On Singular Points of Ordinary Differential Equations of the First Order[1]

Tizuko Katō (加藤千鶴子)

Department of Mathematics, Faculty of Science,
Ochanomizu University.

We consider the differential equation

$$(1) \qquad x\frac{dy}{dx} = yF(x, y),$$

where $F(x, y)$ is regular and developable in the vicinity of the origin such that

$$F(x, y) = \sum_{j,k} a_{jk} x^j y^k$$

$$a_{00} = \dots = a_{0, n-1} = 0, \qquad a_{0n} \neq 0.$$

Then the general solution of the equatiod (1) can be developped in a uniformly convergent series

$$(2) \qquad y = \sum_{j=0}^{\infty} \psi_j(u) x^j$$

for

$$(3) \qquad |x| < \delta, \qquad |\arg(au^n)| < \pi - \varepsilon, \qquad |u| < \delta,$$

where u is the general solution of the differential equation

$$(4) \qquad x\frac{du}{dx} = u^{n+1}(a + a'u^n) \qquad (a = a_{0n} \neq 0)$$

and $\psi_j(u)$ is regular and asymptotically developable into the series

$$\psi_j(u) \sim u \sum_{k=0}^{\infty} p_{jk} u^k$$

for

$$|\arg(au^n)| < \pi - \varepsilon, \qquad |u| < \delta.$$

The above mentioned result has been obtained by Prof. Hukuhara (1937). In the present paper, we shall prove it with the aid of the existence theorem of fixed points Hukuhara (1900).

Puttidg $x = e^t$, the equation (1) becomes

$$(1') \qquad \frac{dy}{dt} = yf(t, y)$$

where $f(t, y) = F(e^t, y)$. The formal solution of the equation (1′) is

[1] Contribution from Department of Mathematics, Faculty of Science, Ochanomizu University No. 5.

$$y = u \sum_{j \cdot k} p_{jk} e^{jt} u^k \qquad (p_{00} = 1),$$

where u is the general solution of the differential equation

$$(4') \qquad \frac{du}{dt} = u^{n+1}(a + a'u^n) \qquad (a = a_{\cdot n} \neq 0).$$

If we put

$$P_N(e^t, u) = \sum_{nj+k \leqq n(N+1)} p_{jk} e^{jt} u^k$$
$$y = u(P_N(e^t, u) + z)$$

and transform the differential equation (1′) into the form

$$(5) \qquad \frac{dz}{dt} = g_N(t, u, z),$$

then $g_N(t, u, z)$ is regular with respect to e^t, u and z at (0, 0, 0) and

$$g_N(t, u, z) = (P_N + z)f - \frac{\partial P_N}{\partial t} - (au^n + a'u^{2n})\left(z + \frac{\partial}{\partial u}(uP_N)\right)$$
$$= \sum b_{jkl} e^{jt} u^k z^l$$

with

$$b_{jk0} = 0 \quad \text{for} \quad nj + k < n(N+1)$$
$$b_{jkl} = 0 \quad \text{for} \quad nj + k < n, \quad l > 0,$$
$$b_{0n1} = na, \qquad b_{101} = a_{10},$$

so that it holds the inequality

$$|g_N(t, u, z)| \leqq A(|e^t| + |u|^n)|z| + B_N(|e^{(N+1)t}| + |u|^{n(N+1)})$$

for

$$\Re t < -R_1, \qquad |u| < \delta_1, \qquad |z| < \delta_1,$$

where A and B_N are constants, A being independent of N.

Let $\mathfrak{F}$ be the family of functions $\varphi(t, u)$ having the following properties

1. $\varphi(t, u)$ is regular with respect to t and u for

$$(7) \qquad \Re t < -R_2, \qquad |\arg(au^n)| < \pi - \varepsilon, \qquad |u| < \delta_2,$$

and has period $2\pi i$ with respect to t.

2. $\varphi(t, u)$ satisfies an inequality

$$(8) \qquad |\varphi(t, u)| \leqq K(|e^{Nt}| + |u|^{nN}),$$

where N, K are independent of φ.

Next we consider a mapping

$$(9) \qquad \Phi(t_0, u_0) = \int_{I_0} g_N(t, u, \varphi) dt,$$

where u denotes the solution of (4′), which takes the value u_0 for $t=t_0$, and where the path L_0 is the half straight line $t=t_0+re^{i\theta}$ $(0\leqq r<\infty)$, in which the value of θ will be determined later. First we prove that $\mathfrak{F}$ includes $\Phi(t,u)$. In order to do this, we have to show that (i) $\Phi(t,u)$ satisfies the inequality

$$(10)\qquad |\Phi(t,u)|\leqq K(|e^{Nt}|+|u^{nN}|)$$

for an arbitrary pair of values t_0, u_0 satisfying (7), (ii) $\Phi(t,u)$ is regular with respect to t, u for (7), and (iii) $\Phi(t,u)$ admits a period $2\pi i$ with respect to t. Form (6) and (8) we have

$$\begin{aligned}|g_N(t,u,\varphi)|&\leqq AK(|e^t|+|u|^n)(|e^{Nt}|+|u|^{nN})\\&\quad+B_N(|e^{(N+1)t}|+|u|^{n(N+1)})\\&\leqq(3AK+B_N)(|e^{(N+1)t}|+|u|^{n(N+1)}).\end{aligned}$$

Then $\Phi(t,u)$ satisfies an inequality

$$|\Phi(t_0,u_0)|\leqq(3AK+B_N)\int_{L_0}(|e^{(N+1)t}|+|u|^{n(N+1)})dt.$$

In order to ascertain the inequality (10), we have only to prove the following

$$(11)\qquad -\frac{d}{dr}(K(|e^{Nt}|+|u|^{nN}))>(3AK+B_N)(|e^{(N+1)t}|+|u|^{n(N+1)}).$$

Now we can determine the constant θ as follows

$$|\pi-\theta|<\frac{\pi}{2}-\frac{\varepsilon}{2},\qquad|\pi-\arg(au_0^n)-\theta|<\frac{\pi}{2}-\frac{\varepsilon}{2}$$

by the assumption $|\arg(au_0^n)|<\pi-\varepsilon$. Then, if $a>a''$ and δ_2 is sufficiently small, we have

$$-\frac{d}{dr}|e^{Nt}|=-\frac{d}{dr}e^{Nr\cos\theta}=N|e^{Nt}||\cos\theta|\geqq N|e^{Nt}|\sin\frac{\varepsilon}{2}$$

$$\begin{aligned}-\frac{d}{dr}|u^{nN}|&=nN|u|^{nN}\frac{d}{dr}\log|u|\\&=-nN|u|^{nN}\Re\{(au^n+a'u^{2n})e^{i\theta}\}\\&\geqq nNa''|u|^{n(N+1)}\sin\frac{\varepsilon}{2}.\end{aligned}$$

Putting

$$\min\left\{\sin\frac{\varepsilon}{2},\ na''\sin\frac{\varepsilon}{2}\right\}=\nu>\mu,$$

it follows

$$(12)\qquad -\frac{d}{dr}(|e^{Nt}|+|u|^{nN})>N\mu(|e^{(N+1)t}|+|u|^{n(N+1)}).$$

If we choose N such that $N\mu > 3A$, and K sufficiently large, we obtain the inequality (11) form (12).

As for the proof of (ii) putting

$$\bar{\Phi}(t, u_0) = \int_{L_0+\overline{t_0, t}} g_N(t, u(t, t_0, u_0), \varphi(t, u(t, t_0, u_0))\, dt\,,$$

it is evident that $\bar{\Phi}(t, u_0)$ is a regular function of t and u_0. And it holds

$$\begin{aligned} &\int_{L_0+\overline{t_0, t}} g_N(t, u(t, t_0, u_0), \varphi(t, u(t, t_0, u_0))\, dt \\ (13) \qquad &= \int_L g_N(t, u(t, t_0, u_0), \varphi(t, u(t, t_0, u_0))\, dt \\ &= \Phi(t, u(t, t_0, u_0))\,. \end{aligned}$$

Further we have $u_0 = u(t_0, t, u)$ from $u = u(t, t_0, u_0)$ whence $\Phi(t, u) = \bar{\Phi}(t, u(t_0, t, u))$ by (13). Hence $\Phi(t, u)$ is a regular function of t and u.

As for the proof of (iii), the relation $\Phi(t_0+2\pi i, u_0) = \Phi(t_0, u_0)$ is equivalent to

$$\begin{aligned} &\int_{L_1} g_N(t', u(t', t_0+2\pi i, u_0), \varphi(t', u(t', t_0+2\pi i, u_0))\, dt' \\ &\qquad = \int_{L_0} g_N(t, u(t, t_0, u_0), \varphi(t, u(t, t_0, u_0))\, dt\,, \end{aligned}$$

where L_1 ; $t' = t_0+2\pi i + r'e^{i\theta}$ $(0 \leqq r' < \infty)$, then we have only to show

$$u(t+2\pi i, t_0+2\pi i, u_0) = u(t, t_0, u_0)\,.$$

But this relation is evident from (4′), since the right side of (4′) does not include the variable t.

Next we prove that the mapping (9) is continuous. $\varphi_n(t, u)$ in $\mathfrak{F}$ converges uniformly to $\varphi(t, u)$ in $\mathfrak{F}$, when $n \to \infty$ for (7), that is $|\varphi - \varphi_n| < \varepsilon$ for a positive number ε and a sufficiently large n. It follows from (8), (9) and (12), that the corresponding functions $\Phi_n(t, u)$ and $\Phi(t, u)$ satisfy following ineqality, for an arbitrary pair of values t_0, u_0,

$$\begin{aligned} |\Phi(t, u) - \Phi_n(t, u)| &\leqq \int_L |g_N(t, u, \varphi) - g_N(t, u, \varphi_n)|\, dt \\ &\leqq G \int_L (|e^t| + |u|^n)\, |\varphi - \varphi_n|\, dt \\ &\leqq G\varepsilon' \int_L (|e^{Nt}| + |u|^{nN})\, dt \\ &\leqq \frac{G\varepsilon'}{(N-1)\mu} (|e^{(N-1)t}| + |u|^{n(N-1)})\,, \end{aligned}$$

where G is constant and ε' tends to zero with ε. Accordingly $\Phi_n(t, u)$ is uniformly convergent for (7).

Now by the existence theorem of fixed points, there exists a function $\varphi(t, u)$ such that $\varphi(t, u) = \Phi(t, u)$. And from $\Phi(t, u) = \bar{\Phi}(t, u(t_0, t, u))$ we have $\Phi(t, u(t, t_0, u_0)) = \bar{\Phi}(t, u_0)$, so that $\varphi(t, u)$ represents the general solution of (5). Obviously, $\varphi(t, u) = \bar{\varphi}(x, u)$ is regular with respect to x and u. As the function $\bar{\varphi}(x, u)$ depends on N, we write $\bar{\varphi}_N(x, u)$ in place of $\bar{\varphi}(x, u)$, and put $\psi(x, u) = P_N(x, u) + \bar{\varphi}_N(x, u)$. Then the function $y = u\psi(x, u)$ represents the general solution of (1) satisfying the condition which we have stated at the beginning.

References

Hukuhara, M. 1937 Mem. Fac. Eng. Kyusyu Imp. Univ. **8**, 233.
——— 1950 常微分方程式: p. 118.