Chapter 6.

Convergence to the limit set of linear cellular automata, II

1 Introduction

A cellular automaton consists of a finite-dimensional lattice of sites, each of which takes an element of a finite set $\mathbb{Z}_{p^r} = \{0, 1, \ldots, p^r - 1\}$ of integers at each time step and the value of each site at any time step is determined as a function of the values of the neighbouring sites at the previous time step.

We introduce the set $\mathcal{P}$ of all configurations $a: \mathbb{Z}^d \to \mathbb{Z}_{p^r}$ with compact support (i.e., $\#\{i \mid a(i) \neq 0\} < \infty$) and define a linear rule $L$ in $\mathcal{P}$ as

$$(La)(x) = \sum_{j=1}^{m} \alpha_j a(x + k_j) \pmod{p^r}.$$  \hfill (1.1)

The configuration of cellular automata at time step $t$ is represented by operating $L$ on the initial configuration by $t$ times.

In case of $p = 2$ and $r = 1$, S. J. Willson [6] investigated the so-called limit set of LCA. For $n \in \mathbb{Z}_+$ and $a \in \mathcal{P}$, he considered the set

$$K(n, a) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq 2^n, \ (L^t a)(x) = 1\},$$

where $L^t$ is the $t$-th power of $L$. He showed that there exists the limit set of $K(n, a)/2^n$ for any nonzero $a \in \mathcal{P}$ in the sense of Kuratowski limit [1, 4] and that the limit set does not depend on an initial configuration. The limit set of LCA for a certain linear rule is a Sierpinski gasket-like pattern.

In case of $\mod p$, the sets $\limsup K(n, \delta)/p^n$ and $\liminf K(n, \delta)/p^n$ in the sense of Kuratowski limit are the same with $\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} K(n, \delta)/p^n$ and $\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} K(n, \delta)/p^n$ respectively, since $\{K(n, \delta)/p^n\}$ is an increasing sequence. So in [3], we considered the limit set in the sense of the set theory, which is defined if the set $\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} K(n, a)/p^n$ coincides with the set $\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} K(n, a)/p^n$ without taking its closure.
As an extension of the result of Willson, S. Takahashi [5] investigated the case of an arbitrary prime number \( p \geq 2 \) and \( r \in \mathbb{N} \) and he considered the set

\[
K(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1, \ (L^j \delta)(x) \neq 0\}
\]

for \( n \in \mathbb{Z}_+ \). By using the set \( K(n, \delta) \), he also defined the limit set as a subset of \( \mathbb{R}^d \times [0, 1] \) in the same way as the case of \( p = 2 \), and showed the existence of the limit set \( \mathcal{Y}_\delta \) of \( \{K(n, \delta)/p^n\} \). Takahashi also investigated the limit set of “\( b \)-state” \( K_b(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1, \ (L^j \delta)(x) = b\} \) for \( j \in \{1, \ldots, p^r - 1\} \).

By defining the metric \( D_f \), we considered the convergence of \( \mathbb{Z}_p \)-valued functions \( \psi_n(\delta) \) on \( \mathbb{R}^d \times [0, 1] \) which corresponds to the values of sites up to the \( p^n \)-th time step of LCA and expresses all the states simultaneously in [3].

In this paper, we extend the result in [3] to the case of mod \( p^r \), where \( p \) is prime and \( r \in \mathbb{N} \). We show that there exists the limit function in the pointwise topology (Theorem 2.3). In Section 3, we define two metric \( d_f, D_f \) in the space USC of \( \mathbb{Z}_p \)-valued upper semi-continuous functions on \( \mathbb{R}^d \times [0, 1] \) and give the result concerning \( d_f \) and \( D_f \) (Theorem 3.1). In Section 4, we investigate the convergence of \( \{\psi_n(\delta)\} \) in these two metrics in the space USC with \( \mathbb{R} \times [0, 1] \). We show that \( \{\psi_n(\delta)\} \) is a Cauchy sequence in the metric \( d_f \) and \( \psi_n(\delta) \) converges to the function \( f_\delta \) in the metric \( D_f \) (Theorem 4.1) and that the similar results hold for any nonzero initial configuration \( a \in \mathcal{P} \) (Theorem 4.14). In Section 5, we consider the relation between the limit function with respect to \( D_f \) and the limit set in the sense of Kuratowski limit. We show that the upper envelope of \( g_a \), which is the limit function of \( \{\psi_n(\delta)\} \) in the pointwise topology, corresponds to the limit sets in the sense of Kuratowski limit and that \( f_\delta \) is the upper envelope of \( g_\delta \) (Theorem 5.2). For a nonzero configuration \( a \in \mathcal{P} \), we show that the upper envelope of \( g_\delta \), which is the limit function of \( \{\psi_n(a)\} \) in the pointwise topology, corresponds to the limit sets in the sense of Kuratowski limit (Theorem 5.3) and this implies that the upper envelope of \( g_\delta \) depends on only the value \( a(0) \). We prove the relation between the upper envelope of \( g_\delta \) and the limit function of \( \{\psi_n(a)\} \) in the metric \( D_f \) and the limit function depends on all values
\( a(x) \ (x \in \mathbb{Z}) \) (Theorem 5.4). This theorem implies that the upper envelope of \( g_a \) is not always equal to the limit function though both are the same in the case of \( \text{mod} \ p \). While the limit function always takes two values in the case of \( \text{mod} \ p \), it occurs the limit function takes more than three values in the case of \( \text{mod} \ p^r \).

2 Convergence in the pointwise topology

We define a \( d \)-dimensional \( p^r \)-state linear cellular automata (LCA) as follows:

Let \( p \) be a prime number and let \( \mathcal{P} \) be the set of all configurations \( a : \mathbb{Z}^d \to \mathbb{Z}_{p^r} \) with compact support. We define \( \delta \in \mathcal{P} \) as

\[
\delta(x) = \begin{cases} 
1 & x = 0 \\
0 & x \neq 0.
\end{cases}
\]

Let \( L : \mathcal{P} \to \mathcal{P} \) mod \( p^r \) be a linear transition rule as follows:

\[
(La)(x) = \sum_{j \in G} \alpha_j a(x + k_j) \quad \text{for} \ a \in \mathcal{P},
\]

(2.1)

where \( G \) is a finite subset of \( \mathbb{Z} \) with \( |G| \geq 2, k_j \in \mathbb{Z}^d \ (j \in G) \) is a neighbouring site of origin, \( \alpha_k \in \mathbb{Z}_{p^r} \backslash \{0\} \) and the summation \( \sum \) is taken as the summation with \( \text{mod} \ p^r \) throughout this paper.

Let

\[
X_n = \{ \left( \frac{x}{p^n}, \frac{t}{p^n} \right) \in \mathbb{R}^d \times [0,1] \mid x \in \mathbb{Z}^d, t \in \mathbb{Z}_+, 0 \leq t \leq p^n \}
\]

for \( n \in \mathbb{Z}_+ \) and put

\[
G_j = \{ \ell \in \mathbb{Z}^d \mid (L^j \delta)(\ell) \neq 0 \}
\]

(2.2)

for \( j \in \mathbb{Z}_+ \).

Define a map \( \psi_n \) from \( \mathcal{P} \) to the function space on \( \mathbb{R}^d \times [0,1] \) for \( a \in \mathcal{P} \) and \( n \in \mathbb{Z}_+ \) by

\[
(\psi_n(a))(\frac{x}{p^n}, \frac{t}{p^n}) = \begin{cases} 
(L^j a)(x) & \text{if } \left( \frac{x}{p^n}, \frac{t}{p^n} \right) \in X_n, \\
0 & \text{if } \left( \frac{x}{p^n}, \frac{t}{p^n} \right) \in \left( \mathbb{R}^d \times [0,1] \right) \backslash X_n
\end{cases}
\]

(2.3)

and a map \( S_{\ell,j} : \mathbb{R}^d \times [0,1] \to \mathbb{R}^d \times [\frac{j}{p}, \frac{j+1}{p}] \) by

\[
S_{\ell,j}(x,t) = \left( \frac{x}{p}, \frac{t}{p} \right) + \left( \frac{\ell}{p^r}, \frac{j}{p} \right).
\]

(2.4)
Figure 1: An example of maps $S_{t,j}$ with $La(x) = a(x - 2) + a(x - 1) + a(x + 1)$ (mod 3).

For a function $g$ on $\mathbb{R}^d \times [0, 1]$, by using maps $S_{t,j}$ define a function $Tg$ on $\mathbb{R}^d \times [0, 1]$ by

$$Tg(y, q) = \sum_{\ell \in G_{jp^q-1}} (L^{jp^q-1} \delta)(\ell) g(S_{t,j}^{-1}(y, q))$$  \hspace{1cm} (2.5)

for $\frac{j}{p} < q \leq \frac{j+1}{p}$ with $0 \leq j \leq p - 1$ and

$$Tg(y, 0) = g(py, 0).$$

**Lemma 2.1.** [5] Let $L$ be a linear cellular automata defined as (2.1) with mod $p^r$. Then for $j, l \in \mathbb{Z}_+$, we have

$$L^{jp^q-1} \delta(x) = \begin{cases} L^{jp^q-1} \delta(y) & \text{if there exists } y \text{ such that } p^l y = x, \\ 0 & \text{otherwise}. \end{cases}$$

We have Lemma 2.2 in a similar way to the case of mod $p$ [3, Lemma 2.3].

**Lemma 2.2.** For $a \in \mathcal{P}$ and $j, n, i \in \mathbb{Z}_+$, we have

$$(L^{jp_{n+r}^{-1}+i} a)(x) = \sum_{\ell \in G_{jp^q-1}} (L^{jp^q-1} \delta)(\ell) (L^i a)(x - \ell p^{n+r-1})$$  \hspace{1cm} (2.6)

Using the above lemmas, we can show the following theorem in a similar way to the case of mod $p$ [3, Theorem 2.5].

**Theorem 2.3.** For $a \in \mathcal{P}$ with $a(0) \neq 0$, we have the following assertions:

1. The sequence $\{\psi_n(a)\}$ converges to a function on $\mathbb{R}^d \times [0, 1]$ in the pointwise topology.
(2) The limit function $g_a$ of the sequence $\{\psi_n(a)\}$ in the pointwise topology is $T$-invariant, that is, $Tg_a = g_a$.

(3) As for the limit functions $g_b$ and $g_a$ of $\{\psi_n(b)\}$ and $\{\psi_n(a)\}$ respectively, we have $a(0)g_b = g_a$.

Proof. (1) For $n \in \mathbb{Z}_+$ satisfying $n > r - 1$, let $X'_n = \{(\frac{x}{p^n}, \frac{j}{p^n}) \mid x \in \mathbb{Z}^d, j = 0, 1, \ldots, p^{n-r+1}\}$. Then we have $\bigcup_{n=1}^{\infty} X'_{n+r-1} = \bigcup_{n=1}^{\infty} X_n$. For $(y, q) \in \mathbb{R}^d \times [0, 1]$ \ \ $\bigcup_{n=1}^{\infty} X_{n+r-1}$,

$$(\psi_n(a))(y, q) = 0$$

by the definition of $\psi_n$.

For $(y, q) \in \bigcup_{n=1}^{\infty} X'_{n+r-1}$, we show there exists $\lim_{n \to \infty}(\psi_n(a))(y, q)$ in the same way as the case of mod $p$. So the sequence $\{\psi_n(a)\}$ converges to a function on $\mathbb{R} \times [0, 1]$ in the pointwise topology.

(2) and (3) are proved in the same way as the case of mod $p$. \hfill \Box

3 The space of $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions

In this section, we shall introduce two metrics $d_f, D_f$ in the space of $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions on a compact subset of $\mathbb{R}^d \times [0, 1]$. Let $USC$ be the space of $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0, 1]$, where $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions mean upper semi-continuous functions embedded in $\mathbb{R}$-valued function spaces. For functions $f, g \in USC$, the order $f \geq g$ is defined by $f(y, q) \geq g(y, q)$ for any $(y, q) \in \mathbb{R}^d \times [0, 1]$ by considering $\mathbb{Z}_{p^r}$ as a subset of $\mathbb{R}$. For functions $\{f_\lambda\}_{\lambda \in \Lambda} \subset USC$ having an upper bound, let

$$g_1(y, q) = \inf\{g(y, q) \mid g \in USC, g \geq f_\lambda \text{ for any } \lambda \in \Lambda\}$$

and

$$g_2(y, q) = \inf\{f_\lambda(y, q) \mid \lambda \in \Lambda\}.$$
Then $g_1$ and $g_2$ belong to $USC$ and $g_1$ is the least upper bound function $\bigvee f_\lambda$ and $g_2$ is the greatest lower bound function $\bigwedge f_\lambda$ in $USC$. So the space $USC$ is an order complete lattice.

Let $K$ be a compact subset of $\mathbb{R}^d \times [0,1]$ and $(y_0, q_0)$ be a point of $(\mathbb{R}^d \times [0,1]) \setminus K$. Let

$$USC|_K = \{ g \in USC \mid \text{support of } g \subseteq K \}.$$  

By using the Hausdorff distance $D(A,B)$ of non-empty compact sets $A$ and $B$ in $\mathbb{R}^d \times [0,1]$, we shall define the pseudodistance $D_0(A,B)$ of $A$ and $B$ in $\mathbb{R}^d \times [0,1]$ by

$$D_0(A,B) = D(A \cup \{(y_0, q_0)\}, B \cup \{(y_0, q_0)\})$$

and metrics $d_f, D_f$ in $USC|_K$ as follows:

$$d_f(g_1, g_2) = \max_{1 \leq i \leq p-1} D_0(\overline{g_1^{-1}(j)}, \overline{g_2^{-1}(j)})$$

$$D_f(g_1, g_2) = \max_{1 \leq s \leq p-1} D_0(g_1^{-1}[s+], g_2^{-1}[s+])$$

for $g_1, g_2 \in USC|_K$, where $g^{-1}[s+] = \{(x, t) \mid g(x, t) \geq s\}$ and $\overline{g^{-1}(j)}$ is the closure of the set $g^{-1}(j) = \{(x, t) \mid g(x, t) = j\}$. It is easy to see that $d_f$ and $D_f$ satisfy the axioms of metric in $USC|_K$. Then we can show the following theorem in a similar way to Theorem 3.5 in [3].

**Theorem 3.1.** For $\{f_n\} \subset USC|_K$, suppose $d_f(f_n, f_m) \to 0$ as $n, m \to \infty$. Let $g = \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} f_n$. Then we have

$$D_f(f_n, g) \to 0 \text{ as } n \to \infty.$$  

Using the metrics $d_f$ and $D_f$, we consider the convergence to the limit set.

4 Convergence of $\psi_n(\delta)$ in case of $\mathbb{R} \times [0,1]$  

In this section, we will consider $\mathbb{Z}_{\omega^r}$-valued upper semi continuous functions on $\mathbb{R} \times [0,1]$ and show $\psi_n(\delta)$ converges to the limit function with respect to the metric $D_f$. We first introduce some notation.
Let \( \alpha_k \) be defined in (2.1) and and suppose \( k_i < k_j (i < j) \) for \( i, j \in G \), which is defined in (2.1). Put
\[
\begin{align*}
k_- &= \min\{ j \mid \alpha_j \neq 0 \text{ for } j \in G \}, \\
k_+ &= \max\{ j \mid \alpha_j \neq 0 \text{ for } j \in G \}
\end{align*}
\]
and
\[
k_0 = k_+ - k_-.
\] (4.1)

For \( j \in \{0, 1, \ldots, p\} \), put
\[
r_j = j + \frac{j(j - 1)p^{r-1}}{2} k_0.
\]

For convenience, we define a map \( S_{\ell}: \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d \times \left[ \frac{j}{p}, \frac{j+1}{p} \right] \), which has the correspondence with some \( S_{\ell, j} \) of (2.4), by
\[
S_{\ell}(x, t) = \left( \frac{x}{p}, \frac{t}{p} \right) + \left( \frac{-j p^{r-1} k_+ + i - 1}{p^r}, \frac{j}{p} \right)
\] (4.2)
with \( \ell = r_j + i (j \in \{0, 1, \ldots, p\}, i \in \{1, 2, \ldots, j p^{r-1} k_0 + 1\}) \) and put
\[
c_{\ell} = L^{j p^{r-1}} \delta(-j p^{r-1} k_+ + i - 1)
\]
and
\[
\Lambda = \{ \ell \in \{1, \ldots, r_p\} \mid c_{\ell} \neq 0 \}.
\]

Then for \( (y, q) \in \mathbb{R} \times [0, 1] \) satisfying \( \frac{j}{p} \leq q \leq \frac{j+1}{p} \) with \( 0 \leq j \leq p - 1 \), we have
\[
(\psi_{n+1}(\delta))(y, q) = \sum_{\ell = r_j + 1}^{r_j + 1} c_{\ell} (\psi_n(\delta))(S_{\ell}^{-1}(y, q)).
\] (4.3)

Let \( X_0 \) be the smallest convex subset of \( \mathbb{R} \times [0, 1] \) containing the support of \( \psi_1(\delta) \), that is,
\[
X_0 = \{(y, q) \in \mathbb{R} \times [0, 1] \mid 0 \leq q \leq 1, -q k_+ \leq y \leq -q k_- \}.
\] (4.4)

Then for any \( n \in \mathbb{Z}_+ \), the support of \( \psi_n(\delta) \) is contained in \( X_0 \) and for \( \ell \in \Lambda \), \( S_{\ell}(X_0) \) is also contained in \( X_0 \). So we consider the space \( \text{USC}_{X_0} \) and the metrics \( d_f, D_f \) in \( \text{USC}_{X_0} \) as in Section 3.
An element \( j \in G \), which is defined in (2.1), is \textit{prime} if \( \alpha_j / p \notin \mathbb{Z}_+ \). In this section, we shall show the following theorem.

**Theorem 4.1.** Let the set \( G \) in (2.1) with \( \text{mod } p^r \) have at least two prime elements. Then we have

1. \( d_f(\psi_n(\delta), \psi_m(\delta)) \to 0 \) as \( n, m \to \infty \).

2. Put \( f_n = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \psi_{n+r-1}(\delta) \), where \( \bigwedge \) and \( \bigvee \) are lattice operations in \( USC \).

Then we have

\[
D_f(\psi_n(\delta), f_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

The way of the proof is a similar to that in the case of \( \text{mod } p \) [3, Theorem 4.1] as shown in the following.

### 4.1 Idea of the proof of Theorem 4.1

In case of \( \text{mod } p \), we proved the lemmas and propositions in [3] by using the property that

\[
(\psi_n(\delta))(y, q) = (\psi_{n+1}(\delta))(y, q) \quad \text{for } (y, q) \in X_n
\]

holds for any \( n \in \mathbb{Z}_+ \). In case of \( \text{mod } p^r \), the equation above does not hold. Therefore we define a function \( H_n \) as follows. For \( n \in \{r, r+1, r+2, \ldots\} \) let

\[
X'_n = \{ (\frac{x}{p^n}, \frac{y j}{p^{n+1}}) \mid x \in \mathbb{Z}, j = 0, 1, \ldots, p^{n-r+1} \}
\]

and

\[
H_n = \psi_n(\delta)1_{X'_n} \quad \text{(see Figure 2). \quad (4.5)}
\]

By Lemma 2.1, we have

\[
H_n(y, q) = H_{n+1}(y, q) \quad \text{for } (y, q) \in X'_n.
\]

In Section 4.2, we shall show

\[
d_f(\psi_n(\delta), H_n) \to 0 \quad \text{(4.6)}
\]

as \( n \to \infty \). Then we shall only show the estimate

\[
d_f(H_{n+1}, H_{m+1}) \leq \frac{1}{p}d_f(H_n, H_m). \quad \text{(4.7)}
\]
\begin{align*}
\psi_3(\delta) & = \psi_3(\delta) \\
H_3 & \equiv \psi_3(\delta) + a(x - 2) + a(x - 1) + a(x + 1) + a(x + 2) \quad \text{(mod } 3^2)\].
\end{align*}

Figure 2: An example $\psi_3(\delta)$ and $H_3$ for $(La)(x) = a(x - 2) + a(x - 1) + a(x + 1) + a(x + 2) \quad \text{(mod } 3^2)$. 

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The inequality (4.7) can easily be verified if \( \{(S_{\ell}(X_0))^\circ\}_{\ell \in \Lambda} \) is mutually disjoint, where \( (S_{\ell}(X_0))^\circ \) is the interior of \( S_{\ell}(X_0) \). However, the equation (4.7) is not easily obtained if \( \{(S_{\ell}(X_0))^\circ\}_{\ell \in \Lambda} \) are mutually overlapped. Just as in the case of mod \( p \), we introduce an auxiliary quantity \( M^{n,n'}_0 \) and show the following estimates:

M-1) \( d_f(H_{n+1}, H_{n'+1}) \leq \frac{1}{p} M^{n,n'}_0 \) (Proposition 4.11);

M-2) \( M^{n+1,n'+1}_0 \leq \frac{1}{p} M^{n,n'}_0 \) (Proposition 4.12).

In order to define \( M^{n,n'}_0 \), we use two divisions \( \{E_s\} \) and \( \{A_{k,j,s}\} \) of \( X_0 \) and functions \( \{h^n_s\} \).

### 4.2 Relation between \( H_n \) and \( \psi_n(\delta) \)

We shall prove the following proposition in this section.

**Proposition 4.2.** For the pseudodistance \( D_0 \) on \( \mathbb{R} \times [0,1] \) and \( \psi_n \in USC|_{X_0} \), we have

\[
D_0(H_n^{-1}[s+],(\psi_n(\delta))^{-1}[s+]) \to 0 \text{ as } n \to \infty \text{ for } s \in \{1, \ldots, p^r - 1\}
\]

and

\[
D_0(H_n^{-1}(j),(\psi_n(\delta))^{-1}(j)) \to 0 \text{ as } n \to \infty \text{ for } j \in \{1, \ldots, p^r - 1\}.
\]

In order to show Proposition 4.2, we need the following

**Proposition 4.3.** For a prime number \( p \) and \( r \in \mathbb{N} \), let \( L \) be defined as (2.1) and the set \( G \) has at least two prime elements. Put \( t(r,j) = j(p^r - p^{r-1}) \) and \( i(r,j) = -(t(r,j) - p^{r-1}k_{j_2} - p^{r-1}k_{j_1}) \), where \( j_1 \) is maximum prime in \( G \) and \( j_2 \) is the maximum prime element next to \( j_1 \) in \( G \) and \( k_j \) is defined in (2.1).

When \( j \) ranges from 1 to \( p^r \), \( L^{t(r,j)}\delta(i(r,j)) \) ranges from 0 to \( p^r - 1 \).

**Proof.** By using the following Lemma 4.4, we can prove in a similar way to that of Theorem 2.7 in [2].

\[\square\]
Lemma 4.4. Suppose \( r \geq 2 \) and the set \( G \) has at least two prime elements. Then
\[
L^{(r,j)}i(r,j)) \equiv L^{(r,m)}i(r,m)) + sL^{(r,p^{-1})}i(r,p^{-1})) \pmod{p^r}
\]
holds for \( j = sp^{r-1} + m \) with \( s \in \{0,1,\ldots,p-1\} \) and \( m \in \{1,2,\ldots,p^{-1}\} \).

We have already proved a similar result to Lemma 4.4 and Proposition 4.3 when we supposed condition (A) in [2]. In this paper, we suppose that the set \( G \) has at least two prime elements instead of condition (A).

In order to verify Lemma 4.4, we need Lemma 4.5, 4.6 and 4.7.

Lemma 4.5 ([2], Lemma 2.2). Suppose \( q \in \mathbb{N} \) with \( q/p \notin \mathbb{N} \), \( t = jp^{r-1} \) with \( j \in \mathbb{N} \) \( v = p^lq \) with \( l \in \{0,1,\ldots,r-2\} \) and \( v < t \).

Then there exists \( q' \in \mathbb{N} \) with \( q'/p \notin \mathbb{N} \) such that \( t - v = p^lq' \).

Lemma 4.6. Put \( a_{a+b}C_a = (a+b)!/(a!b!) \). Then
\[ p^{r-\rho^{-1}C_p^j \equiv 0 \pmod{p^r}} \]
for \( i \in \{1,2,\ldots,p^r-p^{-1}\} \).

Proof. Suppose \( i = qp^j \) with \( q \in \{1,2,\ldots,p-1\} \) and \( \ell \in \{0,1,2,\ldots,r-1\} \). There exists \( b \in \mathbb{N} \) such that \( b/p \notin \mathbb{N} \) and \( p^{r-\rho^{-1}C_p^i} = bp^{r-\ell - \rho^{-1}C_p^q} \) by Lemma 4.5. Since \( r - 1 = \ell + qp^j \geq r \) holds, we obtain the conclusion.

Put \( m_0 = zG \). In order to show the following lemmas, we first note that the value \((L\delta)(x)\) is expressed by
\[
(L\delta)(x) = \sum_{u_1,\ldots,u_{m_0}} \frac{t!}{u_1!\cdots u_{m_0}!} \alpha_{i_1}^{u_1} \cdots \alpha_{i_{m_0}}^{u_{m_0}} \pmod{p^r}, \tag{4.8}
\]
where the summation is taken over \((u_1,\ldots,u_{m_0})\) such that \( u_1 + \cdots + u_{m_0} = t \) and \(-k_i u_1 - \cdots - k_{i_{m_0}} u_{m_0} = -x\). We also recall the relation
\[
\frac{t!}{u_1!\cdots u_{m_0}!} = C_i u_1 \times t - u_1 C_{u_2} \times \cdots \times t - \sum_{i=1}^{m_0} C_{u_{m_0}}
\]
and an element \( j \in G \) is prime if \( \alpha_j/p \notin \mathbb{N} \).
Lemma 4.7. Let the set $G$ in (2.1) have at least two prime elements. Suppose that $j_1$ is maximum prime in $G$ and that $j_2$ is the maximum prime element next to $j_1$ in $G$. Then

$$L^{s(p^r-p^{r-1})} \delta(-k_{j_1}s(p^r-p^{r-1}) + \ell) \equiv 0 \pmod{p^r}$$  \hspace{1cm} (4.9)

for $s \in \{1, 2, \ldots, p-1\}$ and $\ell \in \{1, 2, \ldots, k_{j_1} - k_{j_2} - 1\}$ (see Figure 3).

Proof. We note $k_{j_1}$ is not expressed as a convex linear combination of other $k_j$, where $j \in G$ is prime. If there exists the path from $-k_{j_1}s(p^r-p^{r-1}) + \ell$ to the origin with $s(p^r-p^{r-1})$ time steps, then there exist $n_0 \in \mathbb{Z}_+$, $\{i_j \in \mathbb{N}\}_{j=1}^{n_0}$ and $\{m_j \in G\}_{j=1}^{n_0}$ such that

$$\sum_{j=1}^{n_0} i_j = s(p^r-p^{r-1})$$  \hspace{1cm} (4.10)

and

$$-k_{j_1}s(p^r-p^{r-1}) + \ell = -\sum_{j=1}^{n_0} i_j k_{m_j}.$$  \hspace{1cm} (4.11)

Suppose $m_j$ is prime for all $j \in \{1, \ldots, n_0\}$. From (4.10) and (4.11), $\ell = \sum_{j=1}^{n_0} i_j (k_{j_1} - k_{m_j})$ holds and there exists $j' \in \{1, \ldots, n_0\}$ such that $k_{m_{j'}} \leq k_{j_2}$. By $i_j \geq 1$, we obtain $\ell \geq k_{j_1} - k_{j_2}$, which contradicts $\ell \in \{1, 2, \ldots, k_{j_1} - k_{j_2} - 1\}$.

Therefore there exists $j \in \{1, 2, \ldots, n_0\}$ such that $m_j$ is not prime. The equation (4.9) holds by Lemma 4.6 and (4.8). \qed
When the set $G$ in (2.1) has at least two prime elements, suppose that $j_1$ is maximum prime in $G$ and that $j_2$ is the maximum prime element next to $j_1$ in $G$. Using $k_{j_1}$ and $k_{j_2}$, put

$$t(r, j) = j(p^r - p^{r-1})$$  \hspace{1cm} (4.12)$$

and

$$i(r, j) = -(t(r, j) - p^{r-1})k_{j_1} - p^{r-1}k_{j_2}$$ \hspace{1cm} (4.13)$$

for $j \in \mathbb{N}$ (see Figure 4).

**Proof of Lemma 4.4.**

When we compute $L^{t(r, j)}\delta(i(r, j))$ from the values at time $t(r, j - 1)$, we need the values $L^{t(r, j - 1)}\delta(x)$ with $x \in \{i(r, j) + k_-(p^r - p^{r-1}), \ldots, i(r, j) + k_+(p^r - p^{r-1})\}$ (see Figure 5). We note that the value $L^{t(r, j - 1)}\delta(x)$ with $x \in \{i(r, j) + k_-(p^r - p^{r-1}), \ldots, -k_{j_1}t(r, j - 1) - 1\}$ is a multiple of $p$ by (4.8) and that the path from $i(r, j - 1) + \ell$ to $i(r, j)$ for any $\ell \in \{1, \ldots, k_+(p^r - p^{r-1})\}$ with $p^r - p^{r-1}$ time steps needs at least one $k_j$, where $j \in G$ is not prime. Therefore by Lemma 4.6, the values $L^{t(r, j - 1)}\delta(x)$ with $x \in \{i(r, j) + k_-(p^r - p^{r-1}), \ldots, -k_{j_1}t(r, j - 1) - 1\} \cup \{i(r, j - 1) + 1, \ldots, i(r, j) + k_+(p^r - p^{r-1})\}$ do not effect $L^{t(r, j)}\delta(i(r, j))$. So we
Figure 5: A region which effects the value $L^{(r,j)}\delta(i(r,j))$. However we can ignore the squared regions by Lemma 4.6.

We have

$$L^{(r,j)}\delta(i(r,j)) \equiv p^r - p^{r-1} C_{p^{r-1}} \alpha_j^{p^r-2p^{r-1}} \alpha_j \alpha^{p^r-1}_{j_2} \rho^{-1}(k_{j_1} - k_{j_2})^{-1} \sum_{\ell=1} B(r, \ell)b(r, j - 1, \ell)$$

$$+ \alpha_j^{p^r-1} L^{(r,j-1)}\delta(i(r,j-1)) \pmod{p^r}, \quad (4.14)$$

where $b(r, j, \ell) = L^{(r,j)}\delta(-t(r,j)k_{j_1} + \ell)$ and $B(r, \ell) \in \mathbb{N}$. $B(r, \ell)$ is the number of the path from $-t(r,j)k_{j_1} + \ell$ to $i(r,j)$ with $p^r - p^{r-1}$ time steps, and the number of the path from $-t(r,j)k_{j_1} + \ell$ to $i(r,j)$ with $p^r - p^{r-1}$ time steps is the same as that from $-t(r,j')k_{j_1} + \ell$ to $i(r,j')$ with $p^r - p^{r-1}$ time steps for any $j, j' \in \mathbb{Z}$.

So $B(r, \ell)$ does not depend on $j$.

Using (4.14) and Lemma 4.7, we can show the conclusion in the same way as the case that $L$ satisfies the condition (A). \hfill \Box

**Proof of Proposition 4.2.**

Suppose $n \in \{r, r + 1, r + 2, \ldots\}$. We have

$$D_0(H_n^{-1}[s^+], (\psi_n(\delta))^{-1}[s^+]) \leq \max_{s \leq j \leq p^r - 1} D_0(H_n^{-1}(j), (\psi_n(\delta))^{-1}(j)).$$
So we prove $D_0(H_n^{-1}(j), (\psi_n(\delta))^{-1}(j)) \to 0$ for all $j \in \{1, 2, \ldots, p^r - 1\}$ as $n \to \infty$.

Put

$$D_{0,r}(A, B) = \sup\{d(A \cup \{(y_0, q_0)\}, y) \mid y \in B \cup \{(y_0, q_0)\}\},$$
$$D_{0,\ell}(A, B) = \sup\{d(x, B \cup \{(y_0, q_0)\}) \mid x \in A \cup \{(y_0, q_0)\}\}$$

for compact sets $A$ and $B$. Then we have

$$D_0(H_n^{-1}(j), (\psi_n(\delta))^{-1}(j)) = \max\{D_{0,r}(H_n^{-1}(j), (\psi_n(\delta))^{-1}(j)), D_{0,\ell}(H_n^{-1}(j), (\psi_n(\delta))^{-1}(j))\},$$

and

$$D_{0,r}(H_n^{-1}(j), (\psi_n(\delta))^{-1}(j)) = 0$$

for all $n$ by the definition of $H_n$. So we will show for any $\epsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that

$$D_{0,r}(H_n^{-1}(j), (\psi_n(\delta))^{-1}(j)) < \epsilon$$

for $n > N$.

Put

$$t_0 = p^r(p^r - p^{r-1}) + 1,$$
$$\ell_0 = \min\{\ell \in \mathbb{Z}_+ \mid t_0(k_+ - k_-) < p^r - 1\} \text{ (see Figure 6 (a))}$$

and

$$K(i) = \{x \in \mathbb{Z} \mid L^{i_0p^{r-1+\ell}} \delta(x) \neq 0\} \text{ for } i \in \mathbb{Z}_+.\]$$

For $n > r - 1 + \ell_0$ and $x = (x/p^n, t/p^n) \in X_n \cap X_0$, put

$$K_x = \{x' \in K(i_0) \mid 0 < t - i_0p^{r-1+\ell_0} \leq p^{r-1+\ell_0},$$

$$-k_+(t - i_0p^{r-1+\ell_0}) \leq x - x' \leq -k_-(t - i_0p^{r-1+\ell_0})\}.$$

Then we have

$$\left(\psi_n(\delta)(x/p^n, t/p^n)\right) = \begin{cases} 
\sum_{x' \in K_x} L^{i_0p^{r-1+\ell_0}} \delta(x') L^{-i_0p^{r-1+\ell_0}}(x - x') & K_x \neq \emptyset, \\
0 & K_x = \emptyset 
\end{cases} \quad (4.15)$$

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(a) $\ell_1 = p\ell_0 - 1$, $\ell_2 = |k_1 \ell_0|$ and $\ell_3 = |k_2 \ell_0|$. The regions $A$ and $B$ is disjoint by the definition of $\ell_0$.

(b) $\ell_5 = m_0 p\ell_0 + 1$, $\ell_6 = |k_1 p^{-1} \ell_0|$ and $\ell_7 = |k_2 p^{-1} \ell_0|$.

Figure 6: The sketch of space-time pattern of LCA.
for \( n > r - 1 + \ell_0 \). Put
\[
m_n = \max \{ z | x = (x/p^n, t/p^n) \in X_0 \cap X_0 \}
\] (4.16)
for \( n > r - 1 + \ell_0 \). By the definition of \( K_x \), it is easy to show that there exists \( m_0 \in \mathbb{Z}_+ \) such that \( m_n < m_0 \) for all \( n > r - 1 + \ell_0 \). Put
\[
M = \max \{ k_0 p^{r-1+\ell_0} + m_0 p^{\ell_0} + 1, \sqrt{(|k_+| p^{r-1+\ell_0} + m_0 p^{\ell_0} + 1)^2 + p^{2(r-1+\ell_0)}}, \sqrt{(|k_-| p^{r-1+\ell_0} + m_0 p^{\ell_0} + 1)^2 + p^{2(r-1+\ell_0)}} \} \) (see Figure 6 (b)),
where \( k_0, k_+ \) and \( k_- \) are defined as (4.1). For any \( \epsilon > 0 \), we choose \( N > r - 1 + \ell_0 \) satisfying
\[
\epsilon > M/p^{N+r-1}.
\]
Put \( U_r(x) = \{ y \in X_n \cap X_0 | d(x, y) < \epsilon \} \). Suppose \( n > N \) and \( x = (x/p^n, t/p^n) \in X_n \cap X_0 \). If \( t = 0 \), then \( H_n(x) = (\psi_n(\delta))(x) \). So we consider the case of \( t > 0 \).

For \( i \in \mathbb{Z}_+ \) satisfying \( ip^{r-1+\ell_0} < t \leq (i + 1)p^{r-1+\ell_0} \), suppose \( L^{ip^{r-1+\ell_0}}(x') = 0 \) for all \( (x'/p^n, ip^{r-1+\ell_0}/p^n) \in U_r(x) \). Then \( (\psi_n(\delta))(x) = 0 \).

Suppose \( L^{ip^{r-1+\ell_0}}(x_0)/p \notin \mathbb{N} \) for some \( (x_0/p^n, ip^{r-1+\ell_0}/p^n) \in U_r(x) \). Then we have \( H_n^{-1}(k) \cap U_r(x) \neq \emptyset \) for all \( k \in \{ 0, 1, \ldots, p^{r-1} - 1 \} \) by Proposition 4.3.

Suppose \( L^{ip^{r-1+\ell_0}}(x)/p \in \mathbb{N} \) for all \( (x'/p^n, ip^{r-1+\ell_0}/p^n) \in U_r(x) \). Put \( h_0 = \min \{ h | L^{ip^{r-1+\ell_0}}(x') = kp^h \} \) for all \( k \in \{ 1, 2, \ldots, p^{r-1} \} \) and \( k/p \notin \mathbb{N} \). We have \( H_n^{-1}(kp^{h_0}) \cap U_r(x) \neq \emptyset \) for all \( k \in \{ 0, 1, \ldots, p^{r-1} - 1 \} \) by Proposition 4.3. In the other hand, there exists \( k \in \{ 0, 1, \ldots, p^{r-1} - 1 \} \) such that \( (\psi_n(\delta))(x) = kp^{h_0} \) by (4.15).

So we obtain \( D_{0, \epsilon}(H_n^{-1}(j), (\psi_n(\delta))^{-1}(j)) < \epsilon \) for \( n > N \). \( \square \)

4.3 The definitions of \( \{ E_\gamma \} \) and \( \{ A_{b, j, s} \} \)

We shall divide \( X_0 \) into subsets \( \{ E_\gamma \} \) and \( \{ A_{b, j, s} \} \) as follows (see Figure 7 and 8). Let
\[
\Gamma = \{(1, j, s) | 1 \leq s \leq p^e k_0, 1 \leq j \leq s\} \cup \{(2, j, s) | 2 \leq s \leq p^e k_0, 1 \leq j \leq s - 1\}.
\]

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We define \( \{ E^r_\gamma \}(\gamma \in \Gamma) \) as follows:

In case of \( \gamma = (1, j, s) \in \Gamma \), let

\[
E^r_\gamma = \{(y, q) \mid \frac{s-1}{p^r k_0} \leq q \leq \frac{s}{p^r k_0}, -k_+q + \frac{j-1}{p^r} \leq y \leq -k_-q - \frac{s-j}{p^r}\};
\]
in case of \( \gamma = (2, j, s) \in \Gamma \)

\[
E^r_\gamma = \{(y, q) \mid \frac{s-1}{p^r k_0} \leq q \leq \frac{s}{p^r k_0}, -k_-q - \frac{s-j}{p^r} \leq y \leq -k_+q + \frac{j}{p^r}\}.
\]

Let for \( 1 \leq s \leq k_0 \) and \( 1 \leq j \leq s \),

\[
A^r_{1, j, s} = \{(y, q) \mid \frac{s-1}{p^{r-1} k_0} \leq q \leq \frac{s}{p^{r-1} k_0}, -k_+q + \frac{j-1}{p^{r-1}} \leq y \leq -k_-q - \frac{s-j}{p^{r-1}}\}
\]

and for \( 2 \leq s \leq k_0 \) and \( 1 \leq j \leq s - 1 \),

\[
A^r_{2, j, s} = \{(y, q) \mid \frac{s-1}{p^{r-1} k_0} \leq q \leq \frac{s}{p^{r-1} k_0}, -k_-q - \frac{s-j}{p^{r-1}} \leq y \leq -k_+q + \frac{j}{p^{r-1}}\}.
\]

Then we have the following properties.

**Proposition 4.8.** (1) The sets \( \{ E^r_\gamma \} \) have the following properties.

- **E-1** For \( \gamma = (b, j, s), \gamma' = (b', j', s) \in \Gamma \), \( E^r_\gamma \) is the shift of \( E^r_{\gamma'} \) in the first coordinate direction for any \( s \) and \( b \in \{1, 2\} \).

- **E-2** \( (E^r_\gamma)^{\circ} \cap (E^r_{\gamma'})^{\circ} = \emptyset \) if \( \gamma \neq \gamma' \).

- **E-3** If \( (S_\ell(X_0))^{\circ} \cap (S_{\ell'}(X_0))^{\circ} \neq \emptyset \), then \( S_\ell(X_0) \cap S_{\ell'}(X_0) \) is the union of some \( E^r_\gamma \)'s.

- **E-4** \( X_0 = \bigcup_{\gamma \in \Gamma} E^r_\gamma \).

(2) The sets \( \{ A^r_{b, j, s} \}_{b, j, s} \) have the following properties.

- **A-1** For any \( A^r_{b, j, s} \), there exist \( \gamma \in \Gamma \) and \( \ell \in \{1, \ldots, r_p\} \) such that \( A^r_{b, j, s} = S^\ell_\ell^{-1}(E^r_\gamma) \).

- **A-2** \( X_0 = \bigcup_{b=1}^{k_0} \bigcup_{j=1}^{k_0} A^r_{b, j, s} \).

- **A-3** \( (A^r_{b, j, s})^{\circ} \cap (A^r_{b', j', s'})^{\circ} = \emptyset \) if \( (b, j, s) \neq (b', j', s') \).

**Proof.** By the definition, we can easily get the result. □
Figure 7: $\{E_\gamma\}_\gamma$ for $(La)(x) = a(x - 2) + a(x - 1) + a(x + 1) + a(x + 2) \pmod{3^2}$
Figure 8: \{A_{b,j,s}\}_{b,j,s} for \((La)(x) = a(x - 2) + a(x - 1) + a(x + 1) + a(x + 2) \pmod{3^2}\)
4.4 The definition of \( \{ h^n_v \} \) and their fundamental properties

We shall define the function \( h^n_v \) as follows. Put

\[
V = \{ v = (\gamma_1, \ldots, \gamma_m) \mid m \in \mathbb{Z}_+, \gamma_1 \in \Gamma \text{ with } \# \Lambda_{\gamma_1} \geq 1, \gamma_k \in \Gamma \text{ with } \# \Lambda_{\gamma_k} \geq 1 \text{ and } S_{c_{k-1}}(E_{\gamma_k}^r) \subset E_{\gamma_k-1}^r \text{ for any } k \in \{2, \ldots, m\} \}.
\]

For \( v = (\gamma_1, \ldots, \gamma_m) \in V \) and \( n \in \mathbb{Z}_+ \), define

\[
h^n_v(y, q) = \sum_{\ell_1 \in \Lambda_{\gamma_1}} \cdots \sum_{\ell_m \in \Lambda_{\gamma_m}} c_{\ell_1} \cdots c_{\ell_m} \times H_n(S_{c_{\ell_m}}^{-1}, \ldots, S_{c_{\ell_1}}^{-1}, S_{\gamma_1}(y, q))(y, q)
\]

(4.17)

for \((y, q) \in \mathbb{R} \times [0, 1] \).

When \( v = (\gamma) \), \( h^n_v \) satisfies

\[
h^n_v(y, q) = \sum_{\ell \in \Lambda_{\gamma}} c_{\ell} H_n(S_{c_{\ell}}^{-1}, S_{\gamma}(y, q))(y, q),
\]

and

\[
h^n_v(S_{c_{\ell}}^{-1}(y, q)) = H_n(y, q)1_{c_{\ell}}(y, q)
\]

for \( n \in \mathbb{Z}_+ \). Since the length of \( v \) is one, \( h^n_v \) has the relation with \( H_{n+1} \).

If the length of \( v \) is \( m \), then \( h^n_v \) has the relation with \( H_{n+m} \) and this is useful in estimating the metric \( d_f(h^n_v, h^{n'}_v) \) as shown in the following lemma.

**Lemma 4.9.** For \( v = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in V, k \in \{1, \ldots, m\} \) and \((y, q) \in \mathbb{R} \times [0, 1] \), put

\[
F_k(y, q) = S_{c_{\gamma_k}}(S_{c_{\gamma_{k-1}}}(\ldots(S_{c_{\gamma_1}}(y, q))\ldots)).
\]

Then we have

1. For \((y, q) \in F_{m-1}(E_{\gamma_m}^r) \),

\[
h^n_v(F_{m-1}^{-1}(y, q)) = H_{n+m}(y, q)1_{F_{m-1}(E_{\gamma_m}^r)}(y, q)
\]

and

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(2) if the sets \( \{ j \in \{1, \ldots, p - 1 \} \mid (h^n_v)^{-1}(j) = \emptyset \} \) and \( \{ j \in \{1, \ldots, p - 1 \} \mid (h^{n'}_v)^{-1}(j) = \emptyset \} \) are the same, then
\[
d_f(h^n_v, h^{n'}_v) = p^m d_f(H_{n+m} \downarrow F_{n-1}(\mathcal{E}_{m}^n), H_{n'+m} \downarrow F_{n-1}(\mathcal{E}_{m}^{n'}))
\]
for any \( n, n' \in \mathbb{Z}_+ \).

Proof. The proof is a similar to that of Lemma 4.3 in [3]. \( \square \)

In a similar way to Proposition 4.4 in [3], we can show the following proposition, which means that the sets \( \{ j \in \{1, \ldots, p - 1 \} \mid (h^n_v)^{-1}(j) = \emptyset \} \) and \( \{ j \in \{1, \ldots, p - 1 \} \mid (h^{n'}_v)^{-1}(j) = \emptyset \} \) are the same for sufficiently large \( n, n' \).

**Proposition 4.10.** For sufficiently large \( n \in \mathbb{Z}_+ \), the following assertions are equivalent for any \( v = (\gamma_1, \ldots, \gamma_m) \in V, \ell \in \mathbb{Z}_r \setminus \{0\} \).

1. \( (H_{n+m}^\ell)^{-1}(\ell) \cap (F_{m-1}(E_{\gamma_m}^r))^o \neq \emptyset \).
2. \( (H_{n+m+1})^{-1}(\ell) \cap (F_{m-1}(E_{\gamma_m}^r))^o \neq \emptyset \).

### 4.5 The definition of \( \{ M_{0}^{n,n'} \} \) and their properties

By using \( h^n_v \), we shall define \( M_{0}^{n,n'} \) by
\[
M_{0}^{n,n'} = \sup \{ d_f(h^n_v, h^{n'}_v) \mid v \in V \}.
\]

Then we have the following

**Proposition 4.11.** (1) \( \sup \{ M_{0}^{n,n'} \mid n, n' \in \mathbb{Z}_+ \} < \infty \).

(2) \( d_f(H_{n+1}, H_{n'+1}) \leq \frac{1}{p} M_{0}^{n,n'} \) holds for sufficiently large \( n, n' \in \mathbb{Z}_+ \).

Proof. By using Lemma 4.9 and Proposition 4.10, we get the conclusion in a similar way to [3, Proposition 4.5]. \( \square \)

**Proposition 4.12.** For sufficiently large \( n, n' \), we have
\[
M_{0}^{n+1,n'+1} \leq \frac{1}{p} M_{0}^{n,n'}.
\]

Proof. The proof is a similar to that of Proposition 4.6 in [3]. \( \square \)
4.6 Proof of Theorem 4.1

By using above propositions, we shall prove Theorem 4.1.

(1) By Propositions 4.11 (2) and 4.12, we have

$$\lim_{n,m \to \infty} M_{0,n,m}^{n,m} = 0.$$ 

By Proposition 4.11 (1), we have

$$d_f(H_{n+1}, H_{m+1}) \leq \frac{1}{p} M_{0,n,m}^{n,m}.$$ 

Since we have $d_f(H_n, \psi_n(\delta)) \to 0$ as $n \to \infty$ by Proposition 4.2, we obtain the conclusion.

(2) Since $\{\psi_n(\delta)\} \subseteq \text{USC}_X$ we get the result from (1) and Theorem 3.1. $\square$

4.7 Convergence of $\psi_n(a)(a \in \mathcal{P})$ in case of $\mathbb{R} \times [0, 1]$

We consider convergence of $\psi_n(a)(a \in \mathcal{P})$ in a similar way to $\psi_n(\delta)$. We define a function $H'_n$ as follows. For $n \in \{r, r+1, r+2, \ldots\}$ let $X'_n = \{(\frac{x}{p^n}, \frac{y}{p^n}) \mid x \in \mathbb{Z}, j = 0, 1, \ldots, p^{n-r+1}\}$ and

$$H'_n = \psi_n(a)1_{x'_n}.$$  \hspace{1cm} (4.18)

Then we can show the following theorem in a similar way to the proof of Proposition 4.2.

Proposition 4.13. For the pseudodistance $D_0$ on $\mathbb{R} \times [0, 1]$, we have

$$D_0(H_n^{-1}[s+], (\psi_n(a))^{-1}[s+]) \to 0 \text{ as } n \to \infty \text{ for } s \in \{1, \ldots, p^r - 1\}$$

and

$$D_0(H_n^{-1}(j), (\psi_n(a))^{-1}(j)) \to 0 \text{ as } n \to \infty \text{ for } j \in \{1, \ldots, p^r - 1\}.$$ 

So we can show the following theorem in a similar way to Theorem 4.1.

Theorem 4.14. Let the set $G$ in (2.1) with mod $p^r$ have at least two prime elements. For a nonzero $a \in \mathcal{P}$, we have
(1) \( d_f(\psi_n(a), \psi_m(a)) \to 0 \) as \( n, m \to \infty \).

(2) Put \( f_a = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \psi_{n+r-1}(a) \), where \( \bigwedge \) and \( \bigvee \) are lattice operations in USC. Then we have

\[
D_f(\psi_n(a), f_a) \to 0 \quad \text{as} \quad n \to \infty.
\]

Proof. (1) For \( H'_n \), we can show the following relation in a similar way to the proof of \( \psi_n(a) \) in case of mod \( p \).

\[
d_f(H'_n, H'_m) \to 0
\]
as \( n, m \to \infty \). So by Proposition 4.13, we have (1).

(2) We get the result from (1) and Theorem 3.1. \( \square \)

5 The relation between the limit function and the limit set

In this section, we investigate the relation between the limit function and the limit set of \( \{K^f(n, \delta)/p^n\}_n \), which Takahashi defined in [5]. Put

\[
K^f(n, \delta) = \{(x, t) \in \mathbb{Z} \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1, \ L^f(x) \equiv 0 \pmod{p^f}\}
\]
for \( f \in \{1, 2, \ldots, r\} \) and

\[
K_b(n, \delta) = \{(x, t) \in \mathbb{Z} \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1, \ L^f(x) \equiv b \pmod{p^f}\}
\]
for \( b \in \{1, 2, \ldots, p^r - 1\} \).

Then the following lemma holds.

**Lemma 5.1.** [5] Let \( L \) be defined as (2.1) with mod \( p^r \) and suppose that at least two elements of \( G \) is prime and \( f \in \{1, \ldots, r\} \). Then for \( b \in \mathbb{Z}_{p^r} \) satisfying \( b/p^r-1 \in \mathbb{N} \) and \( b/p^r \notin \mathbb{N} \), we have

\[
\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} K_b(n, \delta)/p^n = \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} K^f(n, \delta)/p^n = \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} K^f(n, \delta)/p^n = \bigcup_{k=1}^{\infty} K_b(n, \delta)/p^n.
\]

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We first show the relation between $Y_f = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} K^f(n, \delta)/p^n$ (Theorem 5.2) and $\lim_{n \to \infty} \psi_n(\delta)$.

Let $\hat{g}$ be the upper envelope of $g$, that is,

$$\hat{g}(x, t) = \inf\{\phi(x, t) | \phi \in \text{USC}, \phi(x, t) \geq g(x, t)\}.$$  

Then the limit function $g_\delta$ in the pointwise topology (Theorem 2.3) has the relation with a limit set in the sense of Kuratowski limit.

**Theorem 5.2.** Suppose the set $G$ in (2.1) has at least two prime elements. Let the function $g_\delta$ be defined by $g_\delta(y, q) = \lim_{n \to \infty} (\psi_n(\delta))(y, q)$.

Then

$$\hat{g}_\delta = \sum_{1 \leq f \leq r} (p^{r+1-f} - 1)p^{f-1}y_1 \cup \cdots \cup y_r$$  \hspace{1cm} (5.1)

and

$$\hat{g}_\delta = \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(\delta).$$  \hspace{1cm} (5.2)

**Proof.** For $f \in \{1, 2, \ldots, r\}$, let $(y, q) \in Y_f \cup \bigcup_{i=1}^{f-1} Y_i$. Then there exists a sequence\{(y_{n_j}, q_{n_j}) \in K^f(n_j, \delta)/p^{n_j}\}_{j=1}^{\infty}$ such that $\lim_{j \to \infty} n_j = \infty$ and $\lim_{j \to \infty} (y_{n_j}, q_{n_j}) = (y, q)$. Since $g_\delta(y_{n_j}, q_{n_j}) \neq 0$, there exists a sequence $\{(y'_{n_j}, q'_{n_j}) \in K^f(n_j, \delta)/p^{n_j}\}$ such that $g_\delta(y'_{n_j}, q'_{n_j}) = (p^{r+1-f} - 1)p^{f-1}$ and $\lim_{j \to \infty} (y'_{n_j}, q'_{n_j}) = (y, q)$. So $\hat{g}_\delta(y, q) = (p^{r+1-f} - 1)p^{f-1}$. If $(y, q) \notin Y_f$ for all $f \in \{1, 2, \ldots, r\}$, then there exists a neighborhood $U$ of $(y, q)$ and $k$ such that $U \cap K^f(n, \delta)/p^n = \emptyset$ for any $n \geq k$. So $\hat{g}_\delta(y, q) = 0$. Therefore we obtain the equation (5.1).

In order to verify (5.2), we will show

$$\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} (\psi_{n+r-1}(\delta))(y, q) =$$  \hspace{1cm} (5.3)

$$\left\{ \begin{array}{ll}
(p^{r+1-f} - 1)p^{f-1} & \text{for } (y, q) \in Y_f \setminus \bigcup_{i=1}^{f-1} Y_i \text{ with } f \in \{1, \ldots, r\}, \\
0 & \text{otherwise.}
\end{array} \right.$$

The equation $\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} (\psi_{n+r-1}(\delta))(y, q) = (p^{r+1-f} - 1)p^{f-1}$ holds if and only if
(i) for any \( k \in \mathbb{Z}_+ \) and \( \epsilon > 0 \) there exist \((y', q') \in \mathbb{R} \times [0, 1]\) and \( n' \geq k \) such that 
\[ |(y', q') - (y, q)| < \epsilon \] 
and 
\[ (\psi_{n'+r-1}(\delta))(y', q') > (p^{r+1-f-1})p^{f-1} - \epsilon \]

(ii) for any \( \epsilon > 0 \) there exists \( k \in \mathbb{N} \) and a neighborhood \( U \) of \((y, q)\) such that 
\[ (\psi_{n'+r-1}(\delta))(y', q') < (p^{r+1-f-1})p^{f-1} + \epsilon \]
for all \( n \geq k \) and all \((y', q') \in U\).

For \( f \in \{1, 2, \ldots, r\} \) and \( b \in \mathbb{N} \) satisfying \( b/p^{f-1} \in \mathbb{N} \) and \( b/p^f \notin \mathbb{N} \), let \((y, q) \in Y_f\). Then for any \( \epsilon > 0 \) there exists \( \{(y_n, q_n) \in K_{b}(n, \delta)/p^{n}\}_{n \in \mathbb{Z}_+} \) such that 
\[ |(y_n, q_n) - (y, q)| < \epsilon \]
by the definition of \( Y_f \) and Lemma 5.1. If \((y, q) \notin \bigcup_{i=1}^{f-1} Y_i\), then for each \( i \in \{1, \ldots, f\} \), there does not exist a sequence 
\[ \{(y_n, q_n) \in K_{b}(n, \delta)/p^{n}\}_{n=1}^{\infty} \]
converging to \((y, q)\), where \( b/p^{f-1-i} \in \mathbb{N} \) and \( b/p^i \notin \mathbb{N} \). By using the fact above, we obtain (5.3). \( \square \)

**Theorem 5.3.** Suppose that the set \( G \) in (2.1) has at least two prime elements. 
For \( a \in \mathbb{P} \) with \( a(0) = kp^l \) for \( k/p \notin \mathbb{Z}_+ \) and \( l \in \{0, 1, \ldots, r-1\} \). Put \( g_a(y, q) = \lim_{n \to \infty}(\psi_n(a))(y, q) \).

Then 
\[ \hat{g}_a = \sum_{1 \leq i \leq r-l} (p^r - p^{r-1+i})1_{Y_f \setminus \bigcup_{i=1}^{r-l} Y_i} \] \quad (5.4)

**Proof.** For \((y, q) \in Y_f \setminus \bigcup_{i=1}^{f-1} Y_i\), there exists a sequence \( \{(y_{n_j}, q_{n_j}) \in K_f(n_j, \delta)/p^{n_j}\}_{j=1}^{\infty} \) such that \( \lim_{j \to \infty} n_j = \infty \), \( \lim_{j \to \infty}(y_{n_j}, q_{n_j}) = (y, q) \) and 
\[
g_a(y_{n_j}, q_{n_j}) = a(0)g_a(y_{n_j}, q_{n_j}) = kp^{r+f-1}\]
for \( 1 \leq b \leq p^{r-f+1} - 1 \) and \( b/p \notin \mathbb{Z}_+ \) by Lemma 5.1 and Theorem 2.3 (3). We have 
\[ \{bk^{p^f-1} \pmod{p^r} \mid 1 \leq b \leq p^{r-f+f+1}\} = \{bp^{f-1} \pmod{p^r} \mid 1 \leq b \leq p^{r-l-f+1}\} \]
by \( k/p \in \mathbb{Z}_+ \). So there exists \( b \in \{1, \ldots, p^{f+r-1}\} \) such that 
\[ kb^{p^{f+r-1}} \equiv p^r - p^{f+l-1} \pmod{p^r} \].

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Therefore there exists a sequence \( \{(y'_{n_j}, q'_{n_j}) \in K^f(n_j, \delta)/p^{n_j}\}_{j=1}^{\infty} \) such that
\[
\lim_{j \to \infty} n_j = \infty, \\
\lim_{j \to \infty} (y'_{n_j}, q'_{n_j}) = (y, q)
\]
and
\[
g_a(y'_{n_j}, q'_{n_j}) = p^r - p^{f-1+l}.
\]

There exists a neighborhood \( U \) of \( (y, q) \) such that \( g_a(y', q') \leq p^r - p^{f-1+l} \) for all \( (y', q') \in U \) by \( (y, q) \notin \cup_{i=1}^{f-1} Y_i \). So \( g_a(y, q) = p^r - p^{f-1+l} \).

If \( (y, q) \notin Y_f \) for all \( f \in \{1, 2, \ldots, r\} \), then there exists a neighborhood \( U \) of \( (y, q) \) and \( k \) such that \( U \cap K^f(n, a)/p^n = \emptyset \) for any \( n \geq k \). So \( g_a(y, q) = 0 \). Therefore we obtain the conclusion. \( \Box \)

For \( a = a(x) \in \mathcal{P} \), put
\[
G_a = \{ x \in \mathbb{Z} \mid a(x) \neq 0 \}.
\]

Let \( \tau_x : \mathcal{P} \to \mathcal{P} \) be a shift operator such that
\[
\tau_x a(y) = a(y - x).
\]

The following theorem shows the relation between \( \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a) \) and \( g_a \) in Theorem 2.3. While the upper envelope of \( g_a \) depends on only the value \( a(0) \),
\[
\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a) \text{ depends on all values } a(x)(x \in \mathbb{Z}).
\]
So \( g_a \) is not necessarily equal to \( \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a) \).

**Theorem 5.4.** Suppose that the set \( G \) in (2.1) has at least two prime elements. Suppose that \( a \in \mathcal{P} \) is nonzero and put \( g_a(y, q) = \lim_{n \to \infty} (\psi_n(a))(y, q) \). Then
\[
\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a) = \bigvee_{x \in G_a} g_{\tau_x(a)}.
\]

Proof. Let \( l_x \in \mathbb{Z}_+ \) satisfy \( \tau_x(a)(0) = kp^x(k/p \notin \mathbb{Z}_+) \) and \( x_0 \in \mathbb{Z} \) satisfy \( l_x \leq l_{x_0} \) for all \( x \in \mathbb{Z} \). Since we have
\[
\hat{g}_{\tau_{x_0}(a)} = \bigvee_{x \in G_a} g_{\tau_x(a)} = \sum_{1 \leq f \leq \tau - l_{x_0}} (p^r - p^{f-1+l_{x_0}})1_{Y_f \setminus \cup_{j=1}^{f-2} Y_i},
\]

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by \( \hat{g}_{\tau(a)} = \sum_{1 \leq f \leq r-t_0} (p^r - p^{r-1+t_0}) Y_{f \cup i_{g-1}} \), we shall show

\[
\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a) = \sum_{1 \leq f \leq r-t_0} (p^r - p^{r-1+t_0}) Y_{f \cup i_{g-1}}.
\]

In order to verify it, we show

\[
\bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} \psi_{n+r-1}(a) = \begin{cases} p^r - p^{r-1+t_0} & (y, q) \in Y_f \cup i_{g-1} Y_i, \\ 0 & \text{otherwise}. \end{cases} \tag{5.5}
\]

For any \( n \in \mathbb{Z}_+ \) and \( (y, q) \in \mathbb{R} \times [0, 1] \), the equation \( \psi_{n+r-1}(\tau(a))(y, q) = \psi_{n+r-1}(a)(y - x/p^{n+r-1}, q) \) holds. Using the relation above, we can show the equation (5.5) in a similar way to the proof of the equation (5.2) in Theorem 5.2. \( \square \)

References


