On a Theorem of the Phragmén-Lindelöf Type¹

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The following theorem is one of the Phragmén-Lindelöf type, which may find some future applications, perhaps in the analytic number-theory. The proof is modelled on that for Satz 405 of Landau’s Vorlesungen über Zahlentheorie (Leipzig, 1927), vol. 2.

**Theorem.** Let \( s \) be a complex variable with \( \sigma = \Re s \) and \( t = \Im s \); \( a, b, c, A \) and \( B \) real constants; \( \alpha < \beta, t_0 > 3. \) Let \( f(s) \) be a function regular in the half-strip \( E \) defined by \( \alpha \leq \sigma \leq \beta \) and \( t \geq t_0, \) such that

\[
\begin{align*}
|f(\alpha + ti)| &< Kt^\alpha (\log t)^A & (t \geq t_0), \\
|f(\beta + ti)| &< Kt^\beta (\log t)^B & (t \geq t_0), \\
|f(s)| &< Kt^\sigma & (s \in E).
\end{align*}
\]

Then we have, for \( s \in E, \)

\[
|f(s)| < Lt^{\beta-\sigma}t^{2(\beta-\sigma)}(\log t)^{A(\beta-\sigma)+B(\beta-\sigma)+E(\sigma-\beta)}.
\]

where \( L \) is a suitable positive number independent of \( s. \)

**Proof.** The function \( \log s \) is regular in the \( s \)-plane cut along the non-positive axis \( (\sigma \leq 0, t = 0), \) if we take the branch which is real for positive \( s. \) Since the unique solution of the equation

\[
\log s - \frac{\pi i}{2} = 0
\]

is \( s = i, \) the function \( \log (\log s - \frac{\pi i}{2}) \) is regular in the half-plane \( t > 2, \)

if we take the branch which is real for \( s = it, t > 2. \)

This being so, let us put

\[
\Phi(s) = A\frac{\beta-s}{\beta-\alpha} + B\frac{s-\alpha}{\beta-\alpha}, \quad \Psi(s) = A\frac{\beta-s}{\beta-\alpha} + B\frac{s-\alpha}{\beta-\alpha}.
\]

Then the function

\[
g(s) = \exp\{\Phi(s)(\log s - \frac{\pi i}{2}) + \Psi(s)(\log s - \frac{\pi i}{2})\}
\]

is regular and vanishes nowhere, in the half-plane \( t > 2, \) by what has been said above.

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Now, we have in the half-strip \( |\sigma| \leq |\alpha| + |\beta|, \ t > 2, \)
\[
| \log s - \log t - \frac{\pi}{2} i | = | \log s - \log (ti) | \\
= \left| \int_{\sigma/t}^{\alpha + \beta} \frac{du}{u} \right| \leq \frac{|\sigma|}{t} \leq \frac{|\alpha| + |\beta|}{t},
\]
the integral being taken along a horizontal path. Hence, in the same half-strip,
\[
\log s - \frac{\pi}{2} i = \log t + \frac{\theta}{t} \quad (|\theta| \leq |\alpha| + |\beta|),
\]
\[
\log (\log s - \frac{\pi}{2} i) = \log \log t + \log \left( 1 + \frac{\theta}{t \log t} \right) \\
= \log \log t + \frac{\xi}{t \log t} \quad (|\xi| \leq C_1),
\]
where \( \log \log t \) is real and \( C_1 \) denotes a positive number independent of \( s \) (similarly for \( C_2, C_3, \ldots \) in the sequel).

Hence, in the given half-strip \( E \), we find
\[
\Psi(s) (\log s - \frac{\pi}{2} i) + \Theta(s) \log (\log s - \frac{\pi}{2} i) \\
= \left( \Phi(\sigma) + \frac{b - \alpha}{\beta - \alpha} ti \right) \left( \log t + \frac{\theta}{t} \right) + \left( \Psi(\sigma) + \frac{B - A}{\beta - \alpha} ti \right) \left( \log \log t + \frac{\xi}{t \log t} \right) \\
= \Phi(\sigma) \log t + \Psi(\sigma) \log \log t + \frac{b - \alpha}{\beta - \alpha} t \log t \\
+ \frac{B - A}{\beta - \alpha} t \log \log t \quad (|\eta| \leq C_2),
\]
\[
|g(s)| = |e^s| \exp \{ \Phi(\sigma) \log t + \Psi(\sigma) \log \log t \} \\
\leq C_{3\sigma}^{\Phi(\sigma)} (\log t)^{\Psi(\sigma)} \quad (C_3 = e^{C_2}),
\]
\[
\frac{1}{|g(s)|} \leq C_{3\sigma}^{-\Phi(\sigma)} (\log t)^{-\Psi(\sigma)}.
\]

We now put \( F(s) = \frac{f(s)}{g(s)} \). Then \( F(s) \) is regular in \( E \), and we have for \( t \geq t_0 \), since \( \Phi(\alpha) = a \) and \( \Psi(\alpha) = A \),
\[
|F(\alpha + ti)| \leq K t^a (\log t)^{AC_3^{-\Phi(\sigma)}} (\log t)^{-\Psi(\sigma)} = C_5 K,
\]
and similarly
\[
|F(\beta + ti)| \leq C_6 K.
\]
Also, in the set \( E \),
\[
|F(s)| \leq K t^a \cdot C_{3\sigma}^{-\Phi(\sigma)} (\log t)^{-\Psi(\sigma)} < C_7 C_{5\sigma}.
\]
Hence we obtain for \( s \in E \), by the Phragmén-Lindelöf theorem (see Satz 404 of Landau's book cited above),
\[ |F(s)| < C_n, \]
\[ |f(s)| = |F(s)| \cdot |g(s)| \leq C_n C_d t^{\theta(\alpha)} (\log t)^{\psi(\alpha)} \]
\[ = Lt^{a(\beta-\alpha)/\beta - a} + a(\gamma-\alpha)/\beta - a} (\log t)^{A(\beta-\alpha)/\beta - a} + F(\alpha-\alpha)/\beta - a} (L = C_n C_d), \]

which proves our theorem.