Chapter 1.

Introduction

This thesis consists of the following five papers.

(1) “Classification of some rules of cellular automata” (with F. Takeo)

(2) “The limit set of cellular automata” (with F. Takeo)

(3) “Convergence to the limit set of linear cellular automata” (with F. Takeo)

(4) “A note on the property of linear cellular automata”

(5) “Convergence to the limit set of linear cellular automata, II”

A cellular automaton consists of a finite-dimensional lattice of sites, each of which takes an element of a finite set \( Z_M = \{0, 1, \ldots, M - 1\} \) \((M \in \mathbb{N})\) of integers at each time step and the value of each site at any time step is determined as a function \( L \) of the values of the neighboring sites at the previous time step. Patterns of cellular automata are investigated in many ways.

From investigation of a large sample of cellular automata, S. Wolfram suggests that many (perhaps all) cellular automata fall into four basic behavior classes.

For a linear rule \( L \) with \( M = 2 \), S. J. Willson considered the set of sites which takes the value 1 when time increases from 0 to \( 2^n \) in a product space \( Z^d \times Z_+ \), that is,

\[
K(n, \omega) = \{(x, t) \in Z^d \times Z_+ \mid 0 \leq t \leq 2^n, (L^t \omega)(x) = 1\}.
\]

He showed the existence of a limit set of \( \{K(n, \omega)/2^n\} \) in the sense of Kuratowski limit and that limit set does not depend on an initial condition.

In case of \( M = p^r \) (\( p \) is prime and \( r \in \mathbb{N} \)), S. Takahashi considered the set of sites which takes the positive integer with an initial configuration \( \delta \), where \( \delta \)
takes 1 at the origin and 0 at other sites, when time increases from 0 to \( p^n - 1 \), that is,

\[
K(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1, \ (L^j \delta)(x) \neq 0\}
\]

and showed the existence of the limit set \( \{K(n, \omega) / p^n\} \). Moreover Takahashi investigated the set of sites which takes the value \( j (j = 1, 2, \ldots, p^r - 1) \) when time increases from 0 to \( p^n - 1 \), that is,

\[
K_h(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1, \ (L^j \delta)(x) = b \pmod{p^r}\}
\]

and showed the existence of a limit set of \( \{K_h(n, \delta)\} \).

Chapter 2 concerns with Wolfram's four classes. Four classes were defined as follows.

Class 1 Evolution leads to a homogeneous state.

Class 2 Evolution leads to a set of separated simple stable or periodic structures.

Class 3 Evolution leads to a chaotic pattern.

Class 4 Evolution leads to complex localized structures, sometimes long-lived.

We redefine four classes mathematically and discuss classification of rules of the simplest cellular automata without simulation.

In Chapter 3, we consider the case \( M = 2 \). Let \( \mathcal{P} \) be the set of all configurations \( \mathbb{Z}^d \to \mathbb{Z}_2 \). A map \( L : \mathcal{P} \to \mathcal{P} \) is a transition rule if (1) \( L(0) = 0 \); and (2) there exist \( v_1, \ldots, v_m \in \mathbb{Z}^d \) and a map \( f : (\mathbb{Z}_2)^m \to \mathbb{Z}_2 \) such that

\[(La)(x) = f(a(x + v_1), \ldots, a(x + v_m)) \quad \text{for all} \ x \in \mathbb{Z}^d, a \in \mathcal{P}.\]

We consider the space \( USC \) of all upper semi continuous functions \( g : \mathbb{R}^d \times [0, 1] \to \mathbb{Z}_2 \) and consider an operator on \( USC \). We define the product space \( \prod E_k \) and an operator \( \tilde{F}_L \) on \( USC \) corresponding to \( L \). We investigate whether the limit set of \( \tilde{F}_k \) as \( k \to \infty \) belongs to a certain subspace \( E_\infty \) and the relation between \( \lim G_k^a \) and \( \lim \tilde{F}_k^g \) [Theorem 1 in Chapter 3]. We consider a quotient
space $\hat{E} = \prod E_k/\sim$ and the operator $\hat{F}_L$ on it and investigate conditions that the $\hat{F}_L$-invariant set belongs to a certain subspace. In Section 3, we consider the case of linear rules. In Section 4, we consider the case that $L$ is non-linear. We show some conditions for $L$ and initial configurations such that there exists a $\hat{F}_L$-invariant set [Theorems 5, 6 and 7 in Chapter 3].

In Chapter 4, we investigate the case $M = p$, where $p$ is prime. Let $\mathcal{P}$ be the set of all configurations $a: \mathbb{Z}^d \to \mathbb{Z}_p$ with compact support. We define $\delta \in \mathcal{P}$ as

$$
\delta(x) = \begin{cases} 
1 & (x = 0), \\
0 & (x \neq 0)
\end{cases}
$$

and a linear cellular automata rule $L: \mathcal{P} \to \mathcal{P}$ as follows:

$$
(La)(x) = \sum_{j \in G} \alpha_j a(x + k_j) \pmod{p} \quad \text{for } a \in \mathcal{P},
$$

where $G$ is a finite subset of $\mathbb{Z}$ with $|G| \geq 2$, $k_j \in \mathbb{Z}^d$ ($j \in G$) is a neighboring site of origin, $\alpha_k \in \mathbb{Z}_p \setminus \{0\}$.

Let

$$
X_n = \{ (x/p^n, t/p^n) \in \mathbb{R}^d \times [0, 1] \mid x \in \mathbb{Z}^d, t \in \mathbb{Z}_+, 0 \leq t \leq p^n \}
$$

for $n \in \mathbb{N}$. For $j \in \mathbb{N}$, put

$$
G_j = \{ \ell \in \mathbb{Z}^d \mid (L^j \delta)(\ell) \neq 0 \}.
$$

Define a map $\psi_n$ from $\mathcal{P}$ to the function space on $\mathbb{R}^d \times [0, 1]$ for $a \in \mathcal{P}$ and $n \in \mathbb{N}$ by

$$
(\psi_n(a))(x/p^n, t/p^n) = \begin{cases} 
(L^j a)(x) & \text{if } (x/p^n, t/p^n) \in X_n, \\
0 & \text{if } (x/p^n, t/p^n) \in (\mathbb{R}^d \times [0, 1]) \setminus X_n.
\end{cases}
$$

$\psi_n(a)$ represents the state $L^j(a)(x)$ for $0 \leq t \leq p^n$ and $x \in \mathbb{Z}^d$. We shall define an operator $T$ as follows.

Define a map $S_{j, j}: \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d \times [j, j+1]$ by

$$
S_{j, j}(x, t) = \left( \frac{x}{p^j}, \frac{t}{p^j} \right),
$$

$$
(\psi_n(a))(x/p^n, t/p^n) = \begin{cases} 
(L^j a)(x) & \text{if } (x/p^n, t/p^n) \in X_n, \\
0 & \text{if } (x/p^n, t/p^n) \in (\mathbb{R}^d \times [0, 1]) \setminus X_n.
\end{cases}
$$

$$
S_{j, j}(x, t) = \left( \frac{x}{p^j}, \frac{t}{p^j} \right).
$$
By using maps $S_{t,j}$ define an operator $T$ on the space of functions on $\mathbb{R}^d \times [0,1]$ by

$$Tg(y,q) = \sum_{\ell \in G_j} (L^\delta(\ell))g(S_{t,j}^{-1}(y,q)) \pmod p$$

for $\frac{j}{p} < q \leq \frac{j+1}{p}$ with $0 \leq j \leq p-1$ and

$$Tg(y,0) = g(py,0)$$

for a function $g$ on $\mathbb{R}^d \times [0,1]$. Then $T(\psi_n) = \psi_{n+1}$ holds for all $n \in \mathbb{N}$.

We have the following theorem which concerns the limit function $\{\psi_n(a)\}$ in the pointwise topology.

**Theorem I (Theorem 2.5 in Chapter 4).** For $a \in \mathcal{A}$ with $a(0) \neq 0$, we have the following assertions:

1. $\psi_n(a)$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.
2. The limit function $g_a$ of the sequence $\{\psi_n(a)\}$ in the pointwise topology is $T$-invariant, that is, $Tg_a = g_a$.
3. As for the limit functions $g_\delta$ and $g_a$ of $\{\psi_n(\delta)\}$ and $\{\psi_n(a)\}$ respectively, we have $a(0)g_\delta = g_a$.

In order to investigate convergence of $\{\psi_n(\omega)\}$, we shall introduce two metrics $d_f, D_f$ in the space of $\mathcal{L}_p$-valued upper semi-continuous functions on a compact subset of $\mathbb{R}^d \times [0,1]$. Let $\text{USC}$ be the space of $\mathcal{L}_p$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$, where $\mathcal{L}_p$-valued upper semi-continuous functions mean upper semi-continuous functions embedded in $\mathbb{R}$-valued function spaces.

Let $K$ be a compact subset of $\mathbb{R}^d \times [0,1]$ and $(y_0,q_0)$ be a point of $(\mathbb{R}^d \times [0,1]) \setminus K$. Let

$$\text{USC}|_K = \{g \in \text{USC} \mid \text{support of } g \subseteq K\}.$$  

By using the Hausdorff distance $D(A,B)$ of non-empty compact sets $A$ and $B$ in $\mathbb{R}^d \times [0,1]$, we shall define the pseudodistance $D_0(A,B)$ of $A$ and $B$ in
$\mathbb{R}^d \times [0, 1]$ by

$$D_0(A, B) = D(A \cup \{(y_0, q_0)\}, B \cup \{(y_0, q_0)\})$$

and metrics $d_f, D_f$ in $USC_K$ as follows:

$$d_f(g_1, g_2) = \max_{1 \leq j \leq p-1} D_0(g_1^{-1}(j), g_2^{-1}(j)),$$

$$D_f(g_1, g_2) = \max_{1 \leq s \leq p-1} D_0(g_1^{-1}[s+], g_2^{-1}[s+]),$$

for $g_1, g_2 \in USC_K$, where $g^{-1}[s+] = \{(x, t) \mid g(x, t) \geq s\}$ and $g^{-1}(j)$ is the closure of the set $g^{-1}(j) = \{(x, t) \mid g(x, t) = j\}$. Then the following theorem holds.

**Theorem II** (Theorem 3.5 in Chapter 4). For $\{f_n\} \subset USC_K$, suppose $d_f(f_n, f_m) \to 0$ as $n, m \to \infty$. Let $g = \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} f_n$. Then we have

$$D_f(f_n, g) \to 0 \text{ as } n \to \infty.$$ 

Using the above theorem, we have

**Theorem III** (Theorem 6.4 in Chapter 4). For a nonzero $a \in \mathcal{P}$, the following assertions hold:

1. $d_f(\psi_n(a), \psi_m(a)) \to 0$ as $n, m \to \infty$.

2. Put $f_a = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \psi_n(a) \in USC$. Then we have

$$D_f(\psi_n(a), f_a) \to 0 \quad \text{as} \quad n \to \infty.$$ 

and

**Theorem IV** (Theorem 6.5 in Chapter 4). For $a \in \mathcal{P}$ with $a(0) \neq 0$, let

$$Y_a = \cap_{k=1}^{\infty} \bigcup_{n \geq k} \frac{K(n, a)}{p^n}$$

and $g_a$ be defined by $g_a(y, q) = \lim_{n \to \infty} (\psi_n(a))(y, q)$ in the pointwise topology. Then the following assertions hold:
(1) The characteristic function \( \mathbf{1}_{Y_a} \) of the set \( Y_a \) satisfies

\[
\hat{g}_a = (p - 1)\mathbf{1}_{Y_a}
\]

and

\[
\hat{g}_a = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \psi_n(a),
\]

where \( \hat{g}(x,t) = \inf\{\phi(x,t) | \phi \in \text{USC}, \phi(x,t) \geq g(x,t)\} \).

(2) Though \( g_a \) is not necessarily the same as \( g_b \) for any \( a \in \mathcal{P} \) as shown in Theorem 1, the upper envelope \( \hat{g}_a \) of \( g_a \) is the same, that is,

\[
\hat{g}_a = \hat{g}_b = f_a = f_b,
\]

where \( f_a = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \psi_n(a) \).

Therefore in case of \( M = p \), the upper envelope of a limit function, \( \hat{g}_a \), takes either 0 or \( p - 1 \) and does not depend on an initial configuration.

In Chapter 5, we investigate the property of linear cellular automata in order to extend the results in Chapter 4 to the case \( M = p^r \) with \( r \in \mathbb{N} \). We show the following theorem for a linear cellular automata rule \( L \) satisfying some condition, which we call it the condition (A) and it will be described in Chapter 5.

**Theorem V (Theorem 2.7 in Chapter 5).** For a prime number \( p \) and \( r \in \mathbb{N} \), let \( L \) be defined as the equation (1.1) and the summation \( \sum \) is taken as the summation with mod \( p^r \) and satisfy the condition (A). Put \( t(r,j) = j(p^r - p^{r-1}) \) and \( i(r,j) = -(t(r,j) - p^{r-1})r_1 - p^{r-1}r_2 \), where \( r_1, r_2 \in G \) satisfy (I),(II),(III) and (IV) of the condition (A).

Then the set \( \{L^{(r,j)}\delta_0(i(r,j)) | 1 \leq j \leq p^r\} \) is a \( p^r \)-set, where the set \( \{a_n | n = 1, \ldots, k\} \) is a \( k \)-set if the set has one-to-one, onto correspondence with the set \( \{0, 1, \ldots, k - 1\} \).

In Chapter 6, we extend the result above to mod \( p^r \), where \( p \) is prime and \( r \in \mathbb{N} \), that is, \( L \) is defined as the equation (1.1) and the summation \( \sum \) is taken as the summation with mod \( p^r \). We have a similar result to the case of \( M = p \).
Theorem VI (Theorem 2.3 in Chapter 6). For $a \in P$ with $a(0) \neq 0$, we have the following assertions:

(1) The sequence $\{\psi_n(a)\}$ converges to a function on $\mathbb{R}^d \times [0, 1]$ in the pointwise topology.

(2) The limit function $g_a$ of the sequence $\{\psi_n(a)\}$ in the pointwise topology is $T$-invariant, that is, $Tg_a = g_a$, where

$$Tg(y, q) = \sum_{\ell \in \mathbb{Z}_{p^{r-1}}} (L^{jp^{r-1}} \delta)(\ell)g(S_{\ell, j}^{-1}(y, q))$$

for $\frac{j}{p} < q \leq \frac{j+1}{p}$ with $0 \leq j \leq p - 1$ and

$$Tg(y, 0) = g(py, 0)$$

and

$$S_{\ell, j}(x, t) = (\frac{x}{p^\ell}, \frac{t}{p^j}), \frac{j}{p}, \frac{j}{p^r}.$$

(3) As for the limit functions $g_\delta$ and $g_{\alpha}$ of $\{\psi_n(\delta)\}$ and $\{\psi_n(a)\}$ respectively, we have $a(0)g_\delta = g_a$.

S. Takahashi investigated the set of "$b$-state"

$$K_b(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1, (L^t \delta)(x) = b \pmod{p^r}\}$$

for $b \in \{1, \ldots, p^r - 1\}$ and the set

$$K_f(n, \delta) = \{(x, t) \in \mathbb{Z}_d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1, (L^t \delta)(x) \neq 0 \pmod{p^f}\}$$

for $f \in \{1, \ldots, r\}$ and for $b \in \mathbb{Z}_{p^r}$ satisfying $b/p^{f-1} \in \mathbb{N}$ and $b/p^f \notin \mathbb{N}$, he showed a limit set of $\{K_f(n, \delta)/p^n\}$ is equal to a limit set of $\{K_b(n, \delta)/p^n\}$ in the sense of Kuratowski limit.

We show the relationship between the limit function and the limit set of $\{\frac{K_f(n, \delta)}{p^n}\}$. We define that an element $j \in G$ is prime if $\alpha_j/p \notin \mathbb{N}$. 

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Theorem VII (Theorem 5.2 in Chapter 6). Suppose that the set $G$ in (1.1) with mod $p^r$ has at least two prime elements. Let the function $g_\delta$ be defined by $g_\delta(y,q) = \lim_{n \to \infty} (\psi_n(\delta))(y,q)$ and $Y_f$ be the limit set of $\{K^n(n,\delta)/p^n\}$ in the sense of Kuratowski limit.

Then

(1) the relation between $\hat{g}_\delta$ and $\{Y_f\}$ is as follows:

$$\hat{g}_\delta = \sum_{1 \leq f \leq r} (p^{r+1-f} - 1)p^{f-1}1_{Y_f \cup \bigcup_{i=1}^{r-1} Y_i},$$

where $\hat{g}(x,t) = \inf\{g(x,t) : g \in USC, g(x,t) \geq g(x,t)\}$.

(2) The relation between $\hat{g}_\delta$ and $\{\psi_n(\delta)\}$ is as follows:

$$\hat{g}_\delta = \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(\delta).$$

Theorem VIII (Theorem 5.3 in Chapter 6). Suppose that the set $G$ in (1.1) with mod $p^r$ has at least two prime elements. Put $g_a(y,q) = \lim_{n \to \infty} (\psi_n(a))(y,q)$ for $a \in P$ with $a(0) = kp^j$ $(k/p \notin \mathbb{N}$ and $j \in \{0,1,\ldots,r-1\})$. Then we have the following assertions.

(1) the relation between $\hat{g}_a$ and $\{Y_f\}$ is as follows:

$$\hat{g}_a = \sum_{1 \leq f \leq r-j} (p^{r+1-f-j} - 1)p^{f-1+j}1_{Y_f \cup \bigcup_{i=1}^{r-j} Y_i},$$

(2) The relation between $\hat{g}_a$ and $\{\psi_n(a)\}$ is as follows:

$$\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a) = \bigvee_{\sigma \in G_a} \hat{g}_{\sigma}(a).$$

Though Takahashi showed the existence of the limit set of each value of $\mathbb{Z}_{p^r}$ separately, we can consider the convergence to the limit sets of some values simultaneously by using the functions $\{\psi_n\}$. 

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