Chapter 5.

A note on the property of linear cellular automata

1 Introduction

Cellular automata are discrete dynamical systems with simple construction. We define a cellular automaton as follows.

Put $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ for $n \in \mathbb{N}$. Let $p$ be a prime number, $r$ a natural number and $\mathcal{P}$ the set of all configurations $w : \mathbb{Z} \to \mathbb{Z}_p^r$ with compact support. Let a linear transition rule $L : \mathcal{P} \to \mathbb{Z}_p^r$ be defined as follows:

$$Lw(x) \equiv \sum_{k \in G} c_kw(x+k) \quad (\text{mod } p^r) \quad \text{for } w \in \mathcal{P},$$

(1.1)

where the set $G$ is a finite subset of $\mathbb{Z}$ with $\# G \geq 2$ and $c_k \in \mathbb{N}$.

Patterns of linear cellular automata were studied by some people. Existence of the limit of a series of space-time patterns is proved [1, 3, 5, 6]. E. Jen showed that a series $\{L^tw(x) | t \in \mathbb{N}\}$ is aperiodic for some $L$ with $p = 2$ in [2]. In [5], S. Takahashi considered the case where $L$ is $p^r$-state linear cellular automata with the initial state $\delta_0$ which is 1 at the origin and 0 at others. He examined the limit set with respect to each non-zero state, by using the fact that every state appears in the set $\{L^t\delta_0(-(t-1)r_1-r_2) | t = 1, \ldots, p^{r+1}\}$. However, when we consider the limit set as a multi-valued function, the set $\{L^t\delta_0(-(t-1)r_1-r_2) | t = 1, \ldots, p^{r+1}\}$ does not work well. Hence we need another set which includes every state and plays a useful role in examining the limit set. So we give a systematic set which has one-to-one, onto correspondence with the set $\{0, 1, 2, \ldots, p^r - 1\}$. This set may play an important role in examining the limit set of space-time patterns as a $\mathbb{Z}_p^r$-valued upper semi continuous function [4].
2 The result

We deal with the specified transition rule $L$, which satisfies some condition. We say that $L$ satisfies the condition (A) if there exist $r_1, r_2 \in G$ satisfying

(I) $c_{r_1}/p, c_{r_2}/p \notin \mathbb{N}$.

(II) $r_1$ is an either maximum or minimum element of the set $G$.

(III) $r_2$ is extreme or $r_2 = \sum_{k \in G, r_2 \neq k} \beta_k k$ with $0 \leq \beta_k < 1$ such that $\sum_{k \in G, r_2 \neq k} \beta_k = 1$ and $p^{-1} \beta_k \notin \mathbb{N}$.

(IV) if $r_1$ is maximum [resp. minimum], then for $s \in \{1, 2, \ldots, p - 1\}$, $l \in \{1, 2, \ldots, |r_1 - r_2| - 1\}$ there does not exist a path from $-r_1 s(p^r - p^{r-1}) + l$ [resp. $-r_1 s(p^r - p^{r-1}) - l$] to the origin, that is, we have

\[-\sum_{k \in G} \alpha_k k \notin \{-r_1 s(p^r - p^{r-1}) + l \mid l \in \{1, 2, \ldots, |r_1 - r_2| - 1\}\}

[resp. $\sum_{k \in G} \alpha_k k \notin \{-r_1 s(p^r - p^{r-1}) - l \mid l \in \{1, 2, \ldots, |r_1 - r_2| - 1\}\}$]

for the set $\{\alpha_k\}_{k \in G} \subset \mathbb{N} \cup \{0\}$ such that $\sum_{k \in G} \alpha_k = s(p^r - p^{r-1})$ with $s \in \{1, 2, \ldots, p - 1\}$.

Here, an element $k \in G$ is extreme if an element $k$ is not expressed as a convex linear combination of other elements of $G$. We note that a maximum or minimum element of the set $G$ is extreme.

By using $r_1, r_2 \in G$ which satisfy (I),(II),(III) and (IV), put

\[t(r, j) = j(p^r - p^{r-1}) \quad (2.1)\]

and

\[i(r, j) = -(t(r, j) - p^{r-1})r_1 - p^{r-1}r_2 \quad (2.2)\]

for $j \in \mathbb{N}$. We define $\delta_0 \in \mathcal{P}$ as

\[
\delta_0(x) = \begin{cases} 
1 & x = 0 \\
0 & x \neq 0.
\end{cases}
\]

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Figure 1: Space-time pattern of $L_0 \delta_0(x) \equiv 3\delta_0(x - 1) + 5\delta_0(x + 1)$ (mod $2^3$)

We shall prove that the set $\{ L^{(r,j)} \delta_0(i(r,j)) | 1 \leq j \leq p^r \}$ has one-to-one, onto correspondence with the set $\{0, 1, 2, \ldots, p^r - 1 \}$. We shall call the set $\{ a_n | n = 1, \ldots, k \}$ a $k$-set, if the set has one-to-one, onto correspondence with the set $\{0, 1, \ldots, k - 1 \}$.

We need the following lemmas later.

**Lemma 2.1.** [5] Suppose $L$ is defined as (1.1). For $j, l \in \mathbb{N}$, we have

$$L^{j + l - 1} \delta_0(x) = \begin{cases} L^{j + l - 1} \delta_0(y) & \text{if there exists } y \text{ such that } p^j y = x, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.2.** Suppose $q \in \mathbb{N}$ with $q/p \notin \mathbb{N}$, $t = j p^{r - 1}$ with $j \in \mathbb{N}$, $v = p^r q$ with $l \in \{0, 1, \ldots, r - 2\}$ and $v < t$.

Then there exists $q' \in \mathbb{N}$ with $q'/p \notin \mathbb{N}$ such that $t - v = p^r q'$.

**Proof.** We have $t - v = p^r(jp^{-r+1} - q)$. Put $q' = j p^{-r+1} - q$. So we obtain $q'/p \notin \mathbb{N}$ by $q/p \notin \mathbb{N}$.

**Lemma 2.3.** Let $L$ be defined as (1.1) and satisfy the condition (A). Suppose $r \in \mathbb{N}$, $r \geq 2$ and $j \in \{1, 2, \ldots, p^r\}$. Let $r_1, r_2 \in G$ satisfy (I),(II),(III) and (IV) of the condition (A). Then the following assertions hold.
(1) \( L^{(r,1)} \delta_0(i(r,1))/p \notin \mathbb{N} \).

(2) The set \( \{ nL^{(1,1)} \delta_0(i(1,1)) \mod p | 1 \leq n \leq p \} \) is a p-set.

(3) Suppose \( |r_1 - r_2| \geq 2 \). If \( r_1 > r_2 \) [resp. \( r_1 < r_2 \)], then

\[
L^{(r,j)} \delta_0(-t(r,j)r_1 + l) = 0 \tag{2.3}
\]

[resp. \( L^{(r,j)} \delta_0(-t(r,j)r_1 - l) = 0 \)]

holds for \( l \in \{1, 2,\ldots, |r_1 - r_2| - 1 \} \) and

\[
L^{(r,spr^{-1})} \delta_0(-t(r,spr^{-1})r_1 + l) = 0
\]

[resp. \( L^{(r,spr^{-1})} \delta_0(-t(r,spr^{-1})r_1 - l) = 0 \)]

holds for \( l \in \{1, 2,\ldots, pr^{-1}|r_1 - r_2| - 1 \} \) and \( s \in \{1, 2,\ldots, p - 1\} \).

Proof. (1) For \( a, b \in \mathbb{N} \) put \( a + bC_b = (a + b)!/(ab!) \). We have

\[
L^{(r,1)} \delta_0(i(r,1)) \equiv p^{r-p-1}C_{p-1}c_{r_1}^{p-r-1}c_{r_2}^{p-r-1} \quad \text{(mod } p \text{)}
\]

by (II) and (III) of the condition (A). Since

\[
p^{r-p-1}C_{p-1} = \frac{(p^r - p^r - 1)(p^r - p^r - 2)\cdots(p^r - p^r - p - 1)}{p^{r-1}!},
\]

\( p \) does not divide \( p^{r-p-1}C_{p-1} \) by Lemma 2.2. Then \( p \) does not divide \( L^{(r,1)} \delta_0(i(r,1)) \) by (I) of the condition (A).

(2) We shall show that \( nL^{(1,1)} \delta_0(i(1,1)) \neq 0 \) (mod \( p \)) holds for all \( n \in \{1, 2,\ldots, p - 1\} \). The proof is by contradiction. Assume that there exists \( n_0 \in \{1, 2,\ldots, p - 1\} \) such that \( n_0L^{(1,1)} \delta_0(i(1,1)) = s_0p \) holds with some \( s_0 \in \mathbb{N} \). Then we have

\[
L^{(1,1)} \delta_0(i(1,1)) = s_0p/n_0.
\]

Since \( n_0 \leq p - 1 \), \( s_0/n_0 \in \mathbb{N} \) holds. Therefore \( p \) divide \( L^{(1,1)} \delta_0(i(1,1)) \), which contradicts assumption.

(3) Suppose \( r_1 > r_2 \). Since \( 1 \leq l \leq r_1 - r_2 - 1 \), there does not exist any path from the origin to the point \( -t(r,j)r_1 + l \) by (IV) of the condition (A). So we have \( L^{(r,j)} \delta_0(-t(r,j)r_1 + l) = 0 \).
Since $t(r, sp^{r-1}) = p^{r-1}t(r, s)$, we have

$$L^{l(r, sp^{r-1})} \delta_0(-t(r, sp^{r-1})r_1) = L^{l(r, s)} \delta_0(-t(r, s)r_1)$$
$$L^{l(r, sp^{r-1})} \delta_0(-t(r, sp^{r-1})r_1 + p^{r-1}(r_1 - r_2)) = L^{l(r, s)} \delta_0(-t(r, s)r_1 + (r_1 - r_2))$$
$$L^{l(r, sp^{r-1})} \delta_0(-t(r, sp^{r-1})r_1 + l) = 0$$

for $l \in \{1, 2, \ldots, p^{r-1}(r_1 - r_2) - 1\}$ and $s \in \{1, 2, \ldots, p - 1\}$ by Lemma 2.1 and the equation (2.3).

In case $r_1 < r_2$, we can prove it in the same way as above. \hfill \Box

**Lemma 2.4.** Suppose $r \geq 2$ and $L$ satisfies the condition (A). Then

$$L^{l(r, j)} \delta_0(i(r, j)) \equiv L^{l(r, m)} \delta_0(i(r, m)) + sL^{l(r, p^{r-1})} \delta_0(i(r, p^{r-1})) \pmod{p^r}$$

holds for $j = sp^{r-1} + m$ with $s \in \{0, 1, \ldots, p - 1\}$ and $m \in \{1, 2, \ldots, p^{r-1}\}$.

**Proof.** Let $r_1, r_2 \in G$ satisfy (I),(II),(III) and (IV) of the condition (A). We first consider the case where $r_1 > r_2$. We compute $L^{l(r, j)} \delta_0(i(r, j))$ from the values at time $t(r, j - 1)$. We have

$$L^{l(r, j)} \delta_0(i(r, j)) \equiv p^{r-1}C_{p^{r-1}(r_1 - r_2) - 1} p^{r-1}B(r_1, j - 1, k)$$
$$+ \sum_{k=1}^{p^{r-1}(r_1 - r_2) - 1} B(r, k) b(r, j - 1, k)$$
$$+ p^{r-1}L^{l(r, j - 1)} \delta_0(i(r, j - 1)) \pmod{p^r},$$

where $b(r, j, k) = L^{l(r, j)} \delta_0(-t(r, j)r_1 + k)$ and $B(r, k) \in \mathbb{N}$ for $k \in \{1, 2, \ldots, p^{r-1}(r_1 - r_2) - 1\}$ and $B(r, k)$ does not depend on $j$.

Put

$$d(j) = \sum_{k=1}^{p^{r-1}(r_1 - r_2) - 1} B(r, k) b(r, j, k)$$

and we rewrite the equation above as follows:

$$L^{l(r, j)} \delta_0(i(r, j)) \equiv L^{l(r, 1)} \delta_0(i(r, 1)) + d(j - 1) + L^{l(r, j - 1)} \delta_0(i(r, j - 1))$$
$$\equiv jL^{l(r, 1)} \delta_0(i(r, 1)) + \sum_{l=1}^{j-1} d(l) \pmod{p^r}$$
from Euler's theorem \((n^{p^r-p^{r-1}} \equiv 1 \pmod{p^r})\).

We have \(b(r, sp^{r-1}, k) = 0\) for all \(k \in \{1, 2, \ldots, p^{r-1}(r_1 - r_2) - 1\}\) and all \(s \in \{1, 2, \ldots, p - 1\}\) by Lemma 2.3(3) and \(L^{(r, sp^{r-1})}\delta_0(-t(r, sp^{r-1})r_1) \equiv 1 \pmod{p^r}\). Since \(B(r, k)\) does not depend on \(j\), we have

\[
d(m + sp^{r-1}) \equiv d(m) \pmod{p^r}
\]

for \(m \in \{1, 2, \ldots, p^{r-1}\}\) and \(s \in \{1, 2, \ldots, p - 1\}\) by (II) of the condition (A).

For \(j = sp^{r-1} + m\) with \(m \in \{1, 2, \ldots, p^{r-1}\}\) and \(s \in \{0, 1, \ldots, p - 1\}\)

\[
L^{(r,j)}\delta_0(i(r, j)) = jL^{(r,1)}\delta_0(i(r, 1)) + \sum_{l=1}^{j-1} d(l)
\]

\[
= sp^{r-1}L^{(r,1)}\delta_0(i(r, 1)) + \sum_{l=1}^{p^{r-1}-1} d(l) + mL^{(r,1)}\delta_0(i(r, 1)) + \sum_{l=1}^{m-1} d(l)
\]

\[
= L^{(r,m)}\delta_0(i(r, m)) + sL^{(r,p^{r-1})}\delta_0(i(r, p^{r-1})) \pmod{p^r}
\]

by (2.4). In case \(r_1 < r_2\), putting \(b(r, j, k) = L^{(r,j)}\delta_0(-t(r, j)r_1 - k)\), we can prove in the same way.

\[
\square
\]

Put

\[
a(r, j) = L^{(r,j)}\delta_0(i(r, j))
\]  

(2.5)

![Figure 2: The relation among \(t(r, j), i(r, j)\) and \(a(r, j)\).](image)

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for convenience. We will prove the set \( \{a(r, j) | 1 \leq j \leq p^r\} \) is a \( p^r \)-set. In order to prove the following lemma, we define a map \( \mathcal{L} : \mathcal{P} \to \mathbb{N} \) as follows:

\[
\mathcal{L} w(x) = \sum_{k \in G} c_k w(x + k),
\]

where the set \( G \) and \( c_k \) are as in the definition of \( L \). We note that there exists \( k(r, j) \in \mathbb{N} \) such that

\[
\mathcal{L}^{(r,j)} \delta_0(i(r, j)) = k(r, j)p^r + a(r, j).
\]

**Lemma 2.5.** Suppose \( a(r, j) \) is defined as (2.5) and the set \( \{a(r, j) | 1 \leq j \leq p^r\} \) is a \( p^r \)-set. Then the following assertions hold:

1. \( a(r + 1, j) \neq a(r + 1, l) \) holds for \( j, l \in \{1, 2, \ldots, p^r\} \) with \( j \neq l \).
2. There exists \( k_0 \in \mathbb{N}(0 \leq k_0 \leq p - 1) \) such that \( a(r, p^r-1) = p^r-1k_0 \).
3. \( a(r+1, j) \neq a(r+1, l) + ka(r+1, p^r) \pmod{p^{r+1}} \) holds for any \( k \in \{1, 2, \ldots, p-1\} \) and \( j, l \in \{1, 2, \ldots, p^r\} \) with \( j \neq l \).
4. \( a(r + 1, j) + k_1a(r + 1, p^r) \equiv a(r + 1, j) + k_2a(r + 1, p^r) \pmod{p^{r+1}} \) holds for any \( k_1, k_2 \in \{1, 2, \ldots, p-1\} \) with \( k_1 \neq k_2 \) and \( j \in \{1, 2, \ldots, p^r\} \).

**Proof.** We have

\[
a(r, j) \equiv \mathcal{L}^{(r,j)} \delta_0(i(r, j)) \equiv \mathcal{L}^{(r+1,j)} \delta_0(i(r + 1, j)) \pmod{p^r}
\]

for \( j \in \{1, 2, \ldots, p^r\} \) by Lemma 2.4, since

\[
\mathcal{L}^{\pi(r,j)} \delta_0(p\pi(r,j)) = \mathcal{L}^{(r+1,j)} \delta_0(i(r + 1, j))
\]

by Lemma 2.1 and

\[
\mathcal{L}^{(r,j)} \delta_0(i(r, j)) \equiv \mathcal{L}^{\pi(r,j)} \delta_0(p\pi(r,j)) \pmod{p^r}.
\]

Therefore by (2.8) there exists \( k'(r, j) \in \mathbb{N} \) for \( j \in \{1, 2, \ldots, p^r\} \) such that

\[
\mathcal{L}^{(r+1,j)} \delta_0(i(r + 1, j)) = k'(r, j)p^r + a(r, j).
\]
So we obtain by (2.7) and (2.9)

\[
k(r + 1, j)p^{r+1} + a(r + 1, j) = k'(r, j)p^r + a(r, j) \tag{2.10}
\]

\[
k(r + 1, l)p^{r+1} + a(r + 1, l) = k'(r, l)p^r + a(r, l). \tag{2.11}
\]

for \( j, l \in \{1, 2, \ldots, p^r\} \).

(1) Assume \( a(r + 1, j) = a(r + 1, l) \) holds for \( j, l \in \{1, 2, \ldots, p^r\} \) with \( j \neq l \). By (2.10) and (2.11) \( a(r, j) - a(r, l) = (k(r + 1, j) - k(r + 1, l))p^{r+1} - (k'(r, j) - k'(r, l))p^r \), which contradicts the assumption.

(2) We will prove it by induction on \( r \).

(a) In case \( r = 1 \), it is clear by definition.

(b) In case \( r > 1 \), assume that it is true for \( r = r' \). Then we have \( a(r', p^{r'}) \equiv pa(r', p^{r'-1}) \equiv pp^{r'-1}k_0 \equiv p^{r'}k_0 \pmod{p^{r'}} \) by Lemma 2.4 and the assumption of induction. So there exists \( k'_0 \in \mathbb{N}(0 \leq k'_0 \leq p-1) \) such that \( a(r' + 1, p^{r'}) = p^{r'}k'_0 \) by (2.10).

(3) The proof is by contradiction. Assume that there exists \( k_1 \in \mathbb{N} \) such that \( a(r + 1, j) \equiv a(r + 1, l) + k_1 a(r + 1, p^r) \pmod{p^{r+1}} \). Then there exists \( s_0 \in \mathbb{N} \) such that \( a(r + 1, j) - a(r + 1, l) - k_1 a(r + 1, p^r) = s_0 p^{r+1} \). There exists \( k_0 \in \{0, 1, \ldots, p-1\} \) such that \( a(r + 1, p^r) = p^r k_0 \) by the assertion (2). By the equations (2.10) and (2.11), we have

\[
a(r, j) - a(r, l) = (k(r + 1, j) - k(r + 1, l))p^{r+1} - (k'(r, j) - k'(r, l))p^r
\]

\[
+ a(r + 1, j) - a(r + 1, l)
\]

\[
= (k(r + 1, j) - k(r + 1, l))p^{r+1} - (k'(r, j) - k'(r, l))p^r
\]

\[
+ s_0 p^{r+1} + k_0 k_1 p^r
\]

\[
= p^r \{(k(r + 1, j) - k(r + 1, l))p - (k'(r, j) - k'(r, l)) + s_0 p + k_0 k_1 \},
\]

which contradicts \( a(r, j) \neq a(r, l) \) by \( 0 \leq a(r, j) - a(r, l) \leq p^r - 1 \).

(4) It is clear by the property of modulus.
Proposition 2.6. Suppose $r \in \mathbb{N}$. If the set $\{a(r, j) \mid 1 \leq j \leq p^r\}$ is a $p^r$-set, then the set $\{a(r+1, j) \mid 1 \leq j \leq p^{r+1}\}$ is a $p^{r+1}$-set.

Proof. We obtain the conclusion by Lemma 2.4 and the assertion (1), (3) and (4) of Lemma 2.5.

Theorem 2.7. For a prime number $p$ and $r \in \mathbb{N}$, let $L$ be defined as (1.1) and satisfy the condition (A). Put $t(r, j) = j(p^r - p^{r-1})$ and $i(r, j) = -(t(r, j) - p^{r-1})r_1 - p^{r-1}r_2$, where $r_1, r_2 \in G$ satisfy (I) (II) (III) and (IV) of the condition (A).

Then the set $\{L(i, r, j) \delta_0(i(r, j)) \mid 1 \leq j \leq p^r\}$ is a $p^r$-set.

Proof. The proof is by induction on $r$.

(1) In case $r = 1$, from Lemma 2.3(3),

$$L^{(1, j)} \delta_0(i(1, j)) \equiv \ell_{e_2}^{(1,1)} c_0 e_{r_2}^{(1,1)} c_{r_2} + c_{r_1}^{(1,1)} L^{(1, j-1)} \delta_0(i(1, j-1)) \equiv L^{(1,1)} \delta_0(i(1, 1)) + L^{(1, j-1)} \delta_0(i(1, j - 1)) \equiv jL^{(1, 1)} \delta_0(i(1, 1)) \pmod{p}$$

for $1 \leq j \leq p$ by Euler’s theorem ($n^{p-1} \equiv 1 \pmod{p}$ for any $n \in \mathbb{N}$) and $L^{(1,1)} \delta_0(i(1, 1)) \equiv \ell_{e_2}^{(1,1)} c_0 e_{r_2}^{(1,1)} c_{r_2} \pmod{p}$. So the assertion holds for $r = 1$ from Lemma 2.3(2).

(2) In case $r \geq 2$, we get the conclusion by the proposition above and the assumption of induction.

References


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