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A New Formulation of Mean Value Theorem[1]

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It is well known that in the elementary treatise of differential and integral calculus, the Mean Value Theorem plays the most fundamental part. Indeed, every global property of a function with respect to its derivative is based upon this theorem, the proof for which is usually reduced to the global property of a continuous function defined on a compact set, though the theorem seems to represent the property of a function defined on a connected set rather than a compact set.

The purpose of the present note is to give a new formulation of the mean value theorem whose proof depends directly on the connectivity of a linear interval, which enables us to generalize the theorem to the case of interval functions defined in spaces of more than one dimension.

1. **Preliminaries.** Though throughout this note we shall use the terminology in 2-spaces, R^2, the results which will be obtained hold true in spaces of any finite number of dimensions. Given an interval function φ defined on a closed or an open domain $J(\subset R^2)$, we shall say φ is additive, if, for any closed interval $I \subset J$ and its decomposition into two intervals I_1 and I_2 not containing common inner points,

$$\varphi(I) = \varphi(I_1) + \varphi(I_2).$$

We shall also say that φ is continuous in J if $\varphi(I) \rightarrow 0$ as the area of $I(\subset J)$, $m(I)$, tends to zero. A sequence of intervals $\{I_n\}$ will be termed regular, according to the standard of S. Saks's 'The Theory of Integral (1917)' if the ratio of the length of non parallel two edges of each interval I_n remains between two fixed positive numbers independent of n. We shall say that a sequence $\{I_n\}$ of intervals tends to x if $I_n \ni x (n = 1, 2, \ldots)$ and the diameter of I_n tends to zero. Then, the upper derivate, $\overline{D}\varphi(x)$, of $\varphi(I)$ at x is the upper bound of the numbers l such that there exists a regular sequence of intervals $\{I_n\}$ tending to x, for which $\lim_{n\to\infty} (\varphi(I_n) / m(I_n)) = l$. The lower derivate of φ at x, $\underline{D}\varphi(x)$, is similarly defined. If the upper and the lower derivates of φ at x are equal, their common value is called the derivative of φ at x and is denoted by $\varphi'(x)$.

In one-dimensional case, it is well known that $\varphi'(x)$ is identical with the limit of $\varphi(I) / m(I)$ when the length of I, having x as one of

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its extremities, converges to zero.

2. **Statement of theorems.** The theorem we are now going to establish is as follows:

Theorem 1. *Let ψ be an additive, continuous function of an interval on a closed interval I_0 satisfying the condition $\psi(I_0) = 0$, then, there exists a point x_0 and a descending sequence of intervals $\{I_j\}$ with the following properties* 1°–4°:

1° $I_0 \supset I_1 \supset I_2 \supset \ldots \ni x_0$,

2° x_0 *is the inner point of each* I_j,

3° *every I_j may be chosen so as to be similar to* I_0.

4° $\psi(I_j) = 0 \quad (j = 1, 2, \ldots)$.

Once established this theorem, the proof for which will be given in the following section, we may state as its corollary the following more general theorem:

Theorem 2. *Let φ and σ be additive, continuous functions of an interval on a closed interval I_0, then there exist a point x_0 and a descending sequence of intervals $\{I_j\}$ satisfying the conditions 1°–4° of the preceding theorem and the condition*

5° $\varphi(I_j)\sigma(I_0) = \varphi(I_0)\sigma(I_j) \quad (j = 1, 2, \ldots)$.

The proof of this theorem is almost immediate, since putting

$$\psi(I) = \varphi(I)\sigma(I_0) - \varphi(I_0)\sigma(I),$$

we find $\psi(I)$ is additive and continuous and satisfies the condition $\psi(I_0) = 0$.

In theorem 1, if we replace $\sigma(I)$ by $m(I)$, we find at once the following theorem:

Theorem 3. *Let φ be an additive, continuous function of an interval on a closed interval I_0, then there exists a point x_0 and a descending sequence of intervals $\{I_j\}$ satisfying the conditions 1°–3° and*

6° $\varphi(I_j) / m(I_j) = \varphi(I_0) / m(I_0) \quad (j = 1, 2, \ldots)$.

Let us remark here that this theorem contains the usual mean value theorem or more generally the following:

Theorom 4. *If φ is an additive, continuous function of an interval on a closed interval I_0, then there exists a point x_0 in the interior of I_0 such that*

$$\underline{D}\varphi(x_0) \leqq \frac{\varphi(I_0)}{m(I_0)} \leqq \overline{D}\varphi(x_0).$$

The proof is also almost immediate, since by the above theorem we observe that $\{I_j\}$ is a regular sequence tending to x_0 for which

$$\underline{D}\varphi(x_0) \leqq \lim \varphi(I_j) / m(I_j) = \varphi(I_0) / m(I_0)$$

and similarly $\overline{D}\varphi(x_0) \geqq \varphi(I_0) / m(I_0)$

If we assume the existence of the derivative of φ at all inner points of I_0, then by $\bar{D}\varphi(x_0) = \underline{D}\varphi(x_0) = \varphi'(x_0)$ we find

$$\varphi(I_0) = m(I_0)\varphi'(x_0), \tag{1}$$

which is the mean value theorem in 2-space.

3. **Proof of Theorem 1.** Denoting by $[a, b]$ a closed interval in R^1 defined by an inequality $a \leqq x \leqq b$, we may write

$$[a, b] \times [c, d]$$

for the closed interval in R^2 defined by inequalities $a \leqq x \leqq b$ and $c \leqq y \leqq d$.

Let $I_0 = [a, b] \times [c, d]$ and $I(\alpha, \beta) = [\alpha, \beta] \times [c, d]$. Then the function

$$g(\alpha, \beta) = \psi(I(\alpha, \beta)),$$

considered as the function of an interval $[\alpha, \beta] \subset [a, b]$ in R^1, is evidently additive and continuous, so that $g(\alpha, \beta)$ tends to 0 as $\alpha \to \beta$ or $\beta \to \alpha$.

Writing

$$f(\xi) = g(\xi, \xi + h)$$

where $h = (b-a)/3$ and $\xi \in [a, a+2h]$, we observe first by the additivity of g that

$$\begin{aligned} &f(a)+f(a+h)+f(a+2h) \\ &= g(a, a+h)+g(a+h, a+2h)+g(a+2h, a+3h) \\ &= g(a, b) = \psi(I_0) = 0. \end{aligned} \tag{2}$$

We also see that by the additivity and continuity of g that f is a continuous function of ξ, since for $\xi = \xi_1, \xi_2$ $(\xi_2 > \xi_1)$

$$\begin{aligned} f(\xi_2)-f(\xi_1) &= g(\xi_2, \xi_1+h)+g(\xi_1+h, \xi_2+h) \\ &\quad -g(\xi_1, \xi_2)-g(\xi_2, \xi_1+h) \\ &= g(\xi_1+h, \xi_2+h)-g(\xi_1, \xi_2) \to 0 \end{aligned}$$

as $\xi_1 \to \xi_2$ or $\xi_2 \to \xi_1$.

Let us distinguish here two cases:

1) $f(a) = f(a+h) = f(a+2h) = 0$

and

2) among the three values $f(a)$, $f(a+h)$ and $f(a+2h)$ there is at least one $\neq 0$.

If 1) occurs, then let us put

$$a_1 = a+h, \quad b_1 = a_1+h = a+2h. \tag{3}$$

If 2) occurs, then by (2), we can find two values, $f(\xi_1)$ and $f(\xi_2)$, among those three, with different signs. Since $f(\xi)$ is continuous on $[\xi_1, \xi_2]$ or $[\xi_2, \xi_1]$, there exists a point $\xi = a_1$ such that

$$f(a_1) = 0 \quad \text{and} \quad \xi_1 < a_1 < \xi_2 \quad \text{or} \quad \xi_2 < a_1 < \xi_1. \tag{4}$$

In any case, we have by (3) or (4)

$$0 = f(a_1) = \psi([a_1, b_1] \times [c, d])$$

where $a < a_1 < a_1 + h = b_1 < b$ and $b_1 - a_1 = (b-a)/3$.

Replace now I_0 by $[a_1, b_1] \times [c, d]$, divide $[c, d]$, instead of $[a_1, b_1]$, into three equal subintervals, apply the above process, and we shall find $[c_1, d_1]$ such that

$$\psi([a_1, b_1] \times [c_1, d_1]) = 0$$

where $c < c_1 < d_1 < d$ and $d_1 - c_1 = (d-c)/3$. Denoting $[a_1, b_1] \times [c_1, d_1]$ by I_1, we find that I_1 is contained in the interior of I_0 and also that I_1 is similar to I_0. If we repeat indefinitely this process of finding out I_1 from I_0, we shall obtain a sequence of intervals $\{I_j\}$ satisfying all the conditions required in our theorem, since the diameter of I_j tends to zero. Thus the proof is completed.

4. Some applications. Among many theorems derived easily from our generalization (1) of the mean value theorem, let us make a special mention about three theorems of which the following first one is the immediate consequence of (1):

Theorem 5. *Suppose φ be an additive, continuous function of an interval defined in an open set G. If φ has a finite derivative at all points of G and the derived function $\varphi'(x)$ is integrable over an interval $I_0 \subset G$ in the sense of Riemann, then*

$$\varphi(I_0) = \int_{I_0} \varphi'(x)\, dm(x).$$

Theorem 6. *Suppose φ be an additive, continuous function of an interval defined in a domain where a finite derivative exists at every point. Then the derived function $\varphi'(x)$ takes every value λ between any two different values of φ'.*

Proof. We have only to show that if $\varphi'(x_1) > 0$ and $\varphi'(x_2) < 0$, then there exists a point $x_0 \in G$ for which $\varphi'(x_0) = 0$ since, otherwise, we may replace $\varphi(I)$ by $\varphi(I) - \lambda m(I)$. Let $Q_\delta(x)$ be a square, of centre x, whose edges are of length δ and parallel to the coordinate axis. Take, in G, a continuous curve C which joins x_1 and x_2. The distance between C and the boundary of G is evidently positive, and consequently holds

$$Q_\delta(x) \subset G$$

for all $x \in C$ and for a sufficiently small $\delta > 0$. Since $\varphi'(x_1) > 0$ and $\varphi'(x_2) < 0$, we may obviously assume that $\varphi(Q_\delta(x_1)) > 0$ and $\varphi(Q_\delta(x_2)) < 0$. But, as is easily seen, $\varphi(Q_\delta(x))$ is a continuous function of x defined along C, whence there is a point $\xi \in C$ for which $\varphi(Q_\delta(\xi)) = 0$. Then by our generalization (1), we can find a point x_0 in the interior of $Q_\delta(\xi) \subset G$ such that $\varphi'(x_0) = 0$ which is the required result.

Theorem 7. *Let φ be an additive, continuous function of an interval in a neighborhood V of x_0. If φ has a finite derivative at every point of V except perhaps at x_0, then it has a derivative also at x_0, provided that $\lim_{x \to x_0} \varphi'(x)$ exists.*

Proof. Let $I \ni x_0$ be an interval in V. If x_0 does not coincide with a vertex of I, we can subdivide I into two intervals I_1 and I_2 by a straight line, passing through x_0, parallel to one of the axis. Since $\varphi(I) / m(I) = (\varphi(I_1) + \varphi(I_2)) / (m(I_1) + m(I_2))$, we have $\max_{j=1,2} \varphi(I_j) / m(I_j) \geqq \varphi(I) / m(I) \geqq \min_{j=1,2} \varphi(I_j) / m(I_j)$. But as φ has a derivative at every inner point of each I_j, there exists a point x_j in the interior of I_j such that $\varphi'(x_j) = \varphi(I_j) / m(I_j)$ $(j = 1, 2)$. Thus $\min_{j=1,2} \varphi'(x_j) \leqq \varphi(I) / m(I) \leqq \max_{j=1,2} \varphi'(x_j)$. If, on the other hand, x_0 coincides with a vertex of I, then there exists an inner point ξ of I for which $\varphi(I) / m(I) = \varphi'(\xi)$. From the two considerations made above and the assumption that $\lim_{x \to x_0} \varphi'(x)$ should exist, we can conclude that if the diameter of I tends to zero, then $\max_j \varphi'(x_j)$, $\min_j \varphi'(x_j)$ and $\varphi'(\xi)$ tend to the same limit $\lim_{x \to x_0} \varphi'(x)$, which shows $\lim_{x \to x_0} \varphi(I) / m(I) = \lim \varphi'(x)$. This completes the proof.

Remark. As the above proof shows, $\lim_{x \to x_0} \varphi'(x) = \varphi'(x_0)$ is the derivative even in the strong sense. (See S. Saks: The theory of the integral 1937 p. 106).