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Remarks on Closed Mapping and Compactness[1]

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Here we shall point out that closed mappings, to which less attention has been paid than to open ones, may be used in proving compactness of certain spaces, and we shall prove several propositions about the relation of closed mappings of compact spaces to open or continuous mappings. We are chiefly concerned with mappings of completely regular spaces; but we begin with the general case of T_{-1} spaces, i.e. spaces in which the closure operation $X \to \bar{X}$ satisfies the conditions

$$\bar{O} = O, \quad X < \bar{X} = \bar{\bar{X}} \quad \text{and} \quad \bar{X} < \bar{Y} \quad \text{when} \quad X < Y,$$

where O denotes the empty set. A mapping f of a T_{-1} space S into a T_{-1} space E *induces a decomposition* of S into mutually disjoint subsets $F = f^{-1}(p)$ $(p \in E)$. Given a mapping f, we shall denote by E^* the class of all non-empty sets $F = f^{-1}(p)$ $(p \in f(S))$.

First we shall prove the following fact:

If a closed mapping f of a T_{-1} space S into a compact T_{-1} space E induces a decomposition of S into compact sets, then S is compact.

It is sufficient to show that, if a family of non-empty closed subsets of S contains, with any two sets, also the intersection of the two, then there is a point in common to all the sets of the family. Let us denote by X the generic element of the family. Then the images $f(X)$ constitute a family of non-empty closed subsets of the compact space E and the family enjoys the same property as the original one. So the images $f(X)$ have a point in common. Let p be the common point and put $F = f^{-1}(p)$. Then, as before, the sets $F \frown X$ have a point in common; this proves our statement.

Slight generalization is possible:

If there is a family of closed mappings f_α of a fixed T_{-1} space S into compact spaces E_α such that, for any selection of one point p_α from each E_α, the intersection of the inverse images $f_\alpha^{-1}(p_\alpha)$ is compact, then S is compact.

In fact, we have only to apply the previous proposition to the product space E of the spaces E_α.

From now on, we shall consider only the case when S is completely regular. A uniform structure can be defined for S by means of "en-

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tourages" (in A. Weil's sense) V_α, V_β, V_γ,... of the diagonal $\{(p,p) \mid p \in S\}$ in the product space $S \times S$; we denote by V the system of entourages, and we put, for any $A \subset S$ and $V_\alpha \in V$,

$$V_\alpha(A) = \{p \mid (p,q) \in V_\alpha \text{ for some } q \in A\}.$$

With respect to a uniform structure, we call *V-continuous* a decomposition of S induced by a mapping f of S into a T_{-1} space E, when (and only when), for any $F_0 \in E^*$ and $V_\alpha \in V$, there exists a $V_\beta \in V$ such that

$$(※) \qquad F \frown V_\beta(F_0) \neq 0 \quad \text{implies} \quad F \subset V_\alpha(F_0) \quad (F \in E^*).$$

For example, the natural mapping of a topological group G onto the space of left cosets of a given closed subgroup induces a V-continuous decomposition with respect to an obvious uniform structure of G; in this example, the mapping is open.

Now we are going to prove the following proposition:

If an open mapping f of a completely regular space S onto a compact T_{-1} space E induces a V-continuous decomposition of S into compact sets, then f is a closed mapping (and consequently S is compact).

Let C be a closed subset of S and let p be a point of E not belonging to $f(C)$. Then the compact set $F_0 = f^{-1}(p)$ has no point in common with the closed set C; hence $V_\alpha(F_0) \frown C = 0$ for some $V_\alpha \in V$. Take a $V_\beta \in V$ so as to satisfy the condition (※). Then the image $f(V_\beta(F_0))$ contains no points of $f(C)$. As f is an open mapping, $f(V_\beta(F_0))$ contains a neighbourhood of $f(F_0) = f(p)$; so the point $f(p)$ does not belong to $\overline{f(C)}$. Thus the image $f(C)$ of a closed set C is closed, q.e.d..

Further we can show that f is continuous. This can be stated more symmetrically as follows:

Let S be compact and completely regular. Then a closed mapping of S into a compact T_{-1} space E, which induces a V-continuous decomposition of S into compact sets, is a continuous mapping, and the image $E_1 = f(S)$ is a Hausdorff space with respect to the topology induced from E. Conversely, a continuous mapping f of S into a Hausdorff space E induces a V-continuous decomposition of S (into compact sets, and such an f is a closed mapping).

The first part may be shown as follows:

A topology can be defined in E^* by the system of neighbonrhoods of $F_0 \in E^*$:

$$U_\alpha(F_0) = \{F \mid F \in E^*,\ F \subset V_\alpha(F_0)\} \quad (V_\alpha \in V),$$

or equivalently by the neighbourhoods of the form

$$\tilde{U}_\alpha(F_0) = \{F \mid F \in E^*,\ F \frown V_\alpha(F_0) \neq 0\},$$

equivalence being obvious from V-continuity. Let $g(p)$ $(p \in S)$ be the set $F \in E^*$ containing p, i.e. $g(p) = f^{-1}(f(p))$. In view of the latter form of neighbourhoods it is obvious that g maps S continuously onto E^*. From the compactness of the sets $F \in E^*$ follows that, for any two $F_1 \neq F_2$ in E^*, there exists a V_α satisfying $V_\alpha(F_1) \frown V_\alpha(F_2) = 0$ and so $U_\alpha(F_1) \frown U_\alpha(F_2) = 0$. Hence E^* is a Hausdorff space. If we put $h(F) = f(g^{-1}(F))$, we obtain a closed and one-to-one mapping h of E^* onto the compact space E_1. We conclude easily that h is a homeomorphism and consequently, E_1 is a Hausdorff space; moreover f is continuous, since $f(p) = h(g(p))$.

The first part is thus proved; let us consider the second part. Suppose that $F_0 = f^{-1}(p) \in E$ and $V_\alpha \in V$ are given. For each $V_\beta \in V$ let us consider the sum of all $F \in E^*$ satisfying $F \frown V_\beta(F_0) \neq 0$, and let G_β be the set of all points contained in the sum but not in $V_\alpha(F_0)$. Let x be a point of S not belonging to F_0. Then $q = f(x) \neq p$ in the Hausdorff space E_1 and we have $U_0 \frown U_1 = 0$ for some neighbourhoods U_0, U_1 of p, q respectively. The closed set F_0 lies in the interior of $f^{-1}(U_0)$. From the compactness of S follows easily that $V_\beta(F_0) \subset f^{-1}(U_0)$ holds for some $V_\beta \in V$. For such a V_β, $F \frown V_\beta(F_0) \neq 0$ implies $F \frown f^{-1}(U_1) = 0$, and, since q is an interior point of $f^{-1}(U_1)$, x is not contained in $\overline{G}_\beta$. Now, if all G_β were non-empty, then they should constitute a family with finite intersection property, and there should be a point x contained in all $\overline{G}_\beta$ but not in F_0, since $G_\beta \frown V_\alpha(F_0) = 0$; this contradicts the fact mentioned above. Therefore, some G_β is empty, that is, some V_β satisfies the condition (※).

In conclusion, let us remark that our first proposition in this note can be applied to locally compact spaces. Namely, a T_1 space S is locally compact, if there is a closed continuous mapping of S into a locally compact T_1 space which induces a decomposition of S into compact sets. Proof is easy and so omitted.