A New Formulation of Mean Value Theorem

Shunzi Kametani (巖谷俊司)

Department of mathematics, Faculty of Science, Ochanomizu University.

It is well known that in the elementary treatise of differential and integral calculus, the Mean Value Theorem plays the most fundamental part. Indeed, every global property of a function with respect to its derivative is based upon this theorem, the proof for which is usually reduced to the global property of a continuous function defined on a compact set, though the theorem seems to represent the property of a function defined on a connected set rather than a compact set.

The purpose of the present note is to give a new formulation of the mean value theorem whose proof depends directly on the connectivity of a linear interval, which enables us to generalize the theorem to the case of interval functions defined in spaces of more than one dimension.

1. Preliminaries. Though throughout this note we shall use the terminology in 2-spaces, $\mathbb{R}^2$, the results which will be obtained hold true in spaces of any finite number of dimensions. Given an interval function $\varphi$ defined on a closed or an open domain $J(\subset \mathbb{R}^2)$, we shall say $\varphi$ is additive, if, for any closed interval $I \subset J$ and its decomposition into two intervals $I_1$ and $I_2$ not containing common inner points,

$$\varphi(I) = \varphi(I_1) + \varphi(I_2).$$

We shall also say that $\varphi$ is continuous in $J$ if $\varphi(I) \to 0$ as the area of $I(\subset J)$, $m(I)$, tends to zero. A sequence of intervals $\{I_n\}$ will be termed regular, according to the standard of S. Saks's 'The Theory of Integral (1917)' if the ratio of the length of non parallel two edges of each interval $I_n$ remains between two fixed positive numbers independent of $n$. We shall say that a sequence $\{I_n\}$ of intervals tends to $x$ if $I_n \ni x (n = 1, 2, \ldots)$ and the diameter of $I_n$ tends to zero. Then, the upper derivate, $\bar{D}\varphi(x)$, of $\varphi(I)$ at $x$ is the upper bound of the numbers $l$ such that there exists a regular sequence of intervals $\{I_n\}$ tending to $x$, for which $\lim_{n \to \infty} (\varphi(I_n)/m(I_n)) = l$. The lower derivate of $\varphi$ at $x$, $\underline{D}\varphi(x)$, is similarly defined. If the upper and the lower derivates of $\varphi$ at $x$ are equal, their common value is called the derivative of $\varphi$ at $x$ and is denoted by $\varphi'(x)$.

In one-dimensional case, it is well known that $\varphi'(x)$ is identical with the limit of $\varphi(I)/m(I)$ when the length of $I$, having $x$ as one of
its extremities, converges to zero.

2. Statement of theorems. The theorem we are now going to establish is as follows:

**Theorem 1.** Let \( \psi \) be an additive, continuous function of an interval on a closed interval \( I_0 \) satisfying the condition \( \psi(I_0) = 0 \), then, there exists a point \( x_0 \) and a descending sequence of intervals \( \{I_j\} \) with the following properties 1°–4°:

1° \( I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq x_0 \),
2° \( x_0 \) is the inner point of each \( I_j \),
3° every \( I_j \) may be chosen so as to be similar to \( I_0 \).
4° \( \psi(I_j) = 0 \) \((j = 1, 2, \ldots)\).

Once established this theorem, the proof for which will be given in the following section, we may state as its corollary the following more general theorem:

**Theorem 2.** Let \( \varphi \) and \( \sigma \) be additive, continuous functions of an interval on a closed interval \( I_0 \), then there exist a point \( x_0 \) and a descending sequence of intervals \( \{I_j\} \) satisfying the conditions 1°–4° of the preceding theorem and the condition

5° \( \varphi(I_j)\sigma(I_j) = \varphi(I_0)\sigma(I_0) \) \((j = 1, 2, \ldots)\).

The proof of this theorem is almost immediate, since putting

\[
\psi(I) = \varphi(I)\sigma(I_0) - \varphi(I_0)\sigma(I),
\]

we find \( \psi(I) \) is additive and continuous and satisfies the condition \( \psi(I_0) = 0 \).

In theorem 1, if we replace \( \sigma(I) \) by \( m(I) \), we find at once the following theorem:

**Theorem 3.** Let \( \varphi \) be an additive, continuous function of an interval on a closed interval \( I_0 \), then there exists a point \( x_0 \) and a descending sequence of intervals \( \{I_j\} \) satisfying the conditions 1°–3° and

6° \( \varphi(I_j) / m(I_j) = \varphi(I_0) / m(I_0) \) \((j = 1, 2, \ldots)\).

Let us remark here that this theorem contains the usual mean value theorem or more generally the following:

**Theorem 4.** If \( \varphi \) is an additive, continuous function of an interval on a closed interval \( I_0 \), then there exists a point \( x_0 \) in the interior of \( I_0 \) such that

\[
D_{\varphi}(x_0) \leq \frac{\varphi(I)}{m(I)} \leq \bar{D}_{\varphi}(x_0).
\]

The proof is also almost immediate, since by the above theorem we observe that \( \{I_j\} \) is a regular sequence tending to \( x_0 \) for which

\[
D_{\varphi}(x_0) \leq \lim_{I \to I_0} \varphi(I) / m(I) = \varphi(I_0) / m(I_0)
\]

and similarly \( \bar{D}_{\varphi}(x_0) \geq \varphi(I_0) / m(I_0) \).
If we assume the existence of the derivative of $\varphi$ at all inner points of $I_0$, then by $D\varphi(x_0) = D\varphi(x) = \varphi'(x_0)$ we find

$$\varphi(I_0) = m(I_0)\varphi'(x_0),$$

which is the mean value theorem in 2-space.

3. Proof of Theorem 1. Denoting by $[a, b]$ a closed interval in $R^1$ defined by an inequality $a \leq x \leq b$, we may write

$$[a, b] \times [c, d]$$

for the closed interval in $R^2$ defined by inequalities $a \leq x \leq b$ and $c \leq y \leq d$.

Let $I_0 = [a, b] \times [c, d]$ and $I(\alpha, \beta) = [\alpha, \beta] \times [c, d]$. Then the function

$$g(\alpha, \beta) = \varphi(I(\alpha, \beta)),$$

considered as the function of an interval $[\alpha, \beta] \subset [a, b]$ in $R^1$, is evidently additive and continuous, so that $g(\alpha, \beta)$ tends to 0 as $\alpha \to \beta$ or $\beta \to \alpha$.

Writing

$$f(\xi) = g(\xi, \xi + h)$$

where $h = (b - a)/3$ and $\xi \in [a, a + 2h]$, we observe first by the additivity of $g$ that

$$f(a) + f(a + h) + f(a + 2h) = g(a, a + h) + g(a + h, a + 2h) + g(a + 2h, a + 3h) = g(a, b) = \varphi(I_0) = 0.$$

We also see that by the additivity and continuity of $g$ that $f$ is a continuous function of $\xi$, since for $\xi = \xi_1, \xi_2 (\xi_2 > \xi_1)$

$$f(\xi_2) - f(\xi_1) = g(\xi_2, \xi_1 + h) + g(\xi_1 + h, \xi_2 + h) - g(\xi_1, \xi_2 + h) - g(\xi_1 + h, \xi_2 + h) - g(\xi_1 + h, \xi_2 + h) + g(\xi_1, \xi_2 + h) \to 0$$

as $\xi_1 \to \xi_2$ or $\xi_2 \to \xi_1$.

Let us distinguish here two cases:

1) $f(a) = f(a + h) = f(a + 2h) = 0$

and

2) among the three values $f(a)$, $f(a + h)$ and $f(a + 2h)$ there is at least one $\neq 0$.

If 1) occurs, then let us put

$$a_1 = a + h, \quad b_1 = a_1 + h = a + 2h.$$

If 2) occurs, then by (2), we can find two values, $f(\xi_1)$ and $f(\xi_2)$, among those three, with different signs. Since $f(\xi)$ is continuous on $[\xi_1, \xi_2]$ or $[\xi_2, \xi_1]$, there exists a point $\xi = a_i$ such that

$$f(a_i) = 0 \text{ and } \xi_1 < a_1 < \xi_2 \text{ or } \xi_2 < a_1 < \xi_1.$$
In any case, we have by (3) or (4)

\[ 0 = f(a_i) = \psi([a_i, b_i] \times [c, d_i]) \]

where \( a < a_i < a_i + h = b_i < b \) and \( b_i - a_i = (b-a)/3 \).

Replace now \( I_0 \) by \([a_i, b_i] \times [c, d_i]\), divide \([c, d]\), instead of \([a_i, b_i]\), into three equal subintervals, apply the above process, and we shall find \([c_i, d_i]\) such that

\[ \psi([a_i, b_i] \times [c_i, d_i]) = 0 \]

where \( c < c_i < d_i < d \) and \( d_i - c_i = (d-c)/3 \). Denoting \([a_i, b_i] \times [c_i, d_i]\) by \( I_i \), we find that \( I_i \) is contained in the interior of \( I_0 \) and also that \( I_i \) is similar to \( I_0 \). If we repeat indefinitely this process of finding out \( I_i \) from \( I_0 \), we shall obtain a sequence of intervals \( \{I_i\} \) satisfying all the conditions required in our theorem, since the diameter of \( I_i \) tends to zero. Thus the proof is completed.

4. Some applications. Among many theorems derived easily from our generalization (1) of the mean value theorem, let us make a special mention about three theorems of which the following first one is the immediate consequence of (1):

Theorem 5. Suppose \( \varphi \) be an additive, continuous function of an interval defined in an open set \( G \). If \( \varphi \) has a finite derivative at all points of \( G \) and the derived function \( \varphi'(x) \) is integrable over an interval \( I_0 \subseteq G \) in the sense of Riemann, then

\[ \varphi(I_0) = \int_{I_0} \varphi'(x) \, dm(x). \]

Theorem 6. Suppose \( \varphi \) be an additive, continuous function of an interval defined in a domain where a finite derivative exists at every point. Then the derived function \( \varphi'(x) \) takes every value \( \lambda \) between any two different values of \( \varphi' \).

Proof. We have only to show that if \( \varphi'(x_1) > 0 \) and \( \varphi'(x_2) < 0 \), then there exists a point \( x_0 \in G \) for which \( \varphi'(x_0) = 0 \) since, otherwise, we may replace \( \varphi(I) \) by \( \varphi(I) - \lambda m(I) \). Let \( Q_\delta(x) \) be a square, of centre \( x \), whose edges are of length \( \delta \) and parallel to the coordinate axis. Take, in \( G \), a continuous curve \( C \) which joins \( x_1 \) and \( x_2 \). The distance between \( C \) and the boundary of \( G \) is evidently positive, and consequently holds

\[ Q_\delta(x) \subseteq G \]

for all \( x \in C \) and for a sufficiently small \( \delta > 0 \). Since \( \varphi'(x_1) > 0 \) and \( \varphi'(x_2) < 0 \), we may obviously assume that \( \varphi(Q_\delta(x_2)) > 0 \) and \( \varphi(Q_\delta(x_1)) < 0 \). But, as is easily seen, \( \varphi(Q_\delta(x)) \) is a continuous function of \( x \) defined along \( C \), whence there is a point \( \xi \in C \) for which \( \varphi(Q_\delta(\xi)) = 0 \). Then by our generalization (1), we can find a point \( x_0 \) in the interior of \( Q_\delta(\xi) \subseteq G \) such that \( \varphi'(x_0) = 0 \) which is the required result.
Theorem 7. Let \( \varphi \) be an additive, continuous function of an interval in a neighborhood \( V \) of \( x_0 \). If \( \varphi \) has a finite derivative at every point of \( V \) except perhaps at \( x_0 \), then it has a derivative also at \( x_0 \), provided that 
\[
\lim_{x \to x_0} \varphi'(x)
\]
exists.

Proof. Let \( I \ni x_0 \) be an interval in \( V \). If \( x_0 \) does not coincide with a vertex of \( I \), we can subdivide \( I \) into two intervals \( I_1 \) and \( I_2 \) by a straight line, passing through \( x_0 \), parallel to one of the axis. Since
\[
\frac{\varphi(I)}{m(I)} = \frac{(\varphi(I_1) + \varphi(I_2))}{m(I_1) + m(I_2)},
\]
we have
\[
\max_{j=1,2} \frac{\varphi(I_j)}{m(I_j)} \geq \frac{\varphi(I)}{m(I)} \geq \min_{j=1,2} \frac{\varphi(I_j)}{m(I_j)}.
\]
But as \( \varphi \) has a derivative at every inner point of each \( I_j \), there exists a point \( x_j \) in the interior of \( I_j \) such that
\[
\varphi'(x_j) = \frac{\varphi(I_j)}{m(I_j)} (j = 1, 2).
\]
Thus
\[
\min_{j=1,2} \varphi'(x_j) \leq \frac{\varphi(I)}{m(I)} \leq \max_{j=1,2} \varphi'(x_j).
\]
If, on the other hand, \( x_0 \) coincides with a vertex of \( I \), then there exists an inner point \( \xi \) of \( I \) for which
\[
\varphi'(\xi) = \frac{\varphi(I)}{m(I)}.
\]
From the two considerations made above and the assumption that 
\[
\lim_{x \to x_0} \varphi'(x)
\]
should exist, we can conclude that if the diameter of \( I \) tends to zero, then
\[
\max_{j} \varphi'(x_j), \min_{j} \varphi'(x_j) \text{ and } \varphi'(\xi)
\]
tend to the same limit
\[
\lim_{x \to x_0} \varphi'(x), \lim_{x \to x_0} \varphi(I) = \lim_{x \to x_0} \varphi'(x).
\]
This completes the proof.

Remark. As the above proof shows, \( \lim_{x \to x_0} \varphi'(x) = \varphi'(x_0) \) is the derivative even in the strong sense. (See S. Saks: The theory of the integral 1937 p. 106).