A Theorem of Kakutani on Infinite Product Measures

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In 1948 S. Kakutani [1] has proved the following

Theorem. *The infinite product measures* $m^* = \mathfrak{P}_{n=1}^{\infty} m_n$, *and* $m^{*\prime} = \mathfrak{P}_{n=1}^{\infty} m_n'$, *if each pair* (m_n, m_n') *has absolute continuity one another, are either absolutely continuous or singular one another, according as the infinite product* $\Pi_{n=1}^{\infty} \rho(m_n, m_n')$ *is* >0, *or* $=0$, *where*

$$\rho(m_n, m_n') = \int_{\Omega} \sqrt{m_n(d\omega)\, m_n'(d\omega)}\,. \tag{1}$$ [1)]

In this paper we shall give another proof of the above theorem. This proof is based on the idea of a theorem of Lyapunov [2], [3] which is closely related to our studies in statistics [4], [5], [6]. In the problem of testing simple hypothesis m against m', the power γ of the most powerful test depends only on its size α. This mapping $(\alpha \to \gamma)$ is written as $\gamma(\alpha; m, m')$. (This is named 'separation function' in [4].) We have proved, in [5], that $\lim_{n\to\infty} \gamma(\alpha; m^n, m'^n) = 1$, $(0<\alpha<1)$, where m^n and m'^n are the direct product measures of n measures m and m' respectively (see [4]). This result is a special case of the above theorem, where m and m' are independent of n. The two curves $y = \gamma(x; m, m')$ and $y = 1-\gamma(1-x; m, m')$ form the boundary of the convex and closed set L of the points $(\int \phi(\omega) dm, \int \phi(\omega) dm')$ for all measurable functions $\phi(\omega)$ $(0 \leqq \phi(\omega) \leqq 1)$, and, moreover if m and m' are both non-atomic, L coincides with the set of the points $(m(E), m'(E))$, which has been considered by Lyapunov [2] in a more general way. In the statistical languages, $\phi(\omega)$ is a randomized test, and E is a non-randomized one. In Section 1 of the present paper we shall study properties of the class $\mathfrak{L}$ of all L's. In Section 2, we shall prove Kakutani's theorem stated above by the properties of $\mathfrak{L}$. In the last section, we shall discuss whether the absolute variance of the difference of two measures can be used instead of the function ρ.

1. Definitions and Properties of $\mathfrak{L}$. A measurable space $(\Omega, \mathfrak{B})$ is a set Ω and a σ-algebra[2)] $\mathfrak{B}$ of subsets of Ω, and ω is an element of Ω. If m and m' are two probability measures[2')] both defined on the same

1) This is Hellinger's integral

2) A σ-algebra of sets is defined as a non-empty class of sets closed under the formation of complements and countable unions (see [7])

2') For simplicity, we shall omit the word "probability" from "probability measure" in the rest of this paper space $(\Omega, \mathfrak{B})$, then the plane set L of all the points

$$\left(\int \phi(\omega)\,dm, \quad \int \phi(\omega)\,dm'\right)^{3)}$$

is named an $\mathfrak{L}$-*set* of a pair (m, m'), where $\phi(\omega)$ is a $\mathfrak{B}$-measurable function $(0 \leqq \phi(\omega) \leqq 1)$.

Theorem 1. *The* $\mathfrak{L}$-*set of any pair of measures is* i) *convex,* ii) *closed,* iii) *contained in the square* $O=[(x, y)\,|\,0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1]$, iv) *symmetric to the center* $(\frac{1}{2}, \frac{1}{2})$, *and* v) *contains the segment* $I=[(x, x)\,|\,0 \leqq x \leqq 1]$.

Proof. We shall only show ii), since the others are clear from the properties of $\phi(\omega)$,

By Radon-Nikodym's theorem, there exist a $\mathfrak{B}$-measurable function $f(\omega)$ and a $\mathfrak{B}$-measurable set S of m-measure zero, such that

$$m'(E)=\int_E f(\omega)dm+m'(E \frown S) \tag{2}$$

holds for every $\mathfrak{B}$-measurable set E. By using $f(\omega)$ and S, we can define a $\mathfrak{B}$-measurable function $\phi_{k,c}(\omega)$ for real $k\,(0 \leqq k \leqq \infty)$ and c $(0 \leqq c \leqq 1)$ as follows: for $k<\infty$,

$$\phi_{k,c}(\omega)=\begin{cases} 0, \text{ if } f(\omega)>k \text{ or if } \omega \in S, \\ 1, \text{ if } f(\omega)<k \text{ and if } \omega \notin S, \\ c, \text{ if } f(\omega)=k \text{ and if } \omega \notin S, \end{cases}$$

and for $k=\infty$,

$$\phi_{\infty,c}(\omega)=\begin{cases} c, \text{ if } \omega \in S, \\ 1, \text{ if } \omega \notin S. \end{cases}$$

Since, for any $\mathfrak{B}$-measurable function $\phi(\omega)$ $(0 \leqq \phi(\omega) \leqq 1)$, we have

$$\int\phi(\omega)dm'-k\int\phi(\omega)dm \geqq \int\phi_{k,c}(\omega)dm'-k\int\phi_{k,c}(\omega)dm,^{4)}$$

we can easily show that the boundary of the $\mathfrak{L}$-set of the pair (m, m') is the set of all points

$$\left(\int\phi_{k,c}(\omega)dm, \quad \int\phi_{k,c}(\omega)dm'\right) \text{ and } \left(1-\int\phi_{k,c}(\omega)dm, \quad 1-\int\phi_{k,c}(\omega)dm'\right)$$

for all k and c. Thus we see that $\mathfrak{L}$-set is closed.

Before we discuss the relation between a pair of measures and its $\mathfrak{L}$-set, we shall introduce some concepts related to the $\mathfrak{L}$-set.

The class of all sets L satisfying the conditions i) - v) of Theorem 1 will be denoted by $\mathfrak{L}$. The set $B_k(L)=[(x, y)\,|\,y-kx=\inf_{(\xi,\eta)\in L}(\eta-k\xi)]$ $\frown L$ is a point or a segment, where the line $y-kx=\inf_{(\xi,\eta)\in L}(\eta-k\xi)^{5)}$

3) Integral sign omitting any limits is understood as that over the whole space, on which the measure is defined

4) This inequality is essentially that of Neyman and Pearson in the theory of statistics [8]

5) When $k=\infty$, it means that $x=sup_{(\xi,\eta)\in L}\xi=1$

supports $L\,(0 \leqq k \leqq \infty)$. Especially, $B_0(L)$ and $B_\infty(L)$ are the parts of the lines $y=0$ and $x=1$ respectively, which are contained in L.

Defining

$$F_L(\log k) = \sup_{(x,y)\in B_k(L)} x,$$

and

$$F'_L(\log k) = \sup_{(x,y)\in B_k(L)} y,$$

we can easily see that

$$F'_L(t) = \int_{-\infty}^{t} e^t dF_L(t), \quad -\infty < t < \infty. \tag{3}$$

If L is an $\mathfrak{L}$-set of (m, m'), and if m' is represented as (2), then hold

$$F_L(t) = m([\omega \mid f(\omega) \leqq e^t]) \quad \text{and} \quad F'_L(t) = m'([\omega \mid f(\omega) \leqq e^t] - S).$$

These functions $F_L(t)$ and $F'_L(t)$ are called $\mathfrak{l}$-*functions* of L. Consider the set $X = [-\infty, \infty]$[6] and the smallest σ-algebra $\sum$ containing all intervals[6] in X. On the measurable space $(X, \sum)$, we can define measures m_L and m'_L from $F_L\,(t)$ and $F'_L\,(t)$ such as

$$m_L([-\infty, t]) = F_L(t), \quad m'_L([-\infty, t]) = F'_L(t), \quad \text{for } t < \infty,$$

and $\quad m_L([-\infty, \infty]) = m'_L([-\infty, \infty]) = 1.$

Thus we have obtained

Theorem 2. *For any element L of $\mathfrak{L}$, there exists a pair of measures on a mesurable space, whose $\mathfrak{L}$-set coincides with L.*

Lemma 1. *For any pair of measures m and m' defining L of $\mathfrak{L}$, the length of a segment $B_k(L)$ equals to either $\sqrt{m([\omega \mid f(\omega)=k])^2 + m'([\omega \mid f(\omega)=k]-S)^2}$ or $m'(S)$ accordingly as $k<\infty$ or $=\infty$.*

From this Lemma 1 follows

Theorem 3. *Let L be the $\mathfrak{L}$-set of a pair of measures m and m'. m and m' are* i) *identical,* ii) *singular one another, and* iii) *absolutely continuous one another, if and only if* i) $L=I$, ii) $L=O$, *and* iii) *each of $B_0(L)$ and $B_\infty(L)$ consists of a single point, respectively.*

By the product $\mathfrak{L}$-set $L_1 \cdot L_2$ of two $\mathfrak{L}$-sets L_1 and L_2, we mean the $\mathfrak{L}$-set of a pair $(m_1 \times m_2,\ m'_1 \times m'_2)$ defined on the direct product measurable space $(\Omega_1 \times \Omega_2,\ \mathfrak{B}_1 \times \mathfrak{B}_2)$, if L_1 and L_2 are respectively the $\mathfrak{L}$-sets of m_1 and m'_1 on $(\Omega_1, \mathfrak{B}_1)$ and of m_2 and m'_2 on $(\Omega_2, \mathfrak{B}_2)$.[7]

Theorem 4. *The product $\mathfrak{L}$-set $L_1 \cdot L_2$ of $\mathfrak{L}$-sets L_1 and L_2 is independent of the measurable spaces and the measures which define L_1 and L_2.*

Proof. Suppose that L_i is a $\mathfrak{L}$-set of (m_i, m'_i), and that we have a Radon-Nikodym's representation

6) By interval $[-\infty, t]$ we mean the set of all real numbers $\leqq t$ and a point $-\infty$

7) By the direct product of measurable spaces $(\Omega_1, \mathfrak{B}_1)$ and $(\Omega_2, \mathfrak{B}_2)$, we mean the set of all pairs (ω_1, ω_2), $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$, and the smallest σ-algebra containing all sets $E_1 \times E_2$, $E_1 \in \mathfrak{B}_1$, $E_2 \in \mathfrak{B}_2$, and by $m_1 \times m_2$ we mean the extension of $m_1 \times m_2(E_1 \times E_2) = m_1(E_1) \cdot m_2(E_2)$ onto $\mathfrak{B}_1 \times \mathfrak{B}_2$

$$m_i'(E)=\int_E f_i dm_i + m_i'(E \frown S_i) \quad \text{for all } E \in \mathfrak{B}_i,$$

where f_i is a $\mathfrak{B}_i$-measurable function and S_i is a $\mathfrak{B}_i$-measurable set of m_i-measure zero ($i=1, 2$), then

$$m_1' \times m_2'(E)=\int_E f_1 f_2 d(m_1 \times m_2) + m_1' \times m_2'\{(\Omega_1 \times S_2 \smile S_1 \times \Omega_2) \frown E\}$$

holds for all $E \in \mathfrak{B}_1 \times \mathfrak{B}_2$. Therefore the $\mathfrak{l}$-functions $F_{L_1 \cdot L_2}$ and $F'_{L_1 \cdot L_2}$ are as follows:

$$F_{L_1 \cdot L_2}(t)=m_1 \times m_2([(\omega_1, \omega_2) \mid f_1(\omega_1) f_2(\omega_2) \leqq e^t]),$$
$$F'_{L_1 \cdot L_2}(t)=m_1' \times m_2'([(\omega_1, \omega_2) \mid f_1(\omega_1) f_2(\omega_2) \leqq e^t]-S_1 \times \Omega_2 - \Omega_1 \times S_2),$$

which can be also written in the following forms:

$$(4) \qquad \begin{aligned} F_{L_1 \cdot L_2}(t) &= \int_{-\infty}^{\infty} F_{L_1}(t-s) dF_{L_2}(s), \\ F'_{L_1 \cdot L_2}(t) &= \int_{-\infty}^{\infty} F'_{L_1}(t-s) dF'_{L_2}(s). \end{aligned}$$

These equations show that the product $L_1 \cdot L_2$ by means of $(m_1 \times m_2, m_1' \times m_2')$ coincides with that by means of $(m_{L_1} \times m_{L_2}, m'_{L_1} \times m'_{L_2})$, that is to say, $L_1 \cdot L_2$ is independent of its defining measures.

In the sequels we shall write $L_1 \geqq L_2$, when L_1 is a subset of L_2, and a sequence $L_1, L_2, \cdots$ of $\mathfrak{L}$-sets are called *monotone*, when $L_1 \leqq L_2 \leqq \cdots$ or $L_1 \geqq L_2 \geqq \cdots$ holds. If $L_1 \leqq L_2 \leqq \cdots$, then we shall denote $\bigcap_{n=1}^{\infty} L_n$[8] by $\lim_n L_n$, and on the other hand, if $L_1 \geqq L_2 \geqq \cdots$, then we shall denote $\overline{(\bigcup_{n=1}^{\infty} L_n)}$[9] by $\lim_n L_n$.[10] It is evident that $\lim_n L_n$ of a monotone sequence of $\mathfrak{L}$-sets is also an $\mathfrak{L}$-set,

Theorem 5. *If $m^*=\mathfrak{P}_{n=1}^{\infty} m_n$ and $m^{*\prime}=\mathfrak{P}_{n=1}^{\infty} m_n'$ are infinite product measures,*[11] *and if L^* and L_n are $\mathfrak{L}$-sets of $(m^*, m^{*\prime})$ and (m_n, m_n') respectively, then*

$$\prod_{n=1}^{\infty} L_n = \lim_n L_1 \cdot L_2 \cdot \ldots \cdot L_n = L^*.$$

Proof. It is sufficient to prove that for every E in $\mathfrak{B}^*=\mathfrak{P}_{n=1}^{\infty}\mathfrak{B}_n$ and $\varepsilon>0$ there exist an integer N and an $E^N \in \mathfrak{P}_{n=1}^{N}\mathfrak{B}_n$ such that

$$(5) \qquad |m^*(E)-(\mathfrak{P}_{n=1}^{N} m_n)(E^N)|<\varepsilon,$$

and

$$(5') \qquad |m^{*\prime}(E)-(\mathfrak{P}_{n=1}^{N} m_n')(E^N)|<\varepsilon.$$

From the definition of the infinite product measure, we can choose, for any $\varepsilon>0$, two sequences $\{E_i\}$ and $\{E_j'\}$ of disjoint cylinder sets such that

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8), 9) $\bigcup$ denotes the union, $\bigcap$ the meet, and bar $\overline{}$ the closure operation in the sense of the plane topology

10) This definition of the limit coincides with that given in [9], as a special case

$$\bigcup_{i=1}^{\infty} E_i \supset E, \qquad \bigcup_{j=1}^{\infty} E'_j \supset E,$$

$$\sum_{i=1}^{\infty} \overline{m}(E_i) < m^*(E) + \varepsilon,$$

and

$$\sum_{j=1}^{\infty} \overline{m}'(E'_j) < m^{*\prime}(E) + \varepsilon,$$

where $\overline{m}$ and $\overline{m}'$ are defined in the footnote 11).

Since $E_i \cap E'_j$ are at most enumerable, we shall denote them by E''_k $(k=1, 2, \ldots)$, and see that $\bigcup_{k=1}^{\infty} E''_k \supset E$, $\sum_{k=1}^{\infty} \overline{m}(E''_k) \leqq \sum_{i=1}^{\infty} \overline{m}(E_i)$, and $\sum_{k=1}^{\infty} \overline{m}'(E''_k) \leqq \sum_{j=1}^{\infty} \overline{m}'(E'_j)$. Hence there exists an integer $N_1 > 0$ such that

$$\left| \sum_{k=1}^{N_1} \overline{m}(E''_k) - m^*(E) \right| < \varepsilon,$$

and

$$\left| \sum_{k=1}^{N_1} \overline{m}'(E''_k) - m^{*\prime}(E) \right| < \varepsilon$$

hold. Since E''_k's are cylinder sets, and are disjoint each other, there exist an integer $N > 0$ and $E^N \in \mathfrak{P}_{n=1}^{N} \mathfrak{B}_n$, such that

$$\bigcup_{k=1}^{N_1} E''_k = E^N \times (\mathfrak{P}_{n=N+1}^{\infty} \Omega_n),$$

and hence we have

$$\sum_{k=1}^{N_1} \overline{m}(E''_k) = (\mathfrak{P}_{n=1}^{N} m_n)(E^N)$$

and

$$\sum_{k=1}^{N_1} \overline{m}'(E''_k) = (\mathfrak{P}_{n=1}^{N} m'_n)(E^N),$$

which imply (5) and (5′).

As a preparatory of Theorem 6, we need the following lemmas.

Lemma 2. *If $\mathfrak{B}_1$ is a σ-subalgebra of a σ-algebra $\mathfrak{B}$ of sets in Ω, and if m_1 and m'_1 are measures defined on the measurable space $(\Omega, \mathfrak{B}_1)$ as follows:*

$$m_1(E) = m(E) \quad \textit{and} \quad m'_1(E) = m'(E) \quad \textit{for all} \quad E \in \mathfrak{B}_1,$$

where m and m' are measures defined on $(\Omega, \mathfrak{B})$, then the $\mathfrak{L}$-set L_1 of (m_1, m'_1) is contained in the $\mathfrak{L}$-set L of (m, m'), i.e.

$$L_1 \geqq L.$$

11) The infinite product measure is defined as follows: $\Omega^* = \mathfrak{P}_{n=1}^{\infty} \Omega_n$ is the set of all sequences $(\omega_1, \omega_2, \ldots)$, $\omega_n \in \Omega_n$, $\mathfrak{B}^* = \mathfrak{P}_{n=1}^{\infty} \mathfrak{B}_n$ is the smallest σ-algebra containing all cylinder sets $C = E_1 \times E_2 \times \ldots \times E_n \times \Omega_{n+1} \times \Omega_{n+2} \times \ldots$, and $m^* = \mathfrak{P}_{n=1}^{\infty} m_n$ on $(\Omega^*, \mathfrak{B}^*)$ is a measure

$$m^*(E) = \inf_{\cup E_n \supset E} \Sigma_{n=1}^{\infty} \overline{m}(E_n),$$

where $\overline{m}(E_n)$ is the finitely additive measure such that $\overline{m}(C) = m_1(E_1) \ldots m_n(E_n)$

Proof. This is trivial, since $\mathfrak{B}_1$-measurable function $\phi_1(\omega)(0\leqq\phi_1(\omega)\leqq 1)$ is also $\mathfrak{B}$-measurable, and its integral with respect to m_1 (or m_1') equals to that with respect to m (or m').

Lemma 3. *Let* $L_0=\lim_n L_n$. *If* $B_k(L_0)$ *consists of a single point* (x_0, y_0), *then any sequence of points* (x_n, y_n) *has* (x_0, y_0) *as its limit point, i.e.*

$$\lim_{n\to\infty} x_n=x_0, \quad \textit{and} \quad \lim_{n\to\infty} y_n=y_0\,,$$

where $(x_n, y_n)\in B_k(L_n)$ *for every* n.

The proof of this geometrical lemma will be omitted. Applying this lemma to $\mathfrak{l}$-functions, we can see

Lemma 4. $L_0=\lim_n L_n$, *if and only if*

$$\lim_{n\to\infty} F_{L_n}(t)=F_{L_0}(t)$$

and

$$\lim_{n\to\infty} F'_{L_n}(t)=F'_{L_0}(t)$$

hold for every continuity point of F_{L_0}.

The following theorems in this section are used as the starting-point of discussions in the next section.

Theorem 6. *The class* $\mathfrak{L}$ *has the following properties:*

I. a) *For any pair of elements* L_1 *and* L_2 *of* $\mathfrak{L}$, *there exists one and only one product* $L_1\cdot L_2$ *in* $\mathfrak{L}$.

b) *If* L_1, L_2 *and* $L_3\in\mathfrak{L}$, *then* $(L_1\cdot L_2)\cdot L_3=L_1\cdot(L_2\cdot L_3)$.

c) $L_1\cdot L_2=L_2\cdot L_1$ *for any* L_1 *and* L_2 *in* $\mathfrak{L}$.

d) *There exists one and only one element, denoted by* I, *such that*

$$I\cdot L=L \quad \textit{for every} \quad L\in\mathfrak{L}.$$

e) *There exists one and only one element, denoted by* O *such that*

$$O\cdot L=O \quad \textit{for every} \quad L\in\mathfrak{L}.$$

f) *If* $L_1\cdot L_2=O$, *then* $L_1=O$ *or* $L_2=O$.

II. *There exists a relation (denoted by* $\leqq$*) for some pair* L_1 *and* L_2 *in* $\mathfrak{L}$ *which satisfiies*

a) *If* $L_1\leqq L_2$ *and* $L_2\leqq L_1$, *then* $L_1=L_2$.

b) *If* $L_1\leqq L_2$ *and* $L_2\leqq L_3$, *then* $L_1\leqq L_3$.

c) $L\leqq L$.

d) *If* $L_1\leqq L_2$, *then* $L_1\cdot L\leqq L_2\cdot L$ *for all* $L\in\mathfrak{L}$.

e) $I\geqq L\geqq O$ *for every* L *in* $\mathfrak{L}$.

f) *For* $L_1\neq O$ *and* $L_2\neq I$, $L_1>L_1\cdot L_2$.[12)]

III. *For any monotone sequence of elements* $L_1, L_2, \ldots$ *in* $\mathfrak{L}$, *there exists an element, denoted by* $\lim_n L_n$, *such that*

a) $\lim_n L=L$.

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12) If $L\leqq L'$ holds, but not $L=L'$, then we write $L<L'$

b) $\lim_n (L_0 \cdot L_n) = L_0 \cdot (\lim_n L_n)$.

c) $\lim_i L_{n_i} = \lim_n L_n$, *if* $L_{n_1}, L_{n_2}, \ldots$ *is a subsequence of* $L_1, L_2, \ldots$.

d) *If* $\{L_n\}$ *and* $\{L'_n\}$ *are monotone sequences, and if there exists an integer* $N>0$, *for which* $L_n \geqq L'_n$ *holds when* $n \geqq N$, *then*

$$\lim_n L_n \geqq \lim_n L'_n .$$

Proof. I. a) is obvious by Theorem 4. For b), it is sufficient to see that the $\mathfrak{l}$-functions of $L_1 \cdot L_2 \cdot L_3$ are

$$F_{L_1 \cdot L_2 \cdot L_3}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{L_1}(t-r-s) dF_{L_2}(r) dF_{L_3}(s) ,$$

and

$$F'_{L_1 \cdot L_2 \cdot L_3}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F'_{L_1}(t-r-s) dF'_{L_2}(r) dF'_{L_3}(s).$$

But these are obvious from (4). c), d), e) and f) can be easily seen by Theorem 3 and (4).

II. a), b) and c) are clear. d) can be easily shown by the representation (4). And the definition of I and O gives e).

From Lemma 2 we have $L_1 \geqq L_1 \cdot L_2$ for any $\mathfrak{L}$-sets L_1 and L_2. In fact, if L_1 and L_2 are $\mathfrak{L}$-sets of pairs of measures defined on $(\Omega, \mathfrak{B})$ and $(\Omega', \mathfrak{B}')$ respectively, then $L_1 \cdot L_2$ is the $\mathfrak{L}$-sets of the pair of measures on $(\Omega \times \Omega', \mathfrak{B} \times \mathfrak{B}')$, and L_1 is regarded as the $\mathfrak{L}$-sets of the pair of measures on $(\Omega \times \Omega', \mathfrak{B}'')$, where $\mathfrak{B}'' = [E \times \Omega' \mid E \in \mathfrak{B}] \subset \mathfrak{B} \times \mathfrak{B}'$.

Moreover we shall show that if $L_1 = L_1 \cdot L_2$, then $L_1 = O$ or $L_2 = I$. From (3) and (4), we have

$$F_{L_1}(t) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{t-s} dF_{L_1}(r) \right\} dF_{L_2}(s),$$

and

$$\int_{-\infty}^{t} e^t dF_{L_1}(t) = \int_{-\infty}^{\infty} e^s \left\{ \int_{-\infty}^{t-s} e^r dF_{L_1}(r) \right\} dF_{L_1}(s),$$

that is,

$$(6) \qquad \int_0^{\infty} \left\{ \int_{t-s}^{t} dF_{L_1}(r) \right\} dF_{L_2}(s) = \int_{-\infty}^{0-} \left\{ \int_{t}^{t-s} dF_{L_1}(r) \right\} dF_{L_2}(s),$$

and

$$(7) \qquad \int_0^{\infty} \left\{ \int_{t-s}^{t} e^{r+s} dF_{L_1}(r) \right\} dF_{L_2}(s) = \int_{-\infty}^{0-} \left\{ \int_{t}^{t-s} e^{r+s} dF_{L_1}(r) \right\} dF_{L_2}(s).$$

Remembering the definition of F_L, we can easily see that equations (6) and (7) hold at the same time if and only if F_{L_2} has a jump one at $t=0$ or F_{L_1} is constantly one, i.e. $L_2 = I$ or $L_1 = O$.

III. It is sufficient to prove b). Denote $\lim_n L_n$ by L^*, F_{L_n} by F_n, $(n=0, 1, \ldots)$, and F_{L*} by F_*. From the representation (4) of the product we can see that

$$F_{L_0\cdot L^*}(t)=\int_{-\infty}^{\infty}F_0(t-s)dF_*(s),$$

and

$$F_{L_0\cdot L_n}(t)=\int_{-\infty}^{\infty}F_0(t-s)dF_n(s),$$

and that, by Lemma 4, for every continuity point t of F_*

$$\lim_{n\to\infty}F_n(t)=F_*(t).$$

Therefore, since $F_0(t)$ is a bounded monotone function of t, $\lim_{n\to\infty} F_{L_0\cdot L_n}(t)=\lim_{n\to\infty}\int F_0(t-s)dF_n(s)=\int F_0(t-s)dF_*(s)=F_{L_0\cdot L^*}(t)$, and hence $\lim_{n\to\infty}F'_{L_0\cdot L_n}(t)=F'_{L_0\cdot L^*}(t)$ hold for every continuity point t of $F_{L_0\cdot L^*}(t)$. Thus the proof of our theorem is accomplished.

Suppose that $\mathfrak{L}_s$, $\mathfrak{L}'_s$ and $\mathfrak{L}_a$ are the following subsets of $\mathfrak{L}$: the element of $\mathfrak{L}_s$ is a parallelogram with its vertices $(0,0)$, $(a,0)$, $(1,1)$ and $(1-a,1)$, the element of $\mathfrak{L}'_s$ is that with its vertices $(0,0)$, $(1,a)$, $(1,1)$ and $(0,1-a)$ for $0\leqq a\leqq 1$, and the element L_a of $\mathfrak{L}_a$ is a set whose $B_0(L_a)$ and $B_\infty(L_a)$ are either of length one or zero.

Theorem 6′. IV. *There exist three subsets $\mathfrak{L}_s$, $\mathfrak{L}'_s$, and $\mathfrak{L}_a$ of $\mathfrak{L}$ which satisfies the following conditions:*

a) *Each pair of $\mathfrak{L}_s$, $\mathfrak{L}'_s$ and $\mathfrak{L}_a$ have only two common elements I and O.*

b) *If L, $L'\in\mathfrak{L}_a$, then $L\cdot L'\in\mathfrak{L}_a$. If we take $\mathfrak{L}_s$ or $\mathfrak{L}'_s$ instead of $\mathfrak{L}_a$, the similar propositions hold.*

c) *For any element $L\in\mathfrak{L}$, there exist three elements $L_s(\in\mathfrak{L}_s)$, $L'_s(\in\mathfrak{L}'_s)$ and $L_a(\in\mathfrak{L}_a)$ such that*

$$L=L_s\cdot L'_s\cdot L_a,$$

and such a decomposition is unique.

Proof. Clear.

Now we shall proceed the discussion of the relation of a function $\rho(m,m')$ and the $\mathfrak{L}$-set L of (m,m'). Suppose that m and m' are measures having the relation (2). For these measures we define

$$\rho(m,m')=\int\sqrt{f}\,dm.$$

If two pair (m_1,m_1') and (m,m') of measures define the same $\mathfrak{L}$-set L, then

$$\rho(m_1,m_1')=\rho(m,m')$$

holds, since $\rho(m,m')$ is written as

$$\rho(m,m')=\int e^{t/2}dF_L(t)$$

by the use of the $\mathfrak{f}$-function F_L. Consequently we can regard $\rho(m,m')$ as a function of L only, and denote it by $\rho(L)$.

Theorem 6″. V. *The function $\rho(L)$ on $\mathfrak{L}$ satisfies the following conditions:*

a) $0 \leqq \rho(L) \leqq 1$ *for all* $L \in \mathfrak{L}$,

b) $\rho(L)=1$ *if and only if* $L=I$, *and* $\rho(L)=0$ *if and only if* $L=O$.

c) $\rho(L_1 \cdot L_2)=\rho(L_1)\rho(L_2)$.

d) $\rho(\lim_n L_n)=\lim_{n\to\infty}\rho(L_n)$ *for every monotone sequence* $L_1, L_2, \ldots$.

e) *If* $L'>L$, *then* $\rho(L')>\rho(L)$.

Proof. a), b) and c) are clear. Let $L^*=\lim_n L_n$. By Lemma 4 $\lim_{n\to\infty}F_{L_n}(t)=F_{L^*}(t)$ holds for every continuity point. Hence $\lim_{n\to\infty}\rho(L_n)=\lim_{n\to\infty}\int e^{t/2}dF_{L_n}(t)=\int e^{t/2}dF_{L^*}(t)=\rho(L^*)$. At last we shall prove e). Suppose that the line $y-kx+c=0\,(c>0, >k-1)$ intersects with the boundary of L at two points (x_1, y_1) and (x_2, y_2). We define the *cut-off* $\mathfrak{L}$*-set* of L by the line $y-kx+c=0$ as the common part L' of L and the strip $[(x, y)\,|\,-c \leqq y-kx \leqq 1-c-k]$. Let t_1 and t_2 be real numbers such that $F_L(t_1-) \leqq x_1 \leqq F_L(t_1)$ and $F_L(t_2-) \leqq x_2 \leqq F_L(t_2)$ hold for the $\mathfrak{f}$-function F_L of L. Since the $\mathfrak{f}$-function of L' is

$$F_{L'}(t)\begin{cases} =x_1, & \text{if } t_1 \leqq t < \log\dfrac{y_2-y_1}{x_2-x_1} \\ =x_2, & \text{if } \log\dfrac{y_2-y_1}{x_2-x_1} \leqq t < t_2, \\ =F_L(t), & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}\rho(L')&=\int_{-\infty}^{\infty} e^{t/2}dF_{L'}(t)\\ &=\left(\int_{-\infty}^{t_1-}+\int_{t_2+}^{\infty}\right)e^{t/2}dF_L(t)+e^{t_1/2}(x_1-F_L(t_1))+e^{t_2/2}(F_L(t_2)-x_2)+\sqrt{(y_2-y_1)(x_2-x_1)}\\ &>\left(\int_{-\infty}^{t_1-}+\int_{t_1}^{t_2}+\int_{t_2+}^{\infty}\right)e^{t/2}dF_L=\rho(L),\end{aligned}$$

that is to say, if L' is a cut-off $\mathfrak{L}$-set of L, then $\rho(L')>\rho(L)$. Generally, since for any $L'>L$ there exists a monotone descending sequence of L_n such that

$$L \leqq L_n \leqq L',$$
$$\lim_n L_n=L',$$

and L_{n+1} is a cut-off $\mathfrak{L}$-set of L_n, it follows from d) that $\rho(L')>\rho(L)$ holds.

2. The Abstract discussion of the class $\mathfrak{L}$. In this section we shall state three theorems related to the class $\mathfrak{L}$, the last two of which correspond to Kakutani's theorem.

Theorem 7. *Suppose that a set* $\mathfrak{L}$ *satisfies* I–III *of Theorem 6, and that* $L_1, L_2, \ldots$ *are the sequence of element of* $\mathfrak{L}$, *any one of which does not coincide with the element* O. *By writting*

$$M_n=\prod_{i=1}^{n} L_i=L_1\cdot L_2\cdot\ldots\cdot L_n,$$

and

$$N_n = \lim_m \prod_{i=n+1}^{m} L_i = \prod_{i=n+1}^{\infty} L_i ,$$

we have

$$\lim_n M_n \neq O$$

if and only if

$$\lim_n N_n = I. \tag{8}$$

Proof. Since $\{M_n\}_{n=1,2\ldots}$ and $\{N_n\}_{n=1,2\ldots}$ are monotone sequences in $\mathfrak{L}$, $\lim_n M_n$ and $\lim_n N_n$ exist. From the fact that, for $n<m$,

$$\begin{aligned} M_n \cdot N_m &= L_1 \cdot L_2 \cdot \ldots \cdot L_n \cdot (\lim_l \prod_{i=m+1}^{l} L_i) \\ &= \lim_l L_1 \cdot L_2 \cdot \ldots \cdot L_n \cdot L_{m+1} \cdot \ldots \cdot L_l && \text{(b) of III)} \\ &\geqq \lim_l M_l , && \text{(f) of II and d) of III)} \end{aligned}$$

it holds that

$$\begin{aligned} M_n \cdot \lim_m N_m &= \lim_m M_n \cdot N_m && \text{(b) of III)} \\ &\geqq \lim_l M_l . && \text{(d) of III)} \end{aligned}$$

Therefore we have

$$(\lim_n N_n) \cdot (\lim_n M_n) \geqq \lim_n M_n .$$

On the other hand, the reciprocal inequality holds from c), d) of I and e), f) of II. Hence

$$(\lim_n N_n) \cdot (\lim_n M_n) = \lim_n M_n$$

holds. From f) of II, we have

$$\lim_n N_n = I \text{ or } \lim_n M_n = O.$$

If $\lim_n M_n = O$, then for any n we have

$$\begin{aligned} M_n \cdot N_n = M_n \cdot (\lim_m \prod_{i=n+1}^{m} L_i) &= \lim_m M_n \cdot (\prod_{i=n+1}^{m} L_i) && \text{(b) of III)} \\ &= \lim_m M_m = O. \end{aligned}$$

However by the assumption of our theorem, there is no n such that $M_n = L_1 \cdot L_2 \cdot \cdots \cdot L_n = O$. Therefore from f) of I we have

$$\lim_m \prod_{i=n+1}^{m} L_i = N_n = O \quad \text{for every } n,$$

that is to say,

$$\lim_n N_n \neq I.$$

Thus we see that $\lim_n N_n = I$ and $\lim_n M_n = O$ are not compatible, which accomplishes the proof of our theorem.

Theorem 8. *Suppose that a set* $\mathfrak{L}$ *has properties* IV *of Theorem 6′ in addition to* I–III *of Theorem 6. If* $L_1, L_2, \dots \in \mathfrak{L}_a$, *then* $\prod_{i=1}^{\infty} L_i \in \mathfrak{L}_a$. (*On taking* $\mathfrak{L}_s$ *or* $\mathfrak{L}'_s$ *instead of* $\mathfrak{L}_a$, *this theorem also holds.*)

Proof. We shall prove only the case of $\mathfrak{L}_a$. We can assume $\Pi_{n=1}^{\infty} L_n \neq O$ without any loss of generality. In this case we have, by Theorem 7,

$$\lim_n \prod_{i=n+1}^{\infty} L_i = I.$$

From c) of IV, $\prod_{i=n+1}^{\infty} L_i$ can be decomposable into $L_n^s (\in \mathfrak{L}_s)$, $L_n^{s'} (\in \mathfrak{L}'_s)$ and $L_n^a (\in \mathfrak{L}_a)$, which shows, by the fact that $L_n \in \mathfrak{L}_a$,

$$L_{n-1}^a = L_n \cdot L_n^a, \quad L_{n-1}^s = L_n^s \text{ and } L_{n-1}^{s'} = L_n^{s'}.$$

Therefore we have $L_n^s = L_n^{s'} = I$ for every n, that is to say, $\Pi_{i=n+1}^{\infty} L_i \in \mathfrak{L}_a$ holds for every n, especially for $n=0$.

Corollary. *If* $L_n = L_n^s \cdot L_n^{s'} \cdot L_n^a$, *where* $L_n^s \in \mathfrak{L}_s$, $L_n^{s'} \in \mathfrak{L}'_s$ *and* $L_n^a \in \mathfrak{L}_a$, *then*

$$\prod_{n=1}^{\infty} L_n = \Big(\prod_{n=1}^{\infty} L_n^s\Big) \cdot \Big(\prod_{n=1}^{\infty} L_n^{s'}\Big) \cdot \Big(\prod_{n=1}^{\infty} L_n^a\Big).$$

Theorem 9. *Suppose that* $\mathfrak{L}$ *satisfies* I–V. $\Pi_{n=1}^{\infty} L_n = O$ *if and only if* $\Pi_{n=1}^{\infty} \rho(L_n) = 0$.

Proof. From c) and d) of V, we have

$$\rho\Big(\prod_{n=1}^{\infty} L_n\Big) = \rho\Big(\lim_n \prod_{i=1}^{n} L_i\Big) = \lim_{n\to\infty} \rho\Big(\prod_{i=1}^{n} L_i\Big) = \lim_{n\to\infty} \prod_{i=1}^{n} \rho(L_i) = \prod_{n=1}^{\infty} \rho(L_n).$$

This equality implies our theorem.

3. The relation between $\rho(\mathbf{m}, \mathbf{m}')$ **and the absolute variance.** Let L be the $\mathfrak{L}$-set of a pair of measures m and m', where m and m' have the relation (2). And denote the absolute variation of $m - m'$ by $d(m, m')$, that is,

$$d(m, m') = \int |1 - f| \, dm + m'(S).$$

This function $d(m, m')$ of two measures m and m' may be considered as a metric in a set of measures.

Lemma 5. *$d(m, m')$ depends only on* L *and equals to*

$$2 \max_{(x,y) \in L} |x - y|.$$

Proof. Clear.

From this lemma, we may write $d(m, m') = d(L)$. We consider now some examples of $\rho(L)$ and $d(L)$.

Example 1. $\rho(I) = 1$, $d(I) = 0$.

Example 2. $\rho(O) = 0$, $d(O) = 2$.

Example 3. Denote a hexagon with vertices (0, 0), $(u, 0)$, $(1, 1-u)$, (1, 1), $(1-u, 1)$ and $(0, u)$ by $L_h(u)$ $(0 \leqq u \leqq 1)$. Then we have

$$\rho(L_h(u)) = 1-u, \quad \mathrm{d}(L_h(u)) = 2u$$

Example 4. Denote a parallelogram with vertices (0, 0) $(u+v, v)$, (1, 1) and $(1-u-v, 1-v)$ by $L_p(u, v)$ $(0 \leqq u \leqq 1,\ 0 \leqq v \leqq 1-u)$. Then we have

$$\rho(L_p(u, v)) = \sqrt{v(v+u)} + \sqrt{(1-v)(1-u-v)},$$
$$d(L_p(u, v)) = 2u.$$

For $v = \dfrac{1-u}{2}$, we have

$$\rho\left(L_p\left(u, \frac{1-u}{2}\right)\right) = \sqrt{1-u}, \quad d\left(L_p\left(u, \frac{1-u}{2}\right)\right) = 2u$$

Theorem 10. *For any pair of measures m and m′, the inequalities*

$$(9) \qquad 1 - \frac{d(m, m')}{2} \leqq \rho(m, m') \leqq \sqrt{1 - \left\{\frac{d(m, m')}{2}\right\}^2}$$

hold.

Proof. Denote

$$\mathfrak{L}(u) = [L \mid d(L) = 2u].$$

Since, for any L in $\mathfrak{L}(u)$, there exists an $L_p(u, v)$ such that $L \leqq L_p(u, v)$ and since $\sqrt{v(u+v)} + \sqrt{(1-v)(1-u-v)} \leqq \sqrt{1-u}$, $1-u = \rho\left(L_p\left(u, \frac{1-u}{2}\right)\right) \leqq \rho(L)$ holds for every L in $\mathfrak{L}(u)$. On the other hand, since $L_h(u)$ contains every L in $\mathfrak{L}(u)$, $1-u = \rho(L_h(u)) \leqq \rho(L)$ holds for every L in $\mathfrak{L}(u)$.

Theorem 11. 1) *A necessary condition of* $\Pi_{n=1}^{\infty} \rho(m_n, m_n') > 0$ *is that* $\sum_{n=1}^{\infty} \{d(m_n, m_n')\}^2$ *converges.* 2) *A sufficient condition of* $\Pi_{n=1}^{\infty} \rho(m_n, m_n') > 0$ *is that* $\sum_{n=1}^{\infty} d(m_n, m_n')$ *converges.*

Proof. 2) is obvious from the inequality (9). 1) is easily seen from the following inequalities:

$$\begin{aligned} \frac{1}{4}\{d(m_n, m_n')\}^2 &\leqq 1 - \{\rho(m_n, m_n')\}^2 \\ &\leqq (1-\rho(m_n, m_n'))(1+\rho(m_n, m_n')) \\ &\leqq 2(1-\rho(m_n, m_n')). \end{aligned}$$

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