Q/Z, A_t)$.

DEFINITION. A reduced torsion group G is called *closed* if $\mathrm{Pext}(Q/Z, G)=0$.

A reduced torsion group G is closed if and only if p-primary part G_p of G is closed for every prime p in Fuchs' sense, see [1] on page 114. This fact is known as Kulikov-Papp's Theorem in [1]. We shall show this homologically as usual.

Suppose $G_p=(\prod B_n)_t$ where B_n is a direct sum of cyclic groups of order p^n. Using Lemma 1 and Corollary to Theorem 2,

$$\begin{aligned}\mathrm{Pext}(Q/Z, G)&\cong\prod_p\mathrm{Pext}(Z(p^\infty), G)\cong\prod_p\mathrm{Pext}(Z(p^\infty), G_p)\\ &\cong\prod_p\mathrm{Pext}(Q/Z, G_p)\cong\prod_p\mathrm{Pext}(Q/Z, (\prod(B_n)_t))\\ &\cong\prod_p\mathrm{Pext}(Q/Z, \prod B_n)\cong\prod_p\prod_n\mathrm{Pext}(Q/Z, B_n)=0\,.\end{aligned}$$

Conversely, suppose $\mathrm{Pext}(Q/Z, G)=0$. Then $G^1=0$ since the exact sequence: $\mathrm{Hom}(Q, G)=0\to\mathrm{Hom}(Z, G)\cong G\to\mathrm{Ext}(Q/Z, G)$ implies $0\to G^1\to\mathrm{Pert}(Q/Z, G)$. The pure exact sequence (II) at the beginning of this paper with respect to G_p must split, since $\mathrm{Pext}(Q/Z, G_p)=0$. Hence $G_p=(\prod B_n)_t$.

THEOREM 3. *A co-torsion group G is algebraically compact if and only if its torsion part G_t is closed.*

PROOF. This is an immediate consequence of Lemma 2 and Corollary to Theorem 2.

From Propositions 2.2 and 2.3 in [2], we get:

COROLLARY 1. *A reduced algebraically compact group is a direct* *sum of a torsion-free co-torsion group and an adjusted co-torsion group whose torsion part is closed. This decomposition is unique up to isomorphism.*

COROLLARY 2. *Under the one-one correspondence between all adjusted, co-torsion groups and all reduced, torsion groups, all adjusted, reduced, algebraically compact groups correspond with all closed groups.*

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