On the Theorem of Castelnuovo-Enriques

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Introduction. The classical theorem of Castelnuovo-Enriques asserts that the one dimensional Betti number of a sufficiently general divisor on a non-singular algebraic Variety \( V^n(n \geq 3) \) is the same as that of \( V^n \) or the base for 1-cycles on such divisor also forms the base for 1-cycles on \( V \). The algebraic equivalent of the above theorem may be formulated as follows:

\[
V^n(n \geq 3) \text{ and its sufficiently general divisor have the same Picard } \text{(Albanese)} \text{ Variety up to an isomorphism.}
\]

The aim of the present note is to prove the theorem of Castelnuovo-Enriques in the above formulation.

Let \( X \) be a divisor on the product \( \Gamma \times V^n \) of a non-singular Curve \( \Gamma \) with an algebraic Variety \( V^n \) in a projective space and \( k \) be a common field of definition for \( \Gamma \), \( V \) over which \( X \) is rational. We shall say that the totality \( \mathfrak{A} \) of \( V \)-divisors of the form \( X(u) \) defined by

\[
(u \times V) \cdot X = u \times X(u)
\]
is a one-dimensional algebraic family defined by \( \Gamma \) and \( X \) or \( \Gamma \) and \( X \) defines \( \mathfrak{A} \). A field such as \( k \) shall be refered to as a field of definition for \( \mathfrak{A} \) or we shall say that \( \mathfrak{A} \) is defined over \( k \). The one dimensional algebraic family on \( V \) which we shall treat in this paper is a linear pencil of the special kind. When \( V \) is absolutely locally normal, and when the complete linear system on \( V \) is sufficiently ample, then one can extract from the complete linear system a linear pencil such that it contains a Variety. Moreover, when \( V \) is non-singular, we may assume that it contains also a non-singular Variety. Our interest will be concentrated to the linear pencil having this property.

Assume that \( \mathfrak{A} \) is a pencil, \( V \) is an algebraic Surface such that it has a base Point at a simple Point of \( V \) and that a generic divisor \( X(u) \) corresponding to a generic Point \( u \) of \( \Gamma \) over a common field of definition \( k \) for \( V \) and \( \Gamma \) is a non-singular Curve. By Chow’s result on Jacobian Varieties (cf. [C]-1, [C]-2, or [M]-4, § 2)\(^1\), there is a symmetric function \( \mathcal{F} \) defined on the product of sufficiently many factors equal to \( X(u) \) on the Jacobian Variety \( J \) of \( X(u) \) defined over \( k(u) \) immersed

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1) Letters in brackets refer to the bibliography at the end.
into a projective space such that when we write \( \mathcal{T} \) as
\[
\mathcal{T} = \sum \varphi_i \quad (\text{cf. [W]-2 chap. III, cor. of th. 7})
\]
where \( \varphi_i \) is a function defined on \( X(u) \) with values in \( J \), \( \varphi_i \) and \( \varphi \) coincide within an additive constant and \( \varphi_i \) is the canonical function of \( X(u) \). Let \( x_0 \) be a base point of \( \mathcal{U} \). Then \( x_0 \) is algebraic over \( k \) and hence it is rational over \( \bar{k}(u) \). Moreover \( x_0 \) is a simple point of \( X(u) \) and this shows that one of the \( \varphi_i \), say \( \varphi_i \), is defined over \( \bar{k}(u) \) (cf. [W]-2 chap. III, cor. th. 7).

Let \( |X| \) be a linear system, which is not necessarily complete, on a Variety \( V \) and \( k \) be a common field of definition for \( V \) and \( |X| \). Let \( X \) be a generic divisor of \( |X| \) over \( k \) and assume that \( V \) and \( X \) are Varieties having no singular Subvarieties of dimension \( n-2 \) and \( n-3 \) respectively. The induced linear system on \( X \) by \( |X| \) shall be referred to as the characteristic linear system of \( |X| \) on \( X \). "The characteristic linear system of \( |X| \)" will mean the characteristic linear system of \( |X| \) on some generic divisor of \( |X| \) over \( k \).

There is a maximal algebraic family \( \{X\} \) on \( V \) such that when a \( V \)-divisor \( \mathcal{Y} \) is algebraically equivalent to zero, there are two divisors \( X \) and \( X' \) in \( \{X\} \) such that \( \mathcal{Y} \sim X - X' \) (cf. [M]-3, th. 1). We shall call \( \{X\} \) as a total maximal algebraic family\(^2\). We shall say that a field \( k \) of definition for the Picard Variety \( p(V) \) of an algebraic Variety \( V \) having no singular Subvariety of dimension \( n-1 \) is the complete field of definition for \( p(V) \) when the following conditions are satisfied:

when \( X \) is a \( V \)-divisor, algebraically equivalent to zero,

such that it is rational over a field \( K \) containing \( k \), then the class of \( X \) on \( p(V) \) is rational over \( K \) and conversely, when \( \xi \) is a Point on \( p(V) \) rational over a field \( K \) containing \( k \), there is a \( V \)-divisor \( X \) rational over \( K \), algebraically equivalent to zero, such that its class on \( p(V) \) is \( \xi \). (cf. [M]-3, th. 3 and [M]-4, th. 4).

Denote by \( L_m \) the linear system consisting of all the divisors of the form \( H_m \cdot V \) where \( H_m \) is a hypersurface of order \( m \). Let \( C_{m_i} \) be a generic divisor of \( L_{m_i} \) \((i=1, 2, \ldots, s) \) over a common field of definition \( k \) for \( V \) and for every \( L_{m_i} \). Then the intersection-product \( C_{m_1} \cdots C_{m_s} \) is defined on \( V \). We shall say that it is a generic \((n-s)\)-cycle of order \( m_1 \cdots m_s \) of \( V \) over \( k \). When \( V \) has no singular Subvariety of dimension \( n-r \) \((r>0) \), its generic \((n-s)\)-cycle over \( k \) is a Variety and has no singular Subvariety of dimension \( n-s-r \) (cf. [N], [Z]-th. 3). In particular, a generic 1-cycle on \( V \) over \( k \) is a non-singular Curve.

Let \( U \) be a Variety, \( A \) be an Abelian Variety and \( f \) be a function defined on \( U \) with values in \( A \). We shall say that \( U \) (and \( f \)) generates \( A \) when if \( x_1, \ldots, x_n \) are sufficiently many numbers of independent

\(^2\) The writer had used the term "regular," and by the advice of Prof. C. Chevalley, the writer will use the term "total".

"total".
generic Points of $U$ over a common field of definition $k$ for $U$, $A$ and $f$, $\sum f(\omega)$ is a generic Point of $A$ over $k$. When that is so, we shall also say that $A$ is generated by $U$ (and $f$).

1. Proposition. Let $V$ be an algebraic Surface free from singular Points in a projective space, $\mathfrak{A}$ be a linear pencil on $V$ and $W_0$ be a fixed divisor in $\mathfrak{A}$ such that it is a non-singular Curve. Let $X$ be a divisor on $V$ algebraically equivalent to zero such that $X \cdot W_0$ is linearly equivalent to zero on $W_0$. Then $X$ is linearly equivalent to zero within a linear combination of certain numbers of fixed divisors on $V$ independent of $X$. (cf. [W]-3, (A)).

Proof. Since the proof may be essentially the same as that of Weil, we shall sketch briefly the outline of it.

Let $W$ be a generic divisor of $\mathfrak{A}$ over a common field of definition $k$ for $V$ and $\mathfrak{A}$, over which $X$ is rational, then it is a non-singular Curve and it is easy to see that there is a set of finite numbers of Curves $U_1, \ldots, U_s$ all algebraic over $k$ on $V$ having the following properties: let $Z$ be a $V$-divisor rational over $k$ such that $Z \cdot W \sim 0$ on $W$, then $Z$ is linearly equivalent to a certain linear combination of $U_1, \ldots, U_s$. From this, we can derive the following: let $\sum$ be a one-dimensional algebraic family of positive $V$-divisors defined over $k$ such that every divisor of $\sum$ induces on $W$ mutually equivalent $W$-divisors with respect to linear equivalence, then every divisor of $\sum$ is mutually equivalent with respect to linear equivalence.

Let $\{Y\}$ be the total maximal algebraic family of positive $V$-divisors and $U$ be its associated Variety, which is defined over $\bar{k}$. Let $Y$ be a generic divisor of $\{Y\}$ over $\bar{k}(w)$ and $Y_0$ be a rational divisor of it over $\bar{k}$ where $w$ is the Chow-Point of $W$. Let $\varphi_w$ be the canonical function of $W$ defined over $k(w)$ and $h_w$ be the function defined on $U$ with values in the Jacobian Variety $\mathfrak{p}(W)$ defined

$$h_w(y) = \mathfrak{p}(\varphi_w((Y - Y_0) \cdot W)) = \gamma_w$$

where $y$ is the Chow-Point of $Y$. $h_w'(\gamma_w)$ consists of finite numbers of associated Varieties of complete linear systems and the Locus $A_w$ of $\gamma_w$ over $\bar{k}(w)$ is an Abelian Variety isogeneous to the Picard Variety $\mathfrak{p}(V)$ of $V$ (cf. [M]-2). Let $w'$ be the Chow-Point of a generic divisor $W'$ of $\mathfrak{A}$ over $\bar{k}(w, y)$ and put $h_{w'}(y) = \gamma_{w'}$. Then $\gamma_w$ is purely inseparable over $\bar{k}(w, w', \gamma_w)$ since the group of all the Points of given order on an Abelian Variety is a finite group (cf. [W]-2, cor. 1, th. 33).

Let $K$ be the algebraic closure of $k(w')$. Then, there is a homomorphism $\lambda$ defined on $A_w$ with values on $A_w^*$ with a field of definition $K(w)$ (cf. [W]-2, th. 27) such that $\lambda(\gamma_w) = \gamma_w^*$, where $*$ denotes an auto-
morphism of the universal domain defined by
\[ u^* = u^t \]
for a certain non-negative integer \( t \). By using the theorem of complete reducibility for Abelian Varieties (cf. [W]–2, prop. 25, th. 26), we conclude that there is a function \( \tilde{\varphi} \) defined on \( W \) with values in \( A_w \) such that
\[ S[\tilde{\varphi}(Y - Y_0) \cdot W] = \gamma_w^* \]

\( \tilde{\varphi} \) can be extended to a function \( \varphi \) defined on \( V \) with values in \( A_w \), such that \( \varphi_w = \tilde{\varphi} \).

We may assume that \( X = Y' - Y_0 \) where \( Y' \) is in \( \{ Y \} \) and moreover, that \( Y', Y_0 \) are also non-singular Curves (cf. [M]–5, th. 2). Then applying th. 10 of [W]–2, if \( X \cdot W_0 \sim 0 \), we have
\[ 0 = S[\varphi(X \cdot W_0)] = S[\varphi(X \cdot W)] \]
and from this we conclude that \( X \cdot W \sim 0 \) on \( W \).  

q.e.d.

As a corollary of the above proposition, we have

**Corollary.** Let \( V^n \) be a Variety in a projective space, having no singular Subvariety of dimension \( n-2 \) and \( \mathfrak{A} \) be a linear pencil on \( V \) such that \( \mathfrak{A} \) contains a Variety \( W \) free from singular Subvarieties of dimension \( n-3 \). When \( X \) is a \( V \)-divisor which is algebraically equivalent to zero, such that \( W \cdot X \) is defined on \( V \) and that it is linearly equivalent to zero on \( W \), \( X \) is linearly equivalent to zero within a linear combination of certain numbers of \( V \)-divisors which are independent of \( X \).

This corollary follows immediately from prop. above and from [W]–3, lemma.

2. **Theorem.** Let \( V^n \) be a Variety free from singular Subvarieties of dimension \( n-2 \) in a projective space and \( \mathcal{B} \) be a linear system on \( V \) having the following properties:

(i) \( \mathcal{B} \) contains a divisor which is a Variety free from singular Subvarieties of dimension \( n-3 \)

(ii) \( \dim \mathcal{B} = 2 \) and the characteristic linear system of \( \mathcal{B} \) contains a Variety having no singular Subvarieties of dimension \( n-4 \). Then the Picard Variety of a generic divisor of \( \mathcal{B} \) over a common field of definition for \( V \) and \( \mathcal{B} \) is isogeneous to the Picard Variety \( p(V) \) of \( V \).

**Proof.** By the above corollary, it can be easily seen that the Picard Variety \( p(Z) \) of a generic divisor \( Z \) of \( \mathcal{B} \) over a common field of definition \( k \) for \( V \) and \( \mathcal{B} \) contains the Abelian Variety which is isogeneous to the Picard Variety \( p(V) \) of \( V \) (cf. [M]–3, prop. 11).

5) When \( n-4 < 0 \), put 0 instead of \( n-4 \).
Let $Z$ and $\bar{Z}$ be two independent generic divisors of $E$ over $k$ and $f$ be a function on $V$ such that $(f)_0 = Z$, $(f)_\infty = \bar{Z}$. Consider the linear pencil $A$ defined by the function 1 and $f$ on $V$. The base Variety of $A$ is $Z - \bar{Z} = C$ and $C$ has no singular Subvariety of dimension $n$-4 by our assumption (ii). Let $K$ be an algebraically closed field of definition for $f$, $C$ and for the Picard Variety $p(C)$ of $C$ and assume that $K$ is complete as a field of definition for $p(C)$. Let $v$ be a generic Point of the associated Variety of $A$ over $K$ and denote by $Z_v$ the corresponding divisor of $A$. $E$ induces on $Z_v$ a linear pencil $E'$—the characteristic linear system of $E$ on $Z_v$—having $C$ as its divisor. By applying the corollary of our proposition to $Z_v$, to the linear system $E'$ and to a divisor $C$ of $B'$, we see that there is the Abelian Variety isogeneous to $p(Z_v)$ in the Picard Variety $p(C)$ of $C$.

Let $\{X\}$ be a maximal total algebraic family of positive $Z_v$-divisors containing a rational divisor $X_0$ over $K(v)$ and defined over $K(v)$ (cf. [M]-5, prop. 3) such that $C \cdot X_v$ is defined on $Z_v$. Let $M$ be the Chow-Point of the associated Variety $T(X)$ of the complete linear system $[X]$ determined by a generic divisor $X$ of $\{X\}$ over $K(v)$, and $x'$, $x''$ be two independent generic Points of $T(X)$ over $K(v, M)$ corresponding to $X'$, $X''$ respectively. Since $C \cdot (X_v - X')$ is rational over $K(v, M, x')$ (cf. [C]-3), its class $\xi$ at $p(C)$ is rational over $K(v, M, x')$ and in the same way $\eta$ is rational over $K(v, M, x'')$. This shows that $\xi$ is rational over $K(v, M)$ and its Locus $A$ over $K(v)$ is isogeneous to $p(Z_v)$.

Let $\Gamma'$ be a generic 1-cycle of $Z_v$ of order 1 over $K(v, M)$, then $p(Z_v)$ is isogeneous imbedded into the Jacobian Variety $p(\Gamma')$ of $\Gamma$ (cf. [M]-3, prop. 11). We may assume that the degree of $X$ is so large that $\deg(\Gamma' \cdot X') > 2 \cdot \text{genus}(\Gamma') - 2$. Then taking Chow's result on the Jacobian Varieties into account (cf. [C]-2), the same arguments as above show that the class $\eta$ of $\Gamma' \cdot X$ on $p(\Gamma')$ is rational over $K(v, M, t)$ where $t$ is the Chow-Point of $\Gamma$. The Locus $B$ of $\eta$ over $K(v, t)$ is an Abelian Variety in $p(\Gamma')$ isogeneous to $p(Z_v)$ and moreover, $K(v, t, M)$ is a pure inseparable extension of $K(v, t, \eta)$ by Weil's criterion for linear equivalence (cf. [W]-3, (E)). Hence $\xi$ is also purely inseparable over $K(v, t, \eta)$ and when $p$ is the characteristic of our universal domain, there is a positive integer $e$ such that $\xi p^e$ is rational over $K(v, t, \eta)$. Since $K$ is algebraically closed, $\xi p^e$ has also the Locus $A'$ over $K(p^e) \subset K(v)$ and $A'$ is clearly isogeneous to $A$. By what we have observed above, there is a homomorphism $\lambda$ defined over $K(v, t)$ from $B$ onto $A'$ and there is a homomorphism $\mu$ defined over $K(v)$ from $A'$ onto $A$, that is, into $p(C)$. By [W]-2, chap. VII, prop. 25, there is a homomorphism from $p(\Gamma')$ onto $B$ with a field of definition $K(v, t)$ and consequently, there is a symmetric function $\mathcal{F}$ defined on the product
of sufficiently many factors equal to \( \Gamma \) into \( \mathfrak{p}(C) \) such that the image of \( \mathfrak{p}(\Gamma) \) is \( \mathcal{A} \) and that it is defined over \( K(v, t) \). Let \( \Gamma' \) be cut out on \( \mathbb{Z}_v \) by the linear Variety defined by the set of linear equations

\[
\sum l_{ij}X_j - s_iX_0 = 0
\]

where \((l, s)\) is a set of independent variables over \( K(v) \) and put \( \bar{K}(\bar{u}) = K' \). Then \( \Gamma' \) has a Point which is rational over \( K'(v) \) and so, when we write

\[
\mathcal{Y} = \sum_{i=1}^{m} \bar{\varphi}_i,
\]

where \( \bar{\varphi}_i \) is a function defined on \( \Gamma' \) with values in \( \mathfrak{p}(C) \), one of it, say \( \bar{\varphi}_1 = \bar{\varphi} \), may be assumed to be defined over \( K(v, s) \) (cf. [W]-2, chap. III, cor. th.7). This function \( \bar{\varphi} \) can be extended to a function \( \varphi \) defined on \( \mathbb{Z}_v \) with values in \( \mathfrak{p}(C) \) with a field of definition \( K'(v) \) and further, \( \varphi \) can be extended to a function \( \varphi \) defined on \( V \) with values in \( \mathfrak{p}(C) \) defined over \( K' \) in a natural way such that

\[
\varphi_{x_v} = \varphi, \quad \varphi_{\Gamma'} = \bar{\varphi}.
\]

Let \( x_1, \ldots, x_m \) be \( m \) independent generic Points of \( V \) over \( K' \) and put \( \sum \varphi(x) = \zeta \). \( \zeta \) has the Locus \( \mathcal{A}'' \) over \( K' \) and \( \mathcal{A}'' \) clearly contains \( \mathcal{A} \) as a Subvariety. This implies that \( \mathcal{A} = \mathcal{A}'' \) and hence \( V \) generates the Abelian Variety \( \mathcal{A} \) isogeneous to \( \mathfrak{p}(Z_v) \) and consequently \( \mathfrak{p}(V) \) and \( \mathfrak{p}(Z_v) \) are isogeneous. q.e.d.

**Corollary.** Let \( V^n \) be an absolutely locally normal Variety, free from singular Subvariety of dimension \( n-2 \) in a projective space and \( W^{n-1} \) be its generic (\( n-1 \))-cycle over a certain field of definition for \( V \). Then the Picard Variety \( \mathfrak{p}(V) \) and \( \mathfrak{p}(W) \) of \( V \) and \( W \) are isomorphic.

**Proof.** By Weil's criterion for linear equivalence (cf. [W]-3, (E)), when \( V \)-divisor \( X \) is algebraically equivalent to zero, \( X \cdot W = 0 \) on \( W \) and \( X \sim 0 \) are equivalent (cf. also [W]-1, chap. VIII, th.4). Moreover, when we apply our theorem to this case, \( \mathfrak{p}(V) \) and \( \mathfrak{p}(W) \) are isomorphic as abstract groups. Let \( \{X\} \) be a maximal total algebraic family of positive \( V \)-divisors and \( K \) be a common field of definition for \( V, W, \{X\}, \mathfrak{p}(V), \mathfrak{p}(W) \) over which a certain divisor \( X \) in \( \{X\} \) is rational and assume that \( K \) is a complete field of definition for \( \mathfrak{p}(V) \) and \( \mathfrak{p}(W) \). Let \( M \) and \( M^* \) be the Chow-Points of the associated-Varieties of the complete linear system \( \{X\} \) and \( \{W \cdot X\} \) where \( X \) is a generic divisor of \( \{X\} \) over \( K \). Then the remark made at the top of this proof implies that \( K(M) \supseteq K(M^*) \) and moreover, \( M \) is purely inseparable over \( K(M^*) \) (cf. [M]-3, §4). Hence it is sufficient to prove that when we extend the field of reference \( K \) to the algebraic closure \( \bar{K} \) of it, \( \bar{K}(M) \) is a
separable extension of $\bar{K}(M^*)$. It is easy to see that $|X|$ and $|W \cdot X|$ are both defined respectively over $\bar{K}(M)$ and over $\bar{K}(M^*)$, that is, the base of the modules $L(X)$ and $L(W \cdot X)$ have the basis consisting of functions defined over $\bar{K}(M)$ and over $\bar{K}(M^*)$ respectively (cf. [W]-1, chap. VIII, th.10). Since $V^n$ is absolutely locally normal, and free from singular Subvarieties of dimension $n-2$, $W$ is also absolutely locally normal and free from singular Subvarieties of dimension $n-3$ (cf. [N], [Z]-th. 3). Then in view of the Castelnuovo’s lemma\(^4\) (cf. [M]-6, p. 126), we may assume that $|X|$ induces on $W$ the complete linear system $|X \cdot W|$ (cf. also [M]-5, th. 2). Let $X^*$ be a rational divisor of $|X \cdot W|$ over $\bar{K}(M^*)$. There is a divisor $X'$ in $|X|$ such that

$$X^* = X' \cdot W$$

Let $x'$ and $x^*$ be the Chow-Points of $X'$ and $X^*$ respectively. Then we have $\bar{K}(x^*) = \bar{K}(M^*)$ and $\bar{K}(x') \supset \bar{K}(M)$ (cf. [M]-3 lemma 4). We may assume that $X^*$ and $X'$ are both Varieties. Then $\bar{K}(x')$ is a separable extension of $\bar{K}(x^*)$ by [M]-5, prop. 7 and this completes our proof.

Bibliography


\(^4\) The Castelnuovo’s lemma we need here is the following: let $V^n$ be an absolutely locally normal Variety in a projective space, having no singular Subvariety of dimension $n-2$, defined over a field $k$. Let $C$ be a generic divisor of $L_q$ over $k$. When $X$ is a $V$-divisor, then the complete linear system $|X + hC|$ induces on $C$ a complete-linear system if $h$ is large.


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