On the Divisor-Problem Generated by $\zeta^\alpha(s)$

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We are concerned with a generalization of the Piltz divisor-problem. Let $\zeta(s)$ be the Riemann zeta-function and write, for $\alpha > 0$,

$$\zeta^\alpha(s) = \sum_{n=1}^\infty d_\alpha(n)n^{-s} \quad (\Re s > 1),$$

where we take that branch of $\zeta^\alpha(s)$ which is positive for $s > 1$. That this branch can be expanded, as above, into a Dirichlet series absolutely convergent for $\sigma > 1$ is evident if we consider, for instance, the Euler product expression of $\zeta(s)$. We are interested in the asymptotic behaviour (as $x \to \infty$) of the summatorial function

$$D_{\alpha}(x) = \sum_{n \leq x} d_\alpha(n) \quad (x > 0).$$

For integral values of $\alpha$ our problem reduces to that of Piltz, on which there are numerous investigations. However, it seems to the author that the case of non-integral $\alpha$ has not been treated hitherto. It turns out that in this latter case our problem shows itself much like that of $\pi(x)$, the number of primes $\leq x$. Now the most remarkable result for $\pi(x)$ that can be proved in a comparatively simple way is the prime-number theorem, and it is the analogue to this theorem that we propose to give an outlined proof in this short note. Our result is the following

**Theorem.**

$$D_{\alpha}(x) \sim \frac{1}{\Gamma(\alpha)} x(\log x)^{\alpha - 1} \quad (x \to \infty).$$

This is deduced in a routine way from the following lemma, whose proof is not difficult if we apply the Ikehara-Wiener-Landau method for Tauberian theorems.

**Lemma.** Suppose that $g(s)$ is a function defined and continuous on the half-plane $\sigma = \Re s \geq 1$ and regular for $\sigma > 1$, such that $g(1 + ti)$ is of bounded variation on the interval $-\tau \leq t \leq \tau$ for each $\tau > 0$ ($t = \Im s$). Let $A$ and $\lambda$ be certain constants, $A > 0$, $0 < \lambda < 1$, and let

$$f(s) = g(s) + A(s - 1)^{-\lambda} \quad (\sigma > 1),$$

wherein $(s-1)^{-\lambda}$ represents the principal branch. Further let $F(u)$ be a non-negative function defined and non-decreasing for $u > 0$, such that

$$f(s) = \int_0^\infty F(u)e^{-su}du \quad (\sigma > 1).$$

Then

$$F(u) \sim \frac{A}{\Gamma(\alpha)} e^{u^{\alpha - 1}} \quad (u \to \infty).$$

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