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A Note on My Paper, "Some Theorems on Abelian Varieties."

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In the quoted paper, (Natural Science Report of the Ochanomizu University, vol. 4, 1953), the writer proved certain results concerning Abelian Varieties and he needed there the following generalization of normalization due to Zariski:

Let W be a Variety in a projective space defined over a field k and $\boldsymbol{P}$ be a generic Point of W over k. Let K be a finite algebraic extension of $k(\boldsymbol{P})$. Then one can find a projective model $\boldsymbol{U}$ of K/k and a rational mapping g from $\boldsymbol{U}$ onto $\boldsymbol{W}$ defined over k having the following properties,

(i) *g is defined everywhere on $\boldsymbol{U}$,*
(ii) *g^{-1} has no fundamental Point on $\boldsymbol{W}$,*
(iii) *$\boldsymbol{U}$ is a projective Variety relatively normal with reference to k.*

(cf. Prop. 5 of the quoted paper.)

Immediate consequence of this is that, when $\boldsymbol{W}$ is an Abelian Variety and when K has a normal law of composition defined over k, (this means that a certain model of K/k has a normal law of composition defined over k), then $\boldsymbol{U}$ is an Abelian Variety if one takes Zariski's main theorem on birational transformation into account. The aim of the present note is to give a full proof of it, since the original proof was written briefly, in spite of its importance.

The main difficulty lies, in fact, to imbed a model of K/k having properties (i) and (ii) into a projective space and if one does not require the condition that $\boldsymbol{U}$ is in a projective space, the proof would be much simpler.

Proof of the above statement.

Let $(x_0, \ldots, x_n)$ be a generic point of the representative cone of $\boldsymbol{W}$ over k, then $(1, x_1/x_0, \ldots, x_n/x_0)$ is a generic point of a representative W of $\boldsymbol{W}$ over k. Assume that $\boldsymbol{P}$ is the Point on $\boldsymbol{W}$ whose representative on W is $(1, x_1/x_0, \ldots, x_n/x_0)=P$, then we have $k(1, x_1/x_0, \ldots, x_n/x_0)=k(\boldsymbol{P})$. By our assumption, K is a finite algebraic extension of $k(\boldsymbol{P})$ and we can find a module basis $(y_1, \ldots, y_t)$ of K over $k(\boldsymbol{P})$ consisting of integral elements over the ring $k[1, x_1/x_0 \ldots, x_n/x_0]$ which we denote by $k[P]$. Therefore, y_i satisfies the equation of the form

$$y_i^r + f_1(P)y_i^{r-1} + \dots + f_r(P) = 0$$

with $f_i(P) \in k[P]$. This can be written in the form

$$x_0^s y_i^r + g_1(P)y_i^{r-1} + \dots + g_r(P) = 0 ,$$

with $g_i(P) \in k[P]$. Multiplying $x_0^{s(r-1)}$, we get

$$(x_0^s y_i)^r + h_1(P)(x_0^s y_i)^{r-1} + \dots + h_r(P) = 0$$

with $h_i(P) \in k[P]$. This shows that $x_0^s y_i$ is integral over the ring $k[P]$ and so, there is a positive integer m such that $x_0^m y_i (i=1,\dots,t)$ is integral over $k[P]$. Let $(\xi_0,\dots,\xi_N)$ be the set of monomials of the form

$$x_0^{e_0} x_1^{e_1} \dots x_n^{e_n} , \qquad \sum e_i = m$$

arranged in a suitable order, and put $\xi_{N+i} = x_0^m y_i (i=1,2,\dots,t)$. The ring $k[P]$ is integral over $k[\xi_0,\dots,\xi_N]$ and hence ξ_{N+i}'s are integral over $k[\xi_0,\dots,\xi_N]$. The projective Variety W' whose representative cone is the locus of $(\xi_0,\dots,\xi_N)$ over k is in everywhere biregular birational correspondence with W over k. $(\xi_0,\dots,\xi_{N+t})$ may be regarded as a homogeneous coordinate of a Point $\boldsymbol{R}$ in a projective space and it can be seen easily that we have

$$k(\boldsymbol{R}) = K .$$

Let $\boldsymbol{Q}$ be a Point in a projective space whose homogeneous coordinate is $(\xi_0,\dots,\xi_N)$ and Z' be the Locus of the Point $\boldsymbol{R} \times \boldsymbol{Q}$ over k. Z' is the graph of a certain rational map g' from the Locus U' of $\boldsymbol{R}$ over k onto W' such that

$$g'(\boldsymbol{R}) = \boldsymbol{Q} .$$

We shall show that when we replace W by W', then g' and U' satisfy the conditions (i) and (ii).

In order to do so, we remark the following. Denote by U_j' the representative of U' whose generic point over k is

$$(\xi_0/\xi_j,\dots,\xi_N/\xi_j,\dots,\xi_{N+t}/\xi_j) .$$

Then every Point $\boldsymbol{R}'$ on U' has the representative on some U_j' for

$$j = 0, 1, \dots, N .$$

In fact, extend the specialization $\boldsymbol{R} \to \boldsymbol{R}'$ over k to a specialization of a set of quantities

$$(\xi_0/\xi_j,\dots,\xi_N/\xi_j)(j=0,1,\dots,N) .$$

Since W' is complete, and since the above is a representative of $\boldsymbol{Q}$, there must exist at least one index, say a, such that $0 \leqq a \leqq N$ and that a specialization of

$$(\xi_0/\xi_a,\dots,\xi_N/\xi_a)$$

over $\boldsymbol{R} \to \boldsymbol{R}'$ with reference to k is finite. Since ξ_{N+i} is integral over the ring $k[\xi_0,\dots,\xi_N]$, and since $(\xi_0,\dots,\xi_{N+t})$ is a generic point of the representative ray of $\boldsymbol{R}$ over $k(\boldsymbol{R})$, we can see easily, by the well known device, that each ξ_j/ξ_a is integral over the ring $k[\xi_0/\xi_a,\dots,\xi_N/\xi_a]$. Hence a specialization of $(\xi_j/\xi_a)(j=N+1,\ldots,N+t)$ over $\boldsymbol{R}\to\boldsymbol{R}'$ with reference to k is finite. This proves our assertion.

Then the conditions (i) and (ii) are satisfied by $\boldsymbol{U}'$ and g'. Let $\boldsymbol{U}$ be the derived normal model of $\boldsymbol{U}'$ in a projective space over k. The birational transformation between $\boldsymbol{U}'$ and $\boldsymbol{U}$ which defines a normalization has no fundamental Point on $\boldsymbol{U}$ and $\boldsymbol{U}'$ and moreover, $\boldsymbol{W}'$ and $\boldsymbol{W}$ are in everywhere biregular birational correspondence over k. Therefore, we can conclude from this the existence of $\boldsymbol{U}$ and g which are as required.

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