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# On the Imbedding of a Projectively Connected Space in a Projective Space

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This paper will be concerned with the dimension of a projective space in which a given projectively connected space with torsion can be imbedded in the sense of O. Galvani.[1] We shall adopt the notations of E. Cartan[2] and agree that the repeated indices imply summation.

1. Let $P_N$ be a projective space of dimension $N=n+q$. To avoid confusion we agree to use the following ranges of indices throughout the paper:

$$1\leqq i,\ j,\ k\leqq n\,, \qquad n+1\leqq \alpha,\ \beta,\ \gamma\leqq N\,,$$
$$1\leqq \lambda\,,\ \mu,\ \nu\leqq N\,.$$

If an $n$-plane $u$ and a $q$-plane $v$ have common with a point $M$ only, the triple $S=(M,\ u,\ v)$ will be called bi-plane element $S$ and $M$, $u$ is denoted by $M(S)$, $u(S)$ respectively. We shall say that a frame $(A,\ A_i,\ A_\alpha)$ is order 0 with respect to $S$ if $A$ is an analytic point of $M$ and $A_i$, $A_\alpha$ are on $u$, $v$ respectively. Denote it $R_0(S)$. Let $\{y\}$ be the parameters on which $S$ depends. An $n$-dimensional variety $V$ of bi-plane elements is the set of $S$ whose $\{y\}$ are functions of $n$ independent variables $\{u\}=(u_1,\ u_2,\ \ldots,\ u_n)$. Such a variety is called analytic if the parameters $\{z\}$ on which $R_0(S)$, $S\varepsilon V$, depends are analytic functions of $\{u\}$, when an $R_0(S)$ is suitably attached to each $S\varepsilon V$. If the locus $U$ of origin $M(S)$ of $S\varepsilon V$ is $n$ dimensional and any line in $v(S)$ does not tangent to $U$, then $V$ is said to be ordinary. In the following we shall only deal with analytic ordinary varieties.

Define the Pfaffian forms $\{\omega\}$ according to the equations

$$dA=\omega A+\omega_\lambda A_\lambda\,, \qquad dA_\lambda=\omega_{\lambda 0}A+\omega_{\lambda\mu}A_\mu\,, \tag{1.}$$

then the fact that $V$ is ordinary equivalents to $[\omega_1,\ \ldots,\ \omega_n]\neq 0$ on $V$ with respect to the family of $R_0(S)$.[3] The structure equations of $P_N$ are

$$\begin{aligned} &d\omega=[\omega_\lambda\omega_{\lambda 0}]\,, \qquad d\omega_\lambda=[\omega\omega_\lambda]+[\omega_\mu\omega_{\mu\lambda}]\,,\\ &d\omega_{\lambda 0}=[\omega_{\lambda 0}\omega]+[\omega_{\lambda\mu}\omega_{\mu 0}]\,,\\ &d\omega_{\lambda\mu}=[\omega_{\lambda 0}\omega_\mu]+[\omega_{\lambda\nu}\omega_{\nu\mu}]\,. \end{aligned} \tag{1.2}$$

Now let $V$ be analytic ordinary and $S$, $S'$ be two infinitely nearby

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1) O. Galvani (6).
2) E. Cartan (3).
3) O. Galvani (6).

elements in $V$. Consider the $(q-1)$-plane $\sum$ in $v(S)$ which $q$ points $A_\alpha$ of $R_0(S)$ generate. Let $\rho(P')$ be the point at which $u(S)$ and the $q$-plane which contains $\sum$ and a point $P'\varepsilon u(S')$ intersect. The 1–1 correspondence $\rho$ in a neighborhood of $S$ defines the induced connection, $(A, A_i)\to(\rho(A'), \rho(A_i'))$, on $V$. The induced connection are determined by $\omega_i$, $\omega_{i0}$, $\omega_{ij}-\delta_{ij}\omega$.

2. Let $P_n$ be a projectively connected space and the defining Pfaffians be $\{\overline{\omega}\}$. Then the equations

$$
\begin{aligned}
&d\overline{A}=\overline{\omega}\overline{A}+\overline{\omega}_i\overline{A}_i, \qquad d\overline{A}_i=\overline{\omega}_{i0}\overline{A}+\overline{\omega}_{ij}\overline{A}_j\,,\\
&d\overline{\omega}=[\overline{\omega}_i\overline{\omega}_{i0}]+\Omega\,,\\
&d\overline{\omega}_i=[\overline{\omega}\,\overline{\omega}_i]+[\overline{\omega}_k\overline{\omega}_{ki}]+\Omega_i\,,\\
&d\overline{\omega}_{i0}=[\overline{\omega}_{i0}\overline{\omega}]+[\overline{\omega}_{ik}\overline{\omega}_{k0}]+\Omega_{i0}\,,\\
&d\overline{\omega}_{ij}=[\overline{\omega}_{i0}\overline{\omega}_{0j}]+[\overline{\omega}_{ik}\overline{\omega}_{kj}]+\Omega_{ij}
\end{aligned}
\tag{2.1}
$$

hold, where

$$
\Omega_i=\frac{1}{2}T_{ihk}[\overline{\omega}_h\overline{\omega}_k]\,, \qquad \Omega_{i0}=\frac{1}{2}R_{i0hk}[\overline{\omega}_h\overline{\omega}_k]\,,
$$

$$
\Omega_{ij}-\delta_{ij}\Omega=\frac{1}{2}R_{ijhk}[\overline{\omega}_h\overline{\omega}_k]
$$

and $T_{ihk}$, $R_{i0hk}$, $R_{ijhk}$ are skew-symmetric with respect to $h$ and $k$. In the next, for a given $P_n$, we shall find the dimension $N$ of a projective space $P_N$ in which the $P_n$ can be imbedded as an ordinary variety of bi-plane elements. But our purpose is the possibility of the existence of the such $N$. Therefore smaller $N$ may be exist. The problem is reduced to the one that whether $N$ exists or not such as the differental system

$$
\begin{aligned}
&\omega_i=\overline{\omega}_i\,, \qquad \omega_{i0}=\overline{\omega}_{i0}\,,\\
&\omega_{ij}-\delta_{ij}\omega=\overline{\omega}_{ij}-\delta_{ij}\overline{\omega}
\end{aligned}
\tag{2.2}
$$

is in involution. We make it closed by adjoining to it the equation obtained by exterior differentiation. Then we have

$$
\begin{aligned}
&[\omega_\alpha\omega_{\alpha i}]=\frac{1}{2}T_{ihk}[\overline{\omega}_h\overline{\omega}_k]\,,\\
&[\omega_{i\alpha}\omega_{\alpha 0}]=\frac{1}{2}R_{i0hk}[\overline{\omega}_h\overline{\omega}_k]\,,\\
&[\omega_{i\alpha}\omega_{\alpha j}]-\delta_{ij}[\omega_\alpha\omega_{\alpha 0}]=\frac{1}{2}R_{ijhk}[\overline{\omega}_h\overline{\omega}_k]\,.
\end{aligned}
\tag{2.3}
$$

Let $I_n$ be an irreducible $n$-integral element of our system,[4] defined by the following equations:

$$
\begin{aligned}
&\omega_\alpha=\pi_{\alpha i}\overline{\omega}_i\,, \qquad \omega_{\alpha 0}=\theta_{\alpha i}\overline{\omega}_i\,,\\
&\omega_{i\alpha}=\pi_{i\alpha j}\overline{\omega}_j\,, \qquad \omega_{\alpha j}=\theta_{\alpha ji}\overline{\omega}_i\,,
\end{aligned}
\tag{2.4}
$$

[4] E. Cartan (3).

then the prarameters $\{\pi, \theta\}$ on which $I_n$ depends must satisfy the following equations, obtained from (2.3) and (2.4),

$$\begin{aligned} &\pi_{\alpha h}\theta_{\alpha i k}-\pi_{\alpha k}\theta_{\alpha i h}=T_{ihk}\,, \\ (2.5)\qquad &\pi_{i\alpha h}\theta_{\alpha k}-\pi_{i\alpha k}\theta_{\alpha h}=R_{i0hk}\,, \qquad (h>k)\,, \\ &\pi_{i\alpha h}\theta_{\alpha jk}-\pi_{i\alpha k}\theta_{\alpha jh}-\delta_{ij}(\pi_{\alpha h}\theta_{\alpha k}-\pi_{\alpha k}\theta_{\alpha h})R_{ijhk}\,. \end{aligned}$$

We suppose $\{\pi, \theta\}$ as vectors in an auxiliary $(N-n)$ dimensional Euclid space E and put

$$\vec{\pi}_h=\{\pi_{\alpha h}\}\,,\qquad \vec{\theta}_h=\{\theta_{\alpha h}\}\,,$$
$$\vec{\pi}_{ih}=\{\pi_{i\alpha h}\}\,,\qquad \vec{\theta}_{ih}=\{\theta_{\alpha ih}\}\,,$$

then (2.5) can be written in the form of inner product as the following:

$$\begin{aligned} &\vec{\pi}_h\cdot\vec{\theta}_{ik}-\vec{\pi}_k\cdot\vec{\theta}_{ih}=T_{ihk}\,, \\ (2.6)\qquad &\vec{\pi}_{ih}\cdot\vec{\theta}_k-\vec{\pi}_{ik}\cdot\vec{\theta}_h=R_{i0hk}\,, \qquad (h>k)\,, \\ &\vec{\pi}_{ih}\cdot\vec{\theta}_{jk}-\vec{\pi}_{ik}\cdot\vec{\theta}_{jh}-\delta_{ij}(\vec{\pi}_h\cdot\vec{\theta}_k-\vec{\pi}_k\cdot\vec{\theta}_h)=R_{ijhk}\,. \end{aligned}$$

Now we assume that $N-n\geqq n^2-1$, then in $E$, it can be chosen such that the vectors $\vec{\pi}_1, \ldots, \vec{\pi}_{n-1}, \vec{\pi}_{ih}, i=1, \ldots, n; h=1, \ldots, n-1$, are linealy independent. For a given $p\leqq n-1$, the first member of (2.5) with $h\leqq p$ are independent, and we can give $\vec{\pi}_n, \vec{\pi}_{in}$ such as they satisfy (2.5) with $h=n$. Hence the first member of (2.5) are independent, so (2.5) are compatible. Therefore if $N=n^2+n-1$, irreducible $n$-integral elements $I_n$ exist. Define $I_p\subset I_n$, $1\leqq p<n$, by (2.3) and $\bar{\omega}_h=0$, $h>p$, then the reduced polar system of $I_p$ is given by the equations,

$$\begin{aligned} &\omega_{\alpha i}\pi_{\alpha h}-\omega_{\alpha}\theta_{\alpha ih}=0\,, \\ &\omega_{\alpha 0}\pi_{i\alpha h}-\omega_{i\alpha}\theta_{\alpha h}=0\,, \\ &\omega_{\alpha j}\pi_{i\alpha h}-\omega_{i\alpha}\theta_{\alpha jh}-\delta_{ij}(\omega_{\alpha 0}\pi_{\alpha h}-\omega_{\alpha}\theta_{\alpha h})=0\,, \\ &(h=1, \ldots, p;\ i,j=1, \ldots, n)\,. \end{aligned}$$

These equations are independent in general, so[5]

$$\sigma_1+\ldots\ldots+\sigma_p=np(n+2)\,,$$

hence $\sigma_p=n(n+2)$, $1\leqq p\leqq n-1$. On the other hand we have from (2.2), $\sigma_0=n(n+2)$. As the number of unknown functions is $2N(n+1)-n^2$, we get $\sigma_n=n^3+n^2-2$. The number of independent parameters $\{\pi, \theta\}$ on which $I_n$ depend is $n(n-1)(3n^2+6n+4)/2$, which equals to $\sigma_1+2\sigma_2+\ldots+n\sigma_n$. Hence our system is in involution.[5] Thus we get the following

*Theorem. An n-dimensional projectively connected space $P_n$ can be imbedded locally in an $(n^2-n+1)$-dimensional projective space as an ordinary variety of bi-plane elements.*

4. In this section we shall deal with the curvatureless spaces, i.e. spaces in which equations $\Omega_{i0}=0$, $\Omega_{ij}-\delta_{ij}\Omega=0$ hold.

From the structure equations, we can see that the differential system

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[5] E. Cartan (3).

(4.1) $$\overline{\omega}_{i0}=0, \qquad \overline{\omega}_{ij}-\delta_{ij}\overline{\omega}=0$$

is completely integrable. Hence, in the $P_n$, we can adopt the frame satisfying (4.1). In this case, our object is to demonstrate the existence of $N$ such as the differential system

(4.2) $$\omega_i=\overline{\omega}_i, \qquad \omega_{i0}=0, \qquad \omega_{ij}-\delta_{ij}\omega=0$$

is in involution. Now we append to (4.2) the additional equations,

(4.3) $$\omega_{i\alpha}=0, \qquad \omega_{\alpha 0}=0,$$

and shall deal with the system (4.2) and (4.3). To make it closed, we adjoin the equations

(4.4) $$[\omega_\alpha\omega_{\alpha i}]=\frac{1}{2}T_{ihk}[\overline{\omega}_h\overline{\omega}_k],$$

which are obtained by exterior differentiation of (4.2) and (4.3).

Define irreducible elements $I_n$ by

(4.5) $$\omega_\alpha=\pi_{\alpha h}\overline{\omega}_h, \qquad \omega_{\alpha i}=\pi_{\alpha ih}\overline{\omega}_h,$$

so $\{\pi\}$ must satisfy

(4.6) $$\pi_{\alpha k}\pi_{\alpha ih}-\pi_{\alpha h}\pi_{\alpha ik}=T_{ihk}, \qquad (h>k),$$

which can be represented in vector form by

(4.7) $$\vec{\pi}_k\cdot\vec{\pi}_{ih}-\vec{\pi}_h\cdot\vec{\pi}_{ik}=T_{ihk}, \qquad (h>k),$$

in the auxiliary $N-n$ dimensional Euclid space.

Now we put $N=2n-1$, so it can be chosen such that $\vec{\pi}_1, \ldots, \vec{\pi}_{n-}$ are linearly independent. Then the first member of (4.7) with $p\geq h>k$ for a fixed $p$, $p<n$, are independent and (4.7) with $h=n$ become the equations giving the projections of $\vec{\pi}_{in}$ on $\vec{\pi}_k$. Hence (4.6) are compatible, so irreducible $n$-integral elements $I_n$ exist.

Let $I_p\subset I_n$, $1\leq p<n$, be the elements defined by (4.5) and $\omega_h=0$, $h>p$. As the reduced polar system is

$$\omega_\alpha\pi_{\alpha ih}-\omega_{\alpha i}\pi_{\alpha h}=0, \qquad (h=1, \ldots, p),$$

we get $\sigma_p=n$, $1\leqslant p<n$. On the other hand $\sigma_0=2n^2+2n-1$, so $\sigma_n=$(number of unknown functions)$-\sum_{t=0}^{n-1}\sigma_t=n-1$. Hence we have $\sigma_1+2\sigma_2+\ldots+n\sigma_n=n(n-1)(n-2)/2$, which equals to the number of independent parameters $\{\pi\}$. Thus the system in consideration is in involution.

*Theorem. An n-dimensional curvatureless projectively connected space $P_n$ can be imbedded in an $2n-1$ dimensional projective space as an ordinary variety of bi-plane elements.*

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