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On the Theory of Quantization for Particle Dynamics in Non-canonical Formalism[1)]

Tsuneto Shimose and Chohko Fujita
(下瀬恒人・藤田長子)
Department of Physics, Faculty of Science,
Ochanomizu University, Tokyo

Résumé

The theory of quantization in non-canonical formalism, recently given by Feynman and Schwinger, is discussed and another new attempt for it is proposed in the case of particle dynamics. The classical relation between Lagrange and Hamilton coordinate variables is the starting point of our discussion. By requiring that the infinitesimal transformation of the above relation is a unitary one, the quantization for particle system is performed. This method of quantization is different from either of the above theories, but closely connected with them.

Introduction

The theory of quantization for mechanical systems of particles or wave field was principally accomplished, in canonical formalism, by Dirac, Heisenberg and others more than twenty years ago. Recently, however, two representative theories have been proposed for quantization in non-canonical formalism: one is Feynman's theory[(1)(2)] fundamentally treating them with the concept of particle's path, the other Schwinger's[(3)(4)] mainly based on a variational principle for Lagrange function. In the course of our investigation, in which mutual relations between Lagrangian and Hamiltonian formalism were treated among classical mechanics and these two theories, we have attempted another new method of quantization in non-canonical formalism. Our theory, which is based on the statement that the transformation of the coordinate of a particle into its parametric expression is a unitary transformation, is closely connected with the above two theories, especially with Schwinger's, but is different from either of them. In this paper we shall discuss the quantization in particle dynamics by our new method, the application of our method to quantization of wave field being postponed to our later paper.

1. Feynman's and Schwinger's quantized theories and our point of view

As the starting point of our theory we shall discuss, in this section, assumptions and procedures of quantization appeared in Feynman's and Schwinger's theory, and make clear our point of view for the quantiza-

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tion, which has a close relation with the above two theories. But as we are interested only to make clear our method, the discussion about quantization is limited to the domain of particle dynamics, in which one particle moves one-dimensionally.

Feynman's theory. In his quantized theory Feynman took up all possible paths of the particle, paths not prescribed by classical equations of motion being contained too. In his theory the following two postulates are assumed for each path.

1. The phase of probability amplitude Φ associated with each path is proportional to the action integral S

$$\Phi = e^{\frac{i}{h}S}, \quad [2) \tag{1}$$

where S is an integral of Lagrange function $L(q, \dot{}, t)$ taken over the arbitrary path from t_1 to t_2

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt . \tag{2}$$

2. The probability amplitude for each path is ruled by the principle of superposition, so that the transition probability P from one point P_1 to another point P_2 is given by

$$P = |\sum_{\text{all possible paths from } P_1 \text{ to } P_2} \Phi|^2 . \tag{3}$$

As a result of his introduction of the idea of particle's path, the relation between quantized theory and classical mechanics was made clearer than before. However, because of difficulties existing in the process to take up the summation for all possible paths appeared in Eq. (3), the whole extent of special problems to be treated by his method was limited to a very much narrower domain. Further, Feynman developed the calculus of ordered operators and expressed concisely the above summation. In the examples of his ordered operators(2) we have had many suggestions for our method of quantization.

In the first place we shall accept this idea of particle's path from Feynman's view, but give up to make each path associate with the phase of an action integral, and derive it from more fundamental principle obtained by modifying Schwinger's theory.

Schwinger's theory In his theory the variational principle for an action integral was applied under the next postulate 1. and the variation was taken up to prescribe the transformation.

1. The variation of an action integral S of the motion for a particle is carried out, considering effects of variations in the boundary domain.

$$\delta S = \delta \int_{t_0}^{t_1} Ldt = \int_{t_0}^{t_1} \delta L dt + |L\delta t|_{t_0}^{t_1}$$

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2) In our paper h means Planck's constant divided by 2π.

$$=\int_{t_0}^{t_1}\left\{\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right\}\delta q dt+|P\Delta q+(L-P\dot{q})\delta t|_{t_0}^{t_1} ,$$

where $\Delta q=\delta q+\dot{q}\delta t$ and $P=\partial L/\partial \dot{q}$,

$$=|P\Delta q+(L-P\dot{q})\delta t|_{t_0}^{t_1}, \tag{4}$$

if the path consists with equation of motion.

2. The variation of the transformation function $\langle q''(t_2)|q'(t_1)\rangle$ and the variation of S are connected as

$$\Delta\langle q''(t_2)|q'(t_1)\rangle=\frac{i}{h}\langle q''(t_2)|\Delta S(t_2, t_1)|q'(t_1)\rangle. \tag{5}$$

The above-mentioned quantization process 1. of Schwinger's, as pointed out by Dyson[5], related closely with the uncertainty of physical quantities in the boundary, but as the uncertainty principle is not fundamental one, his postulate 2. should be replaced by a more fundamental principle (for example by the commutation relation between p and q). So to place the results followed from this uncertainty principle in the position of fundamental assumptions seems not to be appropriate. Moreover, on the occasion of making a variation of the action integral, Schwinger has introduced the substantial variation Δq of the coordinate of a particle in addition to the usual variation of q, and made use of the variation relation

$$\Delta q=\delta q+\dot{q}\delta t , \tag{6}$$

according to the usual view of classical mechanics. This relation is rather to be situated at the starting point of our theory than connected intimately with our point of view about quantization.

As the meaning of this relation he interpreted further Hamiltonian formalism as the following: "the case $\Delta q=0$, i.e., the eigenvalues of the old and new operators are the same, the change in the operator δq being just that to counteract the natural change in the eigenvalue which would result from the system's carrying out its motion." But he did not develop further his interpretation about the relation (6).

Finally, by using the variation in the boundary domain, Schwinger derived the next relation as the infinitesimal unitary transformation of the operator

$$ih\delta q=[q, P\Delta q-(P\dot{q}-L)\delta t] .^{3)} \tag{7}$$

Later it will be seen that this relation corresponds to our result (3.4).

We shall take up the relation (6) as one of the fundamental postulates for quantization and interpret it as the relation to determine the transformation property of Lagrangian formalism to Hamiltonian. In our theory we shall make utmost efforts to the development of this relation, and in the next section make this relation correspond to an

3) By bracket we mean $[A, B]=AB-BA$.

infinitesimal transformation of the coordinate q of a particle into a new coordinate Q, a group of path parameters,

$$\delta q = \delta Q + \dot{q}\delta t \, , \tag{8}$$

where the Q is a function of path parameters $(\alpha, \beta, \cdots)$ or of these parameters and time t, according to our point of view.

This relation (8) is a mere relation of transformation between particle's coordinate and path's parameter, but when we require that "the transformation specified by the relation (8) is an infinitesimal *unitary* transformation," an important change happens for the interpretation of this relation. It will be seen in the next section that at the same time as this requirement is satisfied, the process of quantization is performed and the commutation relation characteristic of quantization (commutation relation between coordinate and momentum) is determined. In our theory as is seen from the above discussion, our whole construction for quantization is made up by assuming that the idea of particle's path is taken up and the transformation of the coordinate from Lagrangian formalism to Hamiltonian is connected with the idea of path, and further this transformation is connected with the concept of unitary transformation.

2. Mutual Relation between Lagrangian and Hamiltonian Formalism in Classical Mechanics.

In order to make clear the distinction between dynamical variables of Lagrangian formalism and those of Hamiltonian in classical mechanics, and to make it a preliminary for the quantization in non-canonical formalism in the next section, first we shall consider one-dimensional motion of one particle by treating its path with the method of space-time descriptions, following Feynman's view for the quantization in non-canonical formalism. For that description of the particle's motion under the equal dynamical condition, which is called, as is well known, Lagrangian formalism, the time-change of the coordinate q for each path may be expressed as below

$$q = q(\alpha, t) \, , \tag{1}$$

where α stands for a parameter distinguishing different classical paths, for example an initial coordinate or initial velocity.

Next we shall consider a transformation of the coordinate into its parameter α

$$(q, t) \rightarrow (\alpha, t) \, . \tag{2}$$

As shown below the basis for all our discussions will be constructed on this transformation. By this transformation a group of path curves on the $(q-t)$—plane is transformed into another group of straight lines parallel to t-axis on the (α, t)-plane, (cf, Fig. 1).

In order to obtain a concrete expression for this transformation to a certain extent, we should make properties of its infinitesimal transformation clear, which is specified by the following expression, by considering the variation of q,

$$\delta q = \frac{\partial q}{\partial \alpha}\delta\alpha + \frac{\partial q}{\partial t}\delta t\,, \tag{3}$$

the meaning of the expression (3) being as follows. On the (q, t)-plane the variation δq denotes the difference of two neighbouring points, $P_1(\alpha, t)$ and $P_2(\alpha+\delta\alpha, t+\delta t)$, which are situated closely on two different paths α and $\alpha+\delta\alpha$ respectively. The first and the second term of Eq. (3) are indicated in Fig. 2. Lately, this analysis will serve to the interpretation of Hamilton variable Q.

In the following we shall show that, when $\partial q/\partial\alpha$ of Eq. (3) does not contain t explicitly, Eq. (3) is equal to Eq. (1.6) of Schwinger's variational case, and when $\partial q/\partial\alpha$ depends on both α and t, Eq. (3) is transformed finally into the form of Eq. (1.7) of Schwinger's.

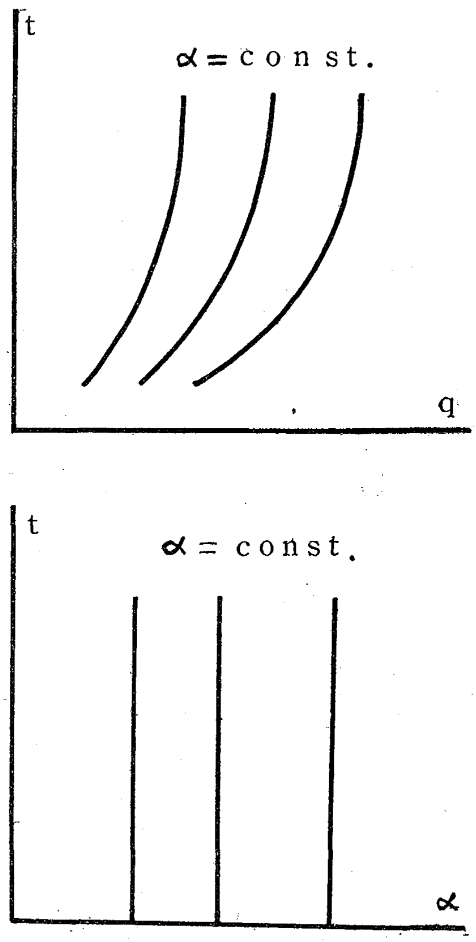

Fig. 1.

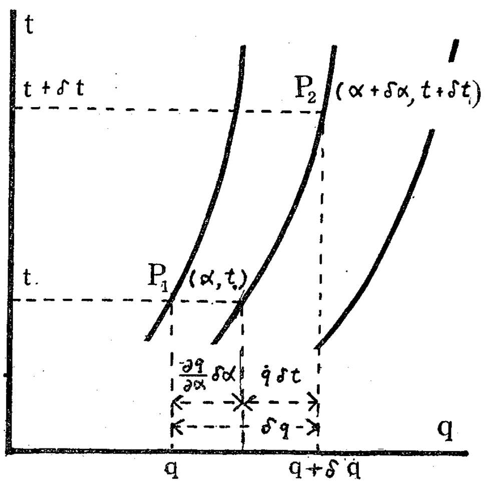

Fig. 2.

Case 1. When $\partial q/\partial\alpha$ is a function of α only and does not contain the time t, we shall introduce a new variable Q by

$$Q = \int^{\alpha} \frac{\partial q}{\partial \alpha} d\alpha\,,$$

then Eq. (3) becomes

$$\delta q = \delta Q + \dot q \delta t, \quad \left(\text{where } \dot q = \frac{\partial q}{\partial t}\right). \tag{4}$$

As compared with Schwinger's variational theory, the relation (1.6) can be made equal to Eq. (4) by replacing as

$$\delta q \to \Delta q \quad \text{and} \quad \delta Q \to \delta q$$

for this equation. In his theory this relation determines the relation between Lagrange and Hamilton variables. So in this paper we shall take up so that this relation has the same meaning as Schwinger's, but expressed by a distinct expression for each variable, Lagrange variable being denoted by small letter q etc., and Hamilton variable by capital letter Q etc.

Case 2. When $\partial q/\partial\alpha$ depends on α and t, we shall introduce a new variable $Q(\alpha, t)$ by taking as

$$\frac{\partial Q}{\partial\alpha}=\frac{\partial q}{\partial\alpha},$$

but $\partial Q/\partial t$ being undetermined. Then the infinitesimal transformation is

$$\delta q(\alpha, t)=\frac{\partial q}{\partial\alpha}\delta\alpha+\dot{q}\delta t,$$

$$=\delta Q+(\dot{q}-\dot{Q})\delta t,$$

or

$$\delta q-\dot{q}\delta t=\delta Q-\dot{Q}\delta t, \tag{5}$$

Eqs. (4) and (5) determining the relation between Lagrange and Hamilton variables in classical mechanics. As we do not so far specify the Hamilton variable P conjugate to Q, these relations do not play an important role in classical mechanics, but by combining the definition of P with unitary transformation the whole situation becomes quite different in quantized theory.

3. Quantized Theory in Non-canonical Formalism

Starting from the classical relation (2.5) between Lagrange variable q and Hamilton variable Q, we shall proceed into the quantized theory of these variables, in which all variables are interpreted as operators. In order to interpret the Q as Hamilton variable we must, further, introduce a Hamilton variable P conjugate to Q. But we do not adopt the usual classical way, in which P is defined as

$$P=\frac{\partial L}{\partial\dot{q}}, \tag{1}$$

at the starting point, but take up another way, the relation (1) being derived as its consequence. In classical mechanics a transformation of Hamilton variables (P_0, Q_0) to another variables (P, Q) is defined as a contact transformation and in quantum mechanics it is specified as unitary transformation between operators. At the same time as defining this unitary transformation of Q we shall be able to introduce correctly the conjugate Hamilton variable P, contrary to the usual way. When a unitary transformation between operators Q_0 and Q is expressed, as is well known, by a transformation function U as

$$Q=UQ_0U^{-1},$$

the infinitesimal unitary transformation of operator Q about unit transformation is expressed as

$$ih\delta Q=[Q, F] \quad \left(\text{where} \quad Q=Q_0-\delta Q \quad \text{and} \quad U=1-\frac{i}{h}F\right), \tag{2}$$

while the relation (2.5) constituting our starting point is expressed as

$$\delta Q=(\delta q-\dot{q}\delta t)+\dot{Q}\delta t \tag{3}$$

In order to proceed into the quantized theory, we must expect that the relation (3) is the same form as the relation (2). So we shall require that *the infinitesimal transformation (3) is an infinitesimal unitary transformation and take the form of (3).* By fulfiling this requirement we shall perform the quantization in Hamiltonian formalism as follows.

By introducing two new operators P and L, which are non-commutative with Q and satisfy the commutation relations

$$[P, Q]=\frac{h}{i}, \quad [L, Q]=\frac{h}{i}\frac{\partial Q}{\partial t}, \tag{4}$$

Eq. (3) may be written in the form of an infinitesimal unitary transformation

$$ih\delta Q=[Q, P\delta q-P\dot{q}\delta t+L\delta t]. \tag{5}$$

Of course we obtain again Eq. (5) by putting as

$$F=P(\delta q-\dot{q}\delta t)+L\delta t \tag{6}$$

in Eq. (2). Then Eqs. (2) and (3) become equivalent. Let us compare Eq. (5) with Eq. (1.7) obtained from Schwinger's variational principle. Then we find that the same relation (5) as Schwinger's is obtained by our method without making use of variational method, except that in our notation the distinction between Lagrange and Hamilton variables are made explicitly.

It is a fundamental assumption for us that L appeared in Eq. (4) must be interpreted as the usual Lagrange function of Lagrange variables $q, \dot{q}$ and time t, which is analogous to the first postulate (1.1) of Feynman's theory. Thus we may assume implicitly that the Q has the same phase as Eq. (1.1) in the non-quantized formalism. To obtain the relation (1) from Eq. (5), we may proceed as below by making use of the method of Feynman's calculus of operators.[2] According to his method Eq. (5) is integrated operationally into the following exponential form

$$Q(\alpha, t)=e^{\frac{i}{h}\int_{\alpha_0}^{\alpha}Pdq-\frac{i}{h}\int_{t_0}^{t}(Pq-L)dt}Q_0(\alpha_0, t_0)e^{-\frac{i}{h}\int_{\alpha_0}^{\alpha}Pdq+\frac{i}{h}\int_{t_0}^{t}(Pq-L)dt} \tag{7}$$

This operational form means that P becomes Eq. (1) by the Feynman's principle of superposition for ordered operator, where L is expressed as a function of q and $\dot{q}$, analogous to the classical principle of stationary phase by Rayleigh. Then Q becomes

$$Q(\alpha, t)=e^{\frac{i}{h}\int_{\alpha_0}^{\alpha}Pdq-\frac{i}{h}\int_{t_0}^{t}Hdt}Q_0(\alpha_0, t_0)e^{-\frac{i}{h}\int_{\alpha_0}^{\alpha}Pdq+\frac{i}{h}\int_{t_0}^{t}Hdt} \tag{8}$$

where H stands for

$$H=\{P\dot{q}-L\}_{\dot{q}=f(P)}\,, \tag{9}$$

if $\dot{q}=f(P)$ is obtained by solving Eq. (1) for $\dot{q}$. Thus the important fact for our theory that the relation (1) should be derived as a result of our assumption, has been verified.

In Schwinger's theory an assumption which makes the system quantized, is prescribed about the connection between the variation of unitary transformation function and that of action integral as Eq. (1.5). As our effort for quantization is concentrated upon the transformation of the coordinate, contrary to Schwinger's view, in our theory, the process of quantization is achieved in Eq. (4), i.e., the sufficient condition to equalize the relation (3) and (5). Even this mere comparison of Schwinger's quantized theory with our's will make it easy to understand the fact that the above-mentioned process is nothing but a quantization process.

Next, we shall discuss some special cases of variations δt, δq and δQ, analogous to Schwinger's treatment.(4)

Case 1. Suppose now $\delta t=0$; this case corresponds to a reshuffling of operators on the surface $t=$constant. In this case we have simply $\delta q=\delta Q$. Then we have again the commutation relation (4) of P and Q from Eq. (5).

Case 2. Let us investigate the case of Heisenberg representation by putting $\delta q=0$. In this case we must change the sign of the left-hand side of Eq. (5), according to the usual meaning of Heisenberg representation. Then we have from Eq. (5)

$$ih\frac{dQ}{dt}=[Q, H]\,, \tag{10}$$

which constitutes the well-known equations of motion in Heisenberg representation. While the equation (10) determines the total variation for the time change of the operator Q, under the condition $\delta q=0$, the latter of Eq. (4) determines the partial derivative of the operator Q.

Case 3. For the expression (5) we shall consider one more special case

$$\delta q=\dot{q}\delta t \quad \text{and} \quad \delta Q=\dot{Q}\delta t\,. \tag{11}$$

For this case we have from Eq. (5)

$$ih\frac{\partial Q}{\partial t}=[Q, L]\,, \tag{12}$$

which is the equation of motion for the Hamilton variable Q in Lagrangian formalism, and expresses again the latter of Eq. (4).

Literature

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