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Combination Matrices and its Application

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Abstract

We study the lower triangular matrices, especially Pascal, Stirling and the sum of powers matrices. Considering these matrices as the linear transformations on 1-variable polynomials, we reveal widely the close relations between them and also with the Vandermonde matrices.

1 Introduction

Based on the previous works [1, 3, 4], we study extensively the lower triangular matrices of the combination numbers, such as Pascal matrices, the first and second Stirling matrices, and the matrices of the coefficients of the sum of powers functions. These matrices are closely related to each other as linear transformations of the space of n-dimensional 1-variable polynomial functions, and are therefore related to Vandermonde matrices. Based on this observation, we propose a general and clear treatment of these matrices with many results and examples.

We introduce the basic notation of matrices and vectors used in this paper, in § 1. For a lower triangular matrix $A = (a_{ij})$, the idea of parametarization $A(r) := (r^{i-j}a_{ij})$ (§ 2) gives us many useful information and properties. Using the idea on the bases $\{1, x, x^2, \cdots, x^{n-1}\}$, we can derive basic useful results and give the precise proofs of well-known properties for well-known lower triangular matrices: Pascal (§ 3), Stirling (§ 4) and also Vandermonde matrices (§ 4).

In § 5, it is shown that the parametarized matrices of the sum of powers have rich connections with Pascal matrices. We also have a construction of Vandermonde matrices using the second Stirling matrices, that gives us the clear visualization of the fact that the determinant of n-dim. Vandermonde matrix is divisible by $\prod_{i=0}^{n-1} i!$ [2].

For educational purposes, we give many examples, hoping many teachers and students will enjoy these topics. We also have several plans to develop the related topics.

2 Basic notation

In this paper, for the convenience and clarity of description, we propose the notation ${}^1_m[a_{ij}]^n$ for the $m \times n$ matrix $(a_{ij})_{(1 \le i \le m,\ 1 \le j \le n)}$, and ${}^0_{m-1}[a_{ij}]^{n-1}$ for $(a_{ij})_{(0 \le i \le m-1,\ 0 \le j \le n-1)}$. Therefore, ${}^1_m[a_{i-1,j-1}]^n$ is denoted by ${}^0_{m-1}[a_{ij}]^{n-1}$. For example,

$$ {}^1_3[a_{ij}]^2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \text{ and } {}^1_4[(i-1)(j-1)]^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{pmatrix} = {}^0_3[ij]^2. $$

The transpose of a matrix A is denoted by ${}^tA$. Therefore, ${}^t\left({}^1_m[a_{ij}]^n\right) = {}^1_n[a_{ji}]^m$.

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Square matrices are represented as $A_n = {}_{n-1}^{\,0}[a_{ij}]^{n-1}$ or $B_m = {}_{m}^{1}[b_{ij}]^{m}$. Here, we use the subscript n of A to indicate the dimension of A, and also for B. For example,

$$A_3 = {}_{2}^{0}[a_{ij}]^2 = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}, \quad B_3 = {}_{3}^{1}[b_{ij}]^3 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

We define $z\left({}_{n-1}^{\,0}[a_{ij}]^{n-1}\right) := {}_{n-1}^{\,1}[a_{ij}]^{n-1}$, i.e., $z(A)$ is the matrix obtained by deleting the first column and the first row of the square matrix A. For example,

$$z\left(\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad z^2\left(\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}\right) = z\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = (a_{22}).$$

For a relational expression $p(i,j)$ of indices i,j of a matrix, we define $\delta_{p(i,j)} := \begin{cases} 1 & \text{if } p(i,j) \\ 0 & \text{if } \neg p(i,j) \end{cases}$.

Examples 2.1 of $p(i,j)$

1. Let $p(i,j)$ be $i=j$. $\delta_{i=j} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$ ${}_{2}^{0}[\delta_{i=j}]^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. Let $p(i,j)$ be $i \geq j$. $\delta_{i\geq j} = \begin{cases} 1 & (i \geq j) \\ 0 & (i < j) \end{cases}$ ${}_{3}^{1}[\delta_{i\geq j}]^3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$$(-1)^{i+j}\delta_{i\geq j} = \begin{cases} (-1)^{i+j} & (i \geq j) \\ 0 & (i < j) \end{cases} \quad {}_{3}^{0}[(-1)^{i+j}\delta_{i\geq j}]^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

3. Let $p(i,j)$ be $i-j=1$. $\delta_{i-j=1} = \begin{cases} 1 & (i-j=1) \\ 0 & (i-j \neq 1) \end{cases}$ ${}_{3}^{1}[\delta_{i-j=1}]^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

The n-dim. identity matrix E_n is defined by $E_n := {}_{n}^{1}[\delta_{i=j}]^n$. Let $O_{mn} := {}_{m}^{1}[0]^n$. The direct sum of A_m and B_n is defined by $A_m \oplus B_n := \begin{pmatrix} A_m & O_{mn} \\ O_{nm} & B_n \end{pmatrix}$.

For n-dim. column vectors, omitting the right-superscript of our matrix notation, we use the notation ${}_{n-1}^{\,0}[a_i]$ or ${}_{n}^{1}[a_i]$. Similarly, we use ${}^{0}[a_i]^{n-1}$ or ${}^{1}[a_i]^n$ for n-dim. row vectors. For example,

$${}_{2}^{0}[a_i] = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \quad {}_{3}^{1}[a_i] = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad {}^{0}[a_i]^2 = (a_0, a_1, a_2), \quad {}^{1}[a_i]^3 = (a_1, a_2, a_3).$$

Let x^i be the x to the ith power. Then ${}_{n}^{1}[x^i] = {}_{n-1}^{\,0}[x^{i+1}] = x\,{}_{n-1}^{\,0}[x^i]$. For example,

$${}_{3}^{1}[x^i] = \begin{pmatrix} x \\ x^2 \\ x^3 \end{pmatrix} = {}_{2}^{0}[x^{i+1}] = x\ {}_{2}^{0}[x^i] = x\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}.$$

For a n-dim. column vector $\boldsymbol{u}$ and a m-dim. column vector $\boldsymbol{v}$, $\boldsymbol{u} \downarrow \boldsymbol{v}$ is defined to be a $(n+m)$-dim. column vector which is arranged vertically $\boldsymbol{u}$ and $\boldsymbol{v}$, i.e., ${}_{n}^{1}[a_j] \downarrow {}_{m}^{1}[a_{n+\ell}] = {}_{n+m}^{\ \ 1}[a_i]$. Therefore, ${}_{n-1}^{\,0}[x^j] = E_1 \downarrow {}_{n-1}^{\,1}[x^i]$.

Examples 2.2

1. $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \downarrow \begin{pmatrix} a_3 \\ a_4 \\ a_5 \end{pmatrix} = {}_{2}^{1}[a_j] \downarrow {}_{3}^{1}[a_{2+\ell}] = {}_{5}^{1}[a_i] = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$

2. $\,_2^0[x^j] = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} = (1) \downarrow \begin{pmatrix} x \\ x^2 \end{pmatrix} = E_1 \downarrow \,_2^1[x^i]$

We divert our notation for row-vectors to represent lists. Thus, the n variable function $f(x_1, x_2, \cdots, x_n)$ is denoted by $f\big({}^1[x_i]^n\big)$ and $f(x_0, x_1, x_2) = f\big({}^0[x_i]^2\big)$.

3 Lower triangular matrices

The set of n-dim. real lower triangular matrices is closed under the usual sum and product of matrices, and scalar product. A lower triangular matrix is regular if and only if each of its diagonal component is non-zero.

Since $a_{ij}\delta_{i\geq j} = \begin{cases} a_{ij} & (i \geq j) \\ 0 & (i < j) \end{cases}$, any n-dim. lower triangular matrix A_n is represented as $A_n = \,_{n-1}^{0}\big[a_{ij}\delta_{i\geq j}\big]^{n-1}$. For example, $A_3 = \,_2^0\big[a_{ij}\delta_{i\geq j}\big]^2 = \begin{pmatrix} a_{00} & 0 & 0 \\ a_{10} & a_{11} & 0 \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$.

In the same way, we define $A_n(r) := \,_{n-1}^{0}\big[r^{i-j}a_{ij}\delta_{i\geq j}\big]^{n-1}$ where $r^{i-j}a_{ij}\delta_{i\geq j} = \begin{cases} a_{ii} & (i = j) \\ r^{i-j}a_{ij} & (i > j) \\ 0 & (i < j) \end{cases}$.

For example, $A_4(r) = \,_3^0\big[r^{i-j}a_{ij}\delta_{i\geq j}\big]^3 = \begin{pmatrix} a_{00} & 0 & 0 & 0 \\ ra_{10} & a_{11} & 0 & 0 \\ r^2a_{20} & ra_{21} & a_{22} & 0 \\ r^3a_{30} & r^2a_{31} & ra_{32} & a_{33} \end{pmatrix}$.

Lemma 3.1 Let A_n, B_n be n-dim. lower triangular matrices.

1. $A_n(1) = A_n$
2. $A_n(0) = \,_{n-1}^{0}\big[a_{ii}\delta_{i=j}\big]^{n-1}$ is diagonal
3. $(A_n + B_n)(r) = A_n(r) + B_n(r)$
4. $\big(A_n(q)\big)(r) = A_n(qr)$
5. $\big(cA_nB_n\big)(r) = cA_n(r)B_n(r) \qquad (c \in \mathbb{R})$
6. $\big(A_n^{-1}\big)(r) = \big(A_n(r)\big)^{-1}$
7. $A_n(r) = O_{nn}\ (\forall r \in \mathbb{R}) \iff A_n = O_{nn}$

Proof. We prove 5 and 6.

5. $A_n(r)B_n(r) = \,_{n-1}^{0}\big[r^{i-j}\ a_{ij}\ \delta_{i\geq j}\big]^{n-1}\,_{n-1}^{0}\big[r^{i-j}\ b_{ij}\ \delta_{i\geq j}\big]^{n-1} = \,_{n-1}^{0}\big[\sum_{k=0}^{n-1} r^{i-k}\ a_{ik}\ \delta_{i\geq k}\ r^{k-j}\ b_{kj}\ \delta_{k\geq j}\big]^{n-1}$

Since $\sum_{k=0}^{n-1}\delta_{i\geq k}\ \delta_{k\geq j} = \sum_{k=j}^{i}\delta_{i\geq j}$, we have $A_n(r)B_n(r) = \,_{n-1}^{0}\big[r^{i-j}\sum_{k=j}^{i}a_{ik}b_{kj}\ \delta_{i\geq j}\big]^{n-1} = (A_nB_n)(r)$.

6. Putting $B_n = A_n^{-1}$ in 5, we have $A_n(r)\big(A_n^{-1}(r)\big) = \big(A_nA_n^{-1}\big)(r) = E(r) = E$, and $A_n^{-1}(r) = \big(A_n(r)\big)^{-1}$. □

By this lemma, for the equations composed of n-dim. lower triangular matrices $A(r), B(r), \ldots,$ it is enough to show the case $r = 1$.

Examples 3.1 Let $f(x, y, z) = xy+3z$. For n-dim. lower triangular matrices A, B, C and $s, t, u \in \mathbb{R}$, we have the followings for any $r \in \mathbb{R}$.

1. $f\big(A(s), B(t), C(u)\big) = A(s)B(t) + 3C(u)$
2. $f\big(A(sr), B(tr), C(ur)\big) = A(sr)B(tr) + 3C(ur) = \big(A(s)B(t) + 3C(u)\big)(r) = f\big(A(s), B(t), C(u)\big)(r)$
3. $f\big(A(sr), B(tr), C(ur)\big) = O \iff f\big(A(s), B(t), C(u)\big) = O$

4 Pascal matrices

The Pascal matrix is defined by $P_n := {}^{0}_{n-1}\left[\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\right]^{n-1} = {}^{0}_{n-1}[{}_iC_j]^{n-1}$, then $P_n(a) = {}^{0}_{n-1}\left[a^{i-j}\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\right]^{n-1}$ $(a \in \mathbb{R})$. For example,

$$P_4 = {}^{0}_{3}\left[\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\right]^{3} = \begin{pmatrix} {}_0C_0 & 0 & 0 & 0 \\ {}_1C_0 & {}_1C_1 & 0 & 0 \\ {}_2C_0 & {}_2C_1 & {}_2C_2 & 0 \\ {}_3C_0 & {}_3C_1 & {}_3C_2 & {}_3C_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \quad P_4(a) = {}^{0}_{3}\left[a^{i-j}\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)\right]^{3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 3a & 1 \end{pmatrix}.$$

Lemma 4.1 For the Pascal matrix P_n, we have the followings.

1. $P_n(a)\ {}^{0}_{n-1}[x^i] = {}^{0}_{n-1}\big[(x+a)^i\big]$ 2. $P_n(0) = E_n$

3. $P_n(a+b) = P_n(a)\ P_n(b)$ 4. $P_n(a)^{-1} = P_n(-a)$ 5. $z(P_{n+1})(a)\ {}^{1}_{n}[x^i] = {}^{1}_{n}\big[(x+a)^i - a^i\big]$

Then $\Big\{P_n(a) | a \in \mathbb{R}\Big\}$ is a one-parameter group. We omit the proof of Lemma 4.1, since it is easy to see from the following Examples 4.1. Similarly, we often omit proofs that are easy to see from examples.

Examples 4.1

1. $P_4(a)\ {}^{0}_{3}[x^i] = {}^{0}_{3}\big[(x+a)^i\big]$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 3a & 1 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ x+a \\ (x+a)^2 \\ (x+a)^3 \end{pmatrix}$

2. $P_4(a+b) = P_4(a)\ P_4(b)$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ a+b & 1 & 0 & 0 \\ (a+b)^2 & 2(a+b) & 1 & 0 \\ (a+b)^3 & 3(a+b)^2 & 3(a+b) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 3a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ b^2 & 2b & 1 & 0 \\ b^3 & 3b^2 & 3b & 1 \end{pmatrix}$

3. $P_4(a)^{-1} = P_4(-a)$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 3a & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ a^2 & -2a & 1 & 0 \\ -a^3 & 3a^2 & -3a & 1 \end{pmatrix}$

4. $z(P_5)(a)\ {}^{1}_{4}[x^i] = {}^{1}_{4}\big[(x+a)^i - a^i\big]$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2a & 1 & 0 & 0 \\ 3a^2 & 3a & 1 & 0 \\ 4a^3 & 6a^2 & 4a & 1 \end{pmatrix}\begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} (x+a)-a \\ (x+a)^2-a^2 \\ (x+a)^3-a^3 \\ (x+a)^4-a^4 \end{pmatrix}$

We define $d_n := {}^{0}_{n-1}\big[\delta_{i=j} + \delta_{i-j=1}\big]^{n-1}$ and $D_n := {}^{0}_{n-1}\big[\delta_{i\geq j}\big]^{n-1}$. For example,

$$d_4 = {}^{0}_{3}\big[\delta_{i=j} + \delta_{i-j=1}\big]^{3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D_4 = {}^{0}_{3}\big[\delta_{i\geq j}\big]^{3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Lemma 4.2 1. $d_n(a)\ D_n(-a) = E_n$ 2. $d_n(0) = D_n(0) = E_n$

Examples 4.2 $d_4(a)D_4(-a) = E_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ a^2 & -a & 1 & 0 \\ -a^3 & a^2 & -a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lemma 4.3

1. $i\binom{i-1}{j} = (i-j)\binom{i}{j}$
2. ${}^{0}_{n-1}\left[(x+1)^i\right] = E_1 \downarrow (x+1)\ {}^{0}_{n-2}\left[(x+1)^i\right]$
3. $d_n(1)\ {}^{0}_{n-1}[x^i] = E_1 \downarrow (x+1)\ {}^{0}_{n-2}[x^i]$
4. $z(D_{n+1})(a) = D_n(a)$
5. $z(d_{n+1})(a) = d_n(a)$

Proof. Since $\frac{i!}{(i-j-1)!} = i\frac{(i-1)!}{(i-j-1)!} = (i-j)\frac{i!}{(i-j)!}$ and $i\frac{(i-1)!}{j!(i-j-1)!} = (i-j)\frac{i!}{j!(i-j)!}$, then we obtain 1. □

Examples 4.3

1. ${}^{0}_{3}\left[(x+1)^i\right] = E_1 \downarrow (x+1)\ {}^{0}_{2}\left[(x+1)^i\right]$

$$\begin{pmatrix} 1 \\ x+1 \\ (x+1)^2 \\ (x+1)^3 \end{pmatrix} = (1) \downarrow (x+1)\begin{pmatrix} 1 \\ x+1 \\ (x+1)^2 \end{pmatrix}$$

2. $d_4(1)\ {}^{0}_{3}[x^i] = E_1 \downarrow (x+1)\ {}^{0}_{2}[x^i]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ x+1 \\ x(x+1) \\ x^2(x+1) \end{pmatrix} = (1) \downarrow (x+1)\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

3. $z(D_5) = D_4$

$$z\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

4. $z(d_5) = d_4$

$$z\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Theorem 4.1 ([1], [3])

1. $P_n(a) = \exp\left(a\ {}^{0}_{n-1}\left[i\delta_{i-j=1}\right]^{n-1}\right)$
2. $P_n(a) = \left(E_1 \oplus P_{n-1}(a)\right)\ d_n(a) = \begin{pmatrix} 1 & {}^{1}[0]^{n-1} \\ {}^{1}_{n-1}[a^i] & P_{n-1}(a)\ d_{n-1}(a) \end{pmatrix} = \prod_{k=n-2}^{0}\left(E_k \oplus d_{n-k}(a)\right)$
3. $P_n(a) = D_n(a)\ \left(E_1 \oplus P_{n-1}(a)\right) = \begin{pmatrix} 1 & {}^{1}[0]^{n-1} \\ {}^{1}_{n-1}[a^i] & D_{n-1}(a)\ P_{n-1}(a) \end{pmatrix} = \prod_{k=0}^{n-2}\left(E_k \oplus D_{n-k}(a)\right)$
4. $D_{n-1}(a)\ P_{n-1}(a) = z(P_n)(a) = P_{n-1}(a)\ d_{n-1}(a)$

Proof. 1. We will first prove $AP_n(a) = \frac{d}{da}P_n(a)$ for $A = {}^{0}_{n-1}\left[i\delta_{i-j=1}\right]^{n-1}$.

$AP_n(a) = {}^{0}_{n-1}\left[i\delta_{i-j=1}\right]^{n-1}\ {}^{0}_{n-1}\left[a^{i-j}\binom{i}{j}\right]^{n-1} = {}^{0}_{n-1}\left[\sum_{k=0}^{n-1} i\delta_{i-k=1}\ a^{k-j}\binom{k}{j}\right]^{n-1}$ and for $k = i-1$, we have $AP_n(a) = {}^{0}_{n-1}\left[ia^{i-1-j}\binom{i-1}{j}\right]^{n-1}$. Since $i\binom{i-1}{j} = (i-j)\binom{i}{j}$,

$AP_n(a) = {}^{\;\;\;0}_{n-1}\left[(i-j)a^{i-j-1}\begin{pmatrix} i \\ j \end{pmatrix}\right]^{n-1} = \dfrac{d}{da}P_n(a)$. Then we obtain 1 from $P_n(0) = E_n$.

2. We will show the case $a = 1$. Since $P_n(1)\ {}^{\;\;\;0}_{n-1}[x^i] = {}^{\;\;\;0}_{n-1}\left[(x+1)^i\right] = E_1 \downarrow (x+1)\ {}^{\;\;\;0}_{n-2}\left[(x+1)^i\right]$,

$\big(E_1 \oplus P_{n-1}(-1)\big)P_n(1)\ {}^{\;\;\;0}_{n-1}[x^i] = \big(E_1 \oplus P_{n-1}(-1)\big)\big(E_1 \downarrow (x+1)\ {}^{\;\;\;0}_{n-2}\left[(x+1)^i\right]\big) = E_1 \downarrow (x+1)\ {}^{\;\;\;0}_{n-2}[x^i]$

$= d_n(1)\ {}^{\;\;\;0}_{n-1}[x^i]$, we obtain $\big(E_1 \oplus P_{n-1}(1)^{-1}\big)P_n(1) = d_n(1)$ and $P_n(1) = \big(E_1 \oplus P_{n-1}(1)\big)\ d_n(1)$.

Next, $P_n(1) = \big(E_1 \oplus P_{n-1}(1)\big)\ d_n(1) = \Big(E_1 \oplus \big(E_1 \oplus P_{n-2}(1)\big)d_{n-1}(1)\Big)\ d_n(1)$

$= \big(E_2 \oplus P_{n-2}(1)\big)\big(E_1 \oplus d_{n-1}(1)\big)\ d_n(1)$ and for $P_2(1) = d_2(1)$, we obtain 2, i.e.,

$P_n(1) = \big(E_{n-2} \oplus d_2(1)\big)\big(E_{n-3} \oplus d_3(1)\big) \cdots \big(E_1 \oplus d_{n-1}(1)\big)\ d_n(1)$.

3. We are able to put from 2, $P_n(-1)^{-1} = P_n(1) = D_n(1)\ \big(E_1 \oplus P_{n-1}(1)\big)$

$= D_n(1)\big(E_1 \oplus D_{n-1}(1)\big) \cdots \big(E_{n-3} \oplus D_3(1)\big)\big(E_{n-2} \oplus D_2(1)\big)$. Then we obtain 3 for $P_2(1) = D_2(1)$. □

Examples 4.4

1. $P_4(a) = \exp\left(a\ {}^{0}_{3}[\delta_{i-j=1}]^3\right)$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 3a & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \end{pmatrix}$

2. $P_5(a) = \big(E_1 \oplus P_4(a)\big)\ d_5(a) = \big(E_2 \oplus P_3(a)\big)\big(E_1 \oplus d_4(a)\big)\ d_5(a)$
$\quad = \big(E_3 \oplus P_2(a)\big)\big(E_2 \oplus d_3(a)\big)\big(E_1 \oplus d_4(a)\big)\ d_5(a)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ a^2 & 2a & 1 & 0 & 0 \\ a^3 & 3a^2 & 3a & 1 & 0 \\ a^4 & 4a^3 & 6a^2 & 4a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & a^2 & 2a & 1 & 0 \\ 0 & a^3 & 3a^2 & 3a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & a^2 & 2a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & a & 1 \end{pmatrix}$$

3. $P_5(a) = D_5(a)\ \big(E_1 \oplus P_4(a)\big) = D_5(a)\ \big(E_1 \oplus D_4(a)\big)\big(E_2 \oplus P_3(a)\big)$
$\quad = D_5(a)\ \big(E_1 \oplus D_4(a)\big)\big(E_2 \oplus D_3(a)\big)\ \big(E_3 \oplus P_2(a)\big)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ a^2 & 2a & 1 & 0 & 0 \\ a^3 & 3a^2 & 3a & 1 & 0 \\ a^4 & 4a^3 & 6a^2 & 4a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ a^2 & a & 1 & 0 & 0 \\ a^3 & a^2 & a & 1 & 0 \\ a^4 & a^3 & a^2 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & a^2 & 2a & 1 & 0 \\ 0 & a^3 & 3a^2 & 3a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ a^2 & a & 1 & 0 & 0 \\ a^3 & a^2 & a & 1 & 0 \\ a^4 & a^3 & a^2 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & a^2 & a & 1 & 0 \\ 0 & a^3 & a^2 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & a^2 & 2a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ a^2 & a & 1 & 0 & 0 \\ a^3 & a^2 & a & 1 & 0 \\ a^4 & a^3 & a^2 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 \\ 0 & a^2 & a & 1 & 0 \\ 0 & a^3 & a^2 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & a^2 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a & 1 \end{pmatrix}$$

4. $D_4(a)\ P_4(a) = z(P_5)(a) = P_4(a)\ d_4(a)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & a & 1 & 0 \\ a^3 & a^2 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 3a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2a & 1 & 0 & 0 \\ 3a^2 & 3a & 1 & 0 \\ 4a^3 & 6a^2 & 4a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 2a & 1 & 0 \\ a^3 & 3a^2 & 3a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \end{pmatrix}$$

5 Stirling matrices and Vandermonde matrices

We define k-dim. polynomial of variable x as $x^{k;a} := \prod_{j=0}^{k-1}(x+ja)$ and $x^{0;a} := 1$. For example, $x^{4;a} = \prod_{j=0}^{3}(x+ja) = x(x+a)(x+2a)(x+3a)$. The $n \times n$ matrix s_n defined by $s_n\ {}^{1}_{n}[x^i] = {}^{1}_{n}[x^{i;1}]$ is called *the first Stirling matrix*, and the $n \times n$ matrix S_n defined by $S_n\ {}^{1}_{n}[x^{i;-1}] = {}^{1}_{n}[x^i]$ is called *the second Stirling matrix*. For example,

$$s_4\ {}^{1}_{4}[x^i] = {}^{1}_{4}[x^{i;1}] \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{pmatrix} \begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} x \\ x(x+1) \\ x(x+1)(x+2) \\ x(x+1)(x+2)(x+3) \end{pmatrix}$$

$$S_4\ {}^{1}_{4}[x^{i;-1}] = {}^{1}_{4}[x^i] \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} x \\ x(x-1) \\ x(x-1)(x-2) \\ x(x-1)(x-2)(x-3) \end{pmatrix} = \begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}.$$

Lemma 5.1

1. $s_n(a)\ {}^{1}_{n}[x^i] = {}^{1}_{n}[x^{i;a}]$
2. $\big(E_1 \oplus s_{n-1}(a)\big)\ {}^{0}_{n-1}[x^i] = {}^{0}_{n-1}[x^{i;a}]$
3. $S_n(a)\ {}^{1}_{n}[x^{i;-a}] = {}^{1}_{n}[x^i]$
4. $\big(E_1 \oplus S_{n-1}(a)\big)\ {}^{0}_{n-1}[x^{i;-a}] = {}^{0}_{n-1}[x^i]$
5. $s_n(a)\ S_n(-a) = E_n$

Examples 5.1

1. $s_4(a)\ {}^{1}_{4}[x^i] = {}^{1}_{4}[x^{i;a}]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 2a^2 & 3a & 1 & 0 \\ 6a^3 & 11a^2 & 6a & 1 \end{pmatrix} \begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} x \\ x(x+a) \\ x(x+a)(x+2a) \\ x(x+a)(x+2a)(x+3a) \end{pmatrix}$$

2. $\big(E_1 \oplus s_3\big)\ {}^{0}_{3}[x^i] = {}^{0}_{3}[x^{i;a}]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 2a^2 & 3a & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ x(x+a) \\ x(x+a)(x+2a) \end{pmatrix}$$

3. $S_4(a)\ {}^1_4[x^{i;-a}] = {}^1_4[x^i]$ $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 3a & 1 & 0 \\ a^3 & 7a^2 & 6a & 1 \end{pmatrix} \begin{pmatrix} x \\ x(x-a) \\ x(x-a)(x-2a) \\ x(x-a)(x-2a)(x-3a) \end{pmatrix} = \begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}$$

4. $\big(E_1 \oplus S_3\big)\ {}^0_3[x^{i;-a}] = {}^0_3[x^i]$ $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & a^2 & 3a & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x(x-a) \\ x(x-a)(x-2a) \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}$$

5. $s_4(1)\ S_4(-1) = E_4$ $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -1 & 7 & -6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$s_4(-1)\ S_4(1) = E_4$ $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lemma 5.2

1. ${}^1_n[x^{i;1}] = x\ {}^0_{n-1}[(x+1)^{i;1}]$
2. ${}^1_n[x^{i;-1}] = x\ {}^0_{n-1}[(x-1)^{i;-1}]$
3. $S_n(1)\ {}^0_{n-1}[(x-1)^{i;-1}] = {}^0_{n-1}[x^i]$
4. $S_n(1)\ {}^0_{n-1}[j^{i;-1}] = {}^0_{n-1}[(j+1)^i]$ for any j
5. $s_n(1)\ {}^0_{n-1}[x^i] = {}^0_{n-1}[(x+1)^{i;1}]$
6. $s_n(1)\ {}^0_{n-1}[(j+1)^i] = {}^0_{n-1}[(j+2)^{i;1}]$ for any j

Examples 5.2

1. ${}^1_4[x^{i;1}] = x\ {}^0_3[(x+1)^{i;1}]$ $$\begin{pmatrix} x \\ x(x+1) \\ x(x+1)(x+2) \\ x(x+1)(x+2)(x+3) \end{pmatrix} = x \begin{pmatrix} 1 \\ x+1 \\ (x+1)(x+2) \\ (x+1)(x+2)(x+3) \end{pmatrix}$$

2. ${}^1_4[x^{i;-1}] = x\ {}^0_3[(x-1)^{i;-1}]$ $$\begin{pmatrix} x \\ x(x-1) \\ x(x-1)(x-2) \\ x(x-1)(x-2)(x-3) \end{pmatrix} = x \begin{pmatrix} 1 \\ x-1 \\ (x-1)(x-2) \\ (x-1)(x-2)(x-3) \end{pmatrix}$$

3. $S_4(1)\ {}^0_3[(x-1)^{i;-1}] = {}^0_3[x^i]$ $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x-1 \\ (x-1)(x-2) \\ (x-1)(x-2)(x-3) \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}$$

4. $S_4(1)\ {}^0_3[j^{i;-1}] = {}^0_3[(j+1)^i]$ $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ j \\ j(j-1) \\ j(j-1)(j-2) \end{pmatrix} = \begin{pmatrix} 1 \\ j+1 \\ (j+1)^2 \\ (j+1)^3 \end{pmatrix}$$

5. $s_4(1)\ {}^0_3[x^i] = {}^0_3[(x+1)^{i;1}]$ $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ x+1 \\ (x+1)(x+2) \\ (x+1)(x+2)(x+3) \end{pmatrix}$$

6. $s_4(1)\ {}^0_3[(j+1)^i] = {}^0_3[(j+2)^{i;1}]$ $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ j+1 \\ (j+1)^2 \\ (j+1)^3 \end{pmatrix} = \begin{pmatrix} 1 \\ j+2 \\ (j+2)(j+3) \\ (j+2)(j+3)(j+4) \end{pmatrix}$$

Theorem 5.1 ([4])

1. $$s_n(a) = \big(E_1 \oplus s_{n-1}(a)\big)P_n(a) = \begin{pmatrix} 1 & {}^1[0]^{n-1} \\ {}^1_{n-1}[i!a^i] & s_{n-1}(a)\ z(P_n)(a) \end{pmatrix} = \prod_{k=n-2}^{0} \big(E_k \oplus P_{n-k}(a)\big)$$

2. $z(s_n)(a) = s_{n-1}(a)\ z(P_n)(a)$

3. $$S_n(a) = P_n(a)\big(E_1 \oplus S_{n-1}(a)\big) = \begin{pmatrix} 1 & {}^1[0]^{n-1} \\ {}^1_{n-1}[a^i] & z(P_n)(a)\ S_{n-1}(a) \end{pmatrix} = \prod_{k=0}^{n-2} \big(E_k \oplus P_{n-k}(a)\big)$$

4. $z(S_n)(a) = z(P_n)(a)\ S_{n-1}(a)$

Proof. 1. Since $s_n(1)\,{}^{1}_{n}[x^i] = {}^{1}_{n}[x^{i;1}]$ and ${}^{1}_{n}[x^{i;1}] = x\,{}^{0}_{n-1}[(x{+}1)^{i;1}]$, we have $\big(E_1{\oplus}S_{n-1}(-1)\big)\, s_n(1)\,{}^{1}_{n}[x^i]$

$$= \big(E_1 \oplus S_{n-1}(-1)\big)\ x\ {}^{0}_{n-1}\big[(x+1)^{i;1}\big] = x\ {}^{0}_{n-1}\big[(x+1)^{i}\big] = x\ P_n(1)\ {}^{0}_{n-1}[x^i] = P_n(1)\ {}^{1}_{n}[x^i],$$

so $\big(E_1 \oplus s_{n-1}(1)^{-1}\big)\ s_n(1) = P_n(1)$ and $s_n(1) = \big(E_1 \oplus s_{n-1}(1)\big)\ P_n(1)$.

Remainders are obtained in the same way as in Theorem 4.1. □

Examples 5.3

1. $s_5(a) = \big(E_1 \oplus s_4(a)\big)\ P_5(a) = \big(E_2 \oplus s_3(a)\big)\big(E_1 \oplus P_4(a)\big)\ P_5(a)$

$$= \big(E_3 \oplus s_2(a)\big)\big(E_2 \oplus P_3(a)\big)\big(E_1 \oplus P_4\big)\ P_5(a)$$

$$\begin{pmatrix} 1&0&0&0&0\\ a&1&0&0&0\\ 2a^2&3a&1&0&0\\ 6a^3&11a^2&6a&1&0\\ 24a^4&50a^3&35a^2&10a&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&a&1&0&0\\ 0&2a^2&3a&1&0\\ 0&6a^3&11a^2&6a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ a&1&0&0&0\\ a^2&2a&1&0&0\\ a^3&3a^2&3a&1&0\\ a^4&4a^3&6a^2&4a&1 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&a&1&0\\ 0&0&2a^2&3a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&a&1&0&0\\ 0&a^2&2a&1&0\\ 0&a^3&3a^2&3a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ a&1&0&0&0\\ a^2&2a&1&0&0\\ a^3&3a^2&3a&1&0\\ a^4&4a^3&6a^2&4a&1 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&a&1&0\\ 0&0&a^2&2a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&a&1&0&0\\ 0&a^2&2a&1&0\\ 0&a^3&3a^2&3a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ a&1&0&0&0\\ a^2&2a&1&0&0\\ a^3&3a^2&3a&1&0\\ a^4&4a^3&6a^2&4a&1 \end{pmatrix}$$

2. $z(s_5)(a) = s_4(a)\ z(P_5)(a)$ $\quad \begin{pmatrix} 1&0&0&0\\ 3a&1&0&0\\ 11a^2&6a&1&0\\ 50a^3&35a^2&10a&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ a&1&0&0\\ 2a^2&3a&1&0\\ 6a^3&11a^2&6a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ 2a&1&0&0\\ 3a^2&3a&1&0\\ 4a^3&6a^2&4a&1 \end{pmatrix}$

3. $S_5(a) = P_5(a)\ \big(E_1 \oplus S_4(a)\big) = P_5(a)\ \big(E_1 \oplus P_4(a)\big)\big(E_2 \oplus S_3(a)\big)$

$$= P_5(a)\ \big(E_1 \oplus P_4(a)\big)\big(E_2 \oplus P_3(a)\big)\big(E_3 \oplus S_2(a)\big)$$

$$\begin{pmatrix} 1&0&0&0&0\\ a&1&0&0&0\\ a^2&3a&1&0&0\\ a^3&7a^2&6a&1&0\\ a^4&15a^3&25a^2&10a&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0\\ a&1&0&0&0\\ a^2&2a&1&0&0\\ a^3&3a^2&3a&1&0\\ a^4&4a^3&6a^2&4a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&a&1&0&0\\ 0&a^2&3a&1&0\\ 0&a^3&7a^2&6a&1 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0&0\\ a&1&0&0&0\\ a^2&2a&1&0&0\\ a^3&3a^2&3a&1&0\\ a^4&4a^3&6a^2&4a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&a&1&0&0\\ 0&a^2&2a&1&0\\ 0&a^3&3a^2&3a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&a&1&0\\ 0&0&a^2&3a&1 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0&0\\ a&1&0&0&0\\ a^2&2a&1&0&0\\ a^3&3a^2&3a&1&0\\ a^4&4a^3&6a^2&4a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&a&1&0&0\\ 0&a^2&2a&1&0\\ 0&a^3&3a^2&3a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&a&1&0\\ 0&0&a^2&2a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&a&1 \end{pmatrix}$$

4. $z(S_5)(a) = z(P_5)(a)S_4(a)$ $\quad \begin{pmatrix} 1&0&0&0\\ 3a&1&0&0\\ 7a^2&6a&1&0\\ 15a^3&25a^2&10a&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ 2a&1&0&0\\ 3a^2&3a&1&0\\ 4a^3&6a^2&4a&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ a&1&0&0\\ a^2&3a&1&0\\ a^3&7a^2&6a&1 \end{pmatrix}$

We call $V\left({}^{0}[x_j]^{n-1}\right) := {}^{\;\;0}_{n-1}\left[x_j^i\right]^{n-1}$ *the Vandermonde matrix* of ${}^{0}[x_j]^{n-1} = (x_0, x_1, \cdots, x_{n-1})$. For example, $V\left({}^{0}[x_j]^{3}\right) = V\left(x_0, x_1, x_2, x_3\right) = {}^{0}_{3}\left[x_j^i\right]^{3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{pmatrix}$ is the Vandermonde matrix of ${}^{0}[x_j]^{3} = (x_0, x_1, x_2, x_3)$.

Lemma 5.3

1. $S_n(1)\ {}^{\;\;0}_{n-1}[j^{i;-1}]^{n-1} = {}^{\;\;0}_{n-1}\left[(j+1)^i\right]^{n-1} = V\left({}^{0}[j+1]^{n-1}\right)$
2. ${}^{\;\;0}_{n-1}[j^{i;-1}]^{n-1} = {}^{\;\;0}_{n-1}\left[i!\binom{j}{i}\right] = {}^{\;\;0}_{n-1}\left[i!\delta_{i=j}\right]^{n-1}\ {}^{t}P_n(1)$
3. $s_n(1)\ {}^{\;\;0}_{n-1}\left[(j+1)^i\right]^{n-1} = {}^{\;\;0}_{n-1}\left[(j+2)^{i;1}\right]^{n-1} = s_n(1)\ V_n\left({}^{0}[j+1]^{n-1}\right)$
4. $P_n(1)\ {}^{t}P_n(1) = {}^{\;\;0}_{n-1}\left[\binom{i+j}{j}\right]^{n-1} = {}^{\;\;0}_{n-1}\left[\binom{i+j}{i}\right]^{n-1}$
5. $z(P_{n+1})(1)\ {}^{t}P_n(1) = D_n(1)\ P_n(1)\ {}^{t}P_n(1) = {}^{\;\;0}_{n-1}\left[\binom{i+j+1}{j+1}\right]^{n-1} = {}^{\;\;0}_{n-1}\left[\binom{i+j+1}{i}\right]^{n-1}$
6. ${}^{\;\;0}_{n-1}\left[(j+2)^{i;1}\right]^{n-1} = {}^{\;\;0}_{n-1}\left[i!\binom{i+j+1}{i}\right]^{n-1} = {}^{\;\;0}_{n-1}\left[i!\delta_{i=j}\right]^{n-1}\ {}^{\;\;0}_{n-1}\left[\binom{i+j+1}{i}\right]^{n-1}$

Examples 5.4

1. $S_4(1)\ {}^{0}_{3}[j^{i;-1}]^{3} = {}^{0}_{3}\left[(j{+}1)^i\right]^{3} = V_4\left({}^{0}[j{+}1]^{3}\right)$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2\cdot 1 & 3\cdot 2 \\ 0 & 0 & 0 & 3\cdot 2\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{pmatrix}$

2. ${}^{0}_{3}[j^{i;-1}]^{3} = {}^{0}_{3}[i!\delta_{i=j}]^{3}\ {}^{t}P_4(1)$ $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2\cdot 1 & 3\cdot 2 \\ 0 & 0 & 0 & 3\cdot 2\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2\cdot 1 & 0 \\ 0 & 0 & 0 & 3\cdot 2\cdot 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. $s_4(1)\ {}^{0}_{3}\left[(j+1)^i\right]^{3} = {}^{0}_{3}\left[(j+2)^{i;1}\right]^{3} = s_4(1)\ V_4\left({}^{0}[j+1]^{3}\right)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2\cdot 3 & 3\cdot 4 & 4\cdot 5 & 5\cdot 6 \\ 2\cdot 3\cdot 4 & 3\cdot 4\cdot 5 & 4\cdot 5\cdot 6 & 5\cdot 6\cdot 7 \end{pmatrix}$$

4. $P_4(1)\ {}^{t}P_4(1) = {}^{0}_{3}\left[\binom{i+j}{j}\right]^{3} = {}^{0}_{3}\left[\binom{i+j}{i}\right]^{3}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}_0C_0 & {}_1C_1 & {}_2C_2 & {}_3C_3 \\ {}_1C_0 & {}_2C_1 & {}_3C_2 & {}_4C_3 \\ {}_2C_0 & {}_3C_1 & {}_4C_2 & {}_5C_3 \\ {}_3C_0 & {}_4C_1 & {}_5C_2 & {}_6C_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$$

5. $z(P_5)(1)\ {}^{t}P_4(1) = D_4(1)\ P_4(1)\ {}^{t}P_4(1) = {}^{0}_{3}\left[\binom{i+j+1}{j+1}\right]^{3} = {}^{0}_{3}\left[\binom{i+j+1}{i}\right]^{3}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 4 & 6 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} {}_1C_1 & {}_2C_2 & {}_3C_3 & {}_4C_4 \\ {}_2C_1 & {}_3C_2 & {}_4C_3 & {}_5C_4 \\ {}_3C_1 & {}_4C_2 & {}_5C_3 & {}_6C_4 \\ {}_4C_1 & {}_5C_2 & {}_6C_3 & {}_7C_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \end{pmatrix}$$

6. ${}^{0}_{3}\big[(j+2)^{i;1}\big]^{3} = {}^{0}_{3}\big[i!\delta_{i=j}\big]^{3}\ {}^{0}_{3}\big[\binom{i+j+1}{i}\big]^{3}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2\cdot3 & 3\cdot4 & 4\cdot5 & 5\cdot6 \\ 2\cdot3\cdot4 & 3\cdot4\cdot5 & 4\cdot5\cdot6 & 5\cdot6\cdot7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \end{pmatrix}$$

From Lemma 5.3, we have the followings.

Theorem 5.2

1. $S_n(1)\ {}^{0}_{n-1}\big[i!\delta_{i=j}\big]^{n-1}\ {}^{t}P_n(1) = V_n\big({}^{0}[j+1]^{n-1}\big)$
2. $s_n(1)\ V_n\big({}^{0}[j+1]^{n-1}\big) = {}^{0}_{n-1}\big[i!\delta_{i=j}\big]^{n-1}\ D_n(1)\ P_n(1)\ {}^{t}P_n(1)$
3. $s_n(a)\ S_n(a) = {}^{0}_{n-1}\big[i!\delta_{i=j}\big]^{n-1}\ D_n(a)\ P_n(a)\ {}^{0}_{n-1}\big[\frac{1}{i!}\delta_{i=j}\big]^{n-1}$

Examples 5.5

1. $S_4(1)\ {}^{0}_{3}\big[i!\delta_{i=j}\big]^{3}\ {}^{t}P_4(1) = V_4\big({}^{0}[j+1]^{n-1}\big)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{pmatrix}$$

2. $s_4(1)\ V_4\big({}^{0}[j+1]^{3}\big) = {}^{0}_{3}\big[i!\delta_{i=j}\big]^{3}\ D_4(1)\ P_4(1)\ {}^{t}P_4(1)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \end{pmatrix}$$

3. $s_4(a)\ S_4(a) = {}^{0}_{3}\big[i!\delta_{i=j}\big]^{3}\ D_4(a)\ P_4(a)\ {}^{0}_{3}\big[\frac{1}{i!}\delta_{i=j}\big]^{3}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 2a^2 & 3a & 1 & 0 \\ 6a^3 & 11a^2 & 6a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^2 & 3a & 1 & 0 \\ a^3 & 7a^2 & 6a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2a & 1 & 0 & 0 \\ 3a^2 & 3a & 1 & 0 \\ 4a^3 & 6a^2 & 4a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}$$

Lemma 5.4

1. ${}^{0}_{n-1}\big[k_j^{i;-1}\big]^{n-1} = {}^{0}_{n-1}\big[i!\binom{k_j}{i}\big]^{n-1} = {}^{0}_{n-1}\big[i!\delta_{i=j}\big]^{n-1}\ {}^{0}_{n-1}\big[\binom{k_j}{i}\big]^{n-1}$ for non-negative integer k_j
2. $S_n(1)\ {}^{0}_{n-1}\big[i!\delta_{i=j}\big]^{n-1}\ {}^{0}_{n-1}\big[\binom{k_j}{i}\big]^{n-1} = V_n\big({}^{0}[k_j+1]^{n-1}\big) = S_n(1){}^{0}_{n-1}\big[k_j^{i;-1}\big]^{n-1}$
3. $\Big|V_n\big({}^{0}[k_j]^{n-1}\big)\Big| = \Big|V_n\big({}^{0}[k_j+b]^{n-1}\big)\Big|\ (b\in\mathbb{Z})$

Examples 5.6 For ${}^{0}[k_j]^{3} = (k_0,k_1,k_2,k_3) = (5,3,7,2)$,

1. ${}^{0}_{3}\big[k_j^{i;-1}\big]^{3} = {}^{0}_{3}\big[i!\binom{k_j}{i}\big]^{3} = {}^{0}_{3}\big[i!\delta_{i=j}\big]^{3}\ {}^{0}_{3}\big[\binom{k_j}{i}\big]^{3}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 7 & 2 \\ 5\cdot4 & 3\cdot2 & 7\cdot6 & 2\cdot1 \\ 5\cdot4\cdot3 & 3\cdot2\cdot1 & 7\cdot6\cdot5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} {}_5C_0 & {}_3C_0 & {}_7C_0 & {}_2C_0 \\ {}_5C_1 & {}_3C_1 & {}_7C_1 & {}_2C_1 \\ {}_5C_2 & {}_3C_2 & {}_7C_2 & {}_2C_2 \\ {}_5C_3 & {}_3C_3 & {}_7C_3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 7 & 2 \\ 10 & 3 & 21 & 1 \\ 10 & 1 & 35 & 0 \end{pmatrix}$$

2. $S_4(1)\ {}^{0}_{3}\left[i!\delta_{i=j}\right]^3\ {}^{0}_{3}\left[\left(\begin{smallmatrix} k_j \\ i \end{smallmatrix}\right)\right]^3 = V_4\left({}^{0}[k_j+1]^3\right) = S_4(1)\ {}^{0}_{3}[k_j^{i;-1}]^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 7 & 2 \\ 10 & 3 & 21 & 1 \\ 10 & 1 & 35 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & 4 & 8 & 3 \\ 6^2 & 4^2 & 8^2 & 3^2 \\ 6^3 & 4^3 & 8^3 & 3^3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 7 & 2 \\ 5\cdot 4 & 3\cdot 2 & 7\cdot 6 & 2\cdot 1 \\ 5\cdot 4\cdot 3 & 3\cdot 2\cdot 1 & 7\cdot 6\cdot 5 & 0 \end{pmatrix}$$

3. $\left|V_4\left({}^{0}[k_j]^3\right)\right| = \left|V_4\left({}^{0}[k_j+1]^3\right)\right| = \left|V_4\left({}^{0}[k_j-1]^3\right)\right|$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 7 & 2 \\ 5^2 & 3^2 & 7^2 & 2^2 \\ 5^3 & 3^3 & 7^3 & 2^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 6 & 4 & 8 & 3 \\ 6^2 & 4^2 & 8^2 & 3^2 \\ 6^3 & 4^3 & 8^3 & 3^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 6 & 1 \\ 4^2 & 2^2 & 6^2 & 1 \\ 4^3 & 2^3 & 6^3 & 1 \end{vmatrix}$$

We obtain the following from Lemma 5.4, 2.

Theorem 5.3 ([2]) The determinant of Vandermonde matrix $\left|V_n\left({}^{0}[k_j]^{n-1}\right)\right|$ is devisible by $\prod_{i=0}^{n-1} i!$ for any $k_j \in \mathbb{Z}$.

Examples 5.7

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 7 & 2 \\ 5^2 & 3^2 & 7^2 & 2^2 \\ 5^3 & 3^3 & 7^3 & 2^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 6 & 4 & 8 & 3 \\ 6^2 & 4^2 & 8^2 & 3^2 \\ 6^3 & 4^3 & 8^3 & 3^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 6 & 1 \\ 4^2 & 2^2 & 6^2 & 1 \\ 4^3 & 2^3 & 6^3 & 1 \end{vmatrix} \text{ is divisible by } \prod_{i=0}^{3} i! = 1\cdot 1\cdot 2\cdot 6 = 0!\cdot 1!\cdot 2!\cdot 3!.$$

6 The matrices of sum of powers

We define $(i+1)$-dim. polynomial of variable $n \in \mathbb{N}$ as $W_i(n) := \sum_{k=1}^{n} k^i$, $(i = 0, 1, 2, \cdots)$. Then we have $W_i(x)$ as the $(n+1)$-dim. polynomial of variable $x \in \mathbb{R}$.

Examples 6.1 of $W_i(x)$

$$W_0 = x, \quad W_1(x) = \frac{x}{2} + \frac{x^2}{2} = \frac{x(x+1)}{2}, \quad W_2(x) = \frac{x}{6} + \frac{x^2}{2} + \frac{x^3}{3} = \frac{x(x+1)(2x+1)}{6}, \cdots$$

For these polynomials, we have the following lemma.

Lemma 6.1

1. $W_i(x) = W_i(x-1) + x^i$ (6.1) 2. $W_i(0) = 0$ (6.2)

3. $W_i(-1) = -\delta_{i=0} = \begin{cases} -1 & (i=0) \\ 0 & (i = 1, 2, \cdots) \end{cases}$ (6.3)

Proof. 1 follows from $\sum_{k=1}^{n} k^i = \sum_{k=1}^{n-1} k^i + n^i$. Since $W_i(1) = 1$ for $i = 0, 1, 2, \cdots$, putting $x = 1$ in (6.1), we obtain 2. From (6.2) and putting $x = 0$ in (6.1), we obtain 3 (since $0^0 = 1$). □

We define *the $n \times n$ matrix of sum of powers* Σ_n as $\Sigma_n\ {}^{1}_{n}[x^i] = {}^{0}_{n-1}[W_i(x)]$. For example,

$$\Sigma_5 \ {}_5^1[x^i] = {}_4^0[W_i(x)] \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} = \begin{pmatrix} W_0(x) \\ W_1(x) \\ W_2(x) \\ W_3(x) \\ W_4(x) \end{pmatrix}.$$

We also define the $n \times n$ matrix σ_n as $\sigma_n := {}_{n-1}^{0}\Big[\binom{i+1}{j}\ \delta_{i \geq j}\Big]^{n-1}$. For example,

$$\sigma_4 = {}_3^0\Big[\binom{i+1}{j}\ \delta_{i \geq j}\Big]^3 \qquad \sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 4 & 6 & 4 \end{pmatrix} = \begin{pmatrix} {}_1C_0 & 0 & 0 & 0 \\ {}_2C_0 & {}_2C_1 & 0 & 0 \\ {}_3C_0 & {}_3C_1 & {}_3C_2 & 0 \\ {}_4C_0 & {}_4C_1 & {}_4C_2 & {}_4C_3 \end{pmatrix}.$$

To show interesting relations between $\Sigma_n(a)$ and $\sigma_n(a)$ (Theorem 6.1) including $\Sigma_n(a)^{-1} = \sigma_n(-a)$, first we see the basic property of $\sigma_n(a)$.

Lemma 6.2 $\quad a\sigma_n(a)\ {}_{n-1}^{0}[x^i] = {}_n^1\big[[x+a]^i - x^i\big]$

For example, $\quad a\sigma_4(a)\ {}_3^0[x^i] = {}_4^1\big[[x+a]^i - x^i\big]$

$$a\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 2 & 0 & 0 \\ a^2 & 3a & 3 & 0 \\ a^3 & 4a^2 & 6a & 4 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ a^2 & 2a & 0 & 0 \\ a^3 & 3a^2 & 3a & 0 \\ a^4 & 4a^3 & 6a^2 & 4a \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} (x+a) & -x \\ (x+a)^2 & -x^2 \\ (x+a)^3 & -x^3 \\ (x+a)^4 & -x^4 \end{pmatrix}.$$

For the proof of Theorem 6.1, we note the following equations.

Lemma 6.3

1. $\sigma_n(1)\ {}_{n-1}^{0}[x^i] = {}_n^1\big[(x+1)^i - x^i\big] \quad (6.4)$ 2. $\sigma_n(-1)\ {}_{n-1}^{0}[x^i] = {}_n^1\big[x^i - (x-1)^i\big] \quad (6.5)$

Then $\Sigma_n(a)$ and $\sigma_n(a)$ have the following interesting properties.

Theorem 6.1

1. $\sigma_n(a)\ \Sigma_n(a) = z(P_{n+1})(a)$ 2. $\sigma_n(-a)\ \Sigma_n(a) = E_n$ 3. $\Sigma_n(a)\ \sigma_n(a) = P_n(a)$

4. $\sigma_n(-a)\ P_n(a) = \sigma_n(a)$ 5. $P_n(a)\ \Sigma_n(-a) = \Sigma_n(a)$ 6. $\Sigma_n(a) - \Sigma_n(-a) = d_n(a) - E_n$

$= {}_{n-1}^{0}a[\delta_{i-j=1}]^{n-1}$

The $(j+2k+1, j)$ component of Σ_n is $\delta_{k=0}/2 \quad (0 \leq j, k \leq n-1)$.

Proof. 1 and 2. We will prove the case $a = 1$. Taking the sum in (6.4) and (6.5), we have

$\sigma_n(1)\ {}_{n-1}^{0}[W_i(x)] = {}_n^1\big[(x+1)^i - 1\big]$, $\sigma_n(-1)\ {}_{n-1}^{0}[W_i(x)] = {}_n^1[x^i]$, $\sigma_n(1)\ \Sigma_n(1)\ {}_n^1[x^i] = {}_n^1\big[(x+1)^i - 1\big]$

$= z(P_{n+1})(1)\ {}_n^1[x^i]$ and $\sigma_n(-1)\ \Sigma_n(1)\ {}_n^1[x^i] = {}_n^1[x^i]$. Then we obtain 1 and 2.

3. Using (6.1), we have $\Sigma_n(1)\ \sigma_n(1)\ {}_{n-1}^{0}[x^i] = \Sigma_n(1)\ {}_n^1\big[(x+1)^i - x^i\big] = {}_{n-1}^{0}\big[W_i(x+1) - W_i(x)\big] = {}_{n-1}^{0}\big[(x+1)^i\big] = P_n(1)\ {}_{n-1}^{0}[x^i]$. Then we obtain 3.

4. Since $\sigma_n(1)\ {}_{n-1}^{0}[x^i] = {}_n^1\big[(x+1)^i - x^i\big] = \sigma_n(-1)\ {}_{n-1}^{0}\big[(x+1)^i\big] = \sigma_n(-1)\ P_n(1)\ {}_{n-1}^{0}[x^i]$, then we obtain 4.

5. From 3, we have $\Sigma_n(1)\ \sigma_n(1)\ \Sigma_n(-1) = P_n(1)\ \Sigma_n(-1)$, then we obtain 5 by 2.

6. From 1, we have $\Sigma_n(-1) = \Sigma_n(1)\ z(P_{n+1})(-1)$. Since $\Sigma_n(1)\ {}_n^1[x^i] = {}_{n-1}^{0}[W_i(x)]$,

$$\Sigma_n(-1)\ {}_n^1[x^i] = \Sigma_n(1)\ z(P_{n+1})(-1)\ {}_n^1[x^i] = \Sigma_n(1)\ {}_n^1\big[(x-1)^i - (-1)^i\big] = {}_{n-1}^0[W_i(x-1) - W_i(-1)].$$

From (6.1) and (6.3), we have $\big(\Sigma_n(1) - \Sigma_n(-1)\big)\ {}_n^1[x^i] = {}_{n-1}^0[W_i(x) - W_i(x-1)] - {}_{n-1}^0[W_i(-1)]$

$= {}_{n-1}^0[x^i] - {}_{n-1}^0[\delta_{i=0}] \ = {}_{n-1}^0[x^i\delta_{i>0}] = {}_{n-1}^0[\delta_{i-j=1}]^{n-1}\ {}_n^1[x^i] \ = \big(d_n(1) - E_n\big)\ {}_n^1[x^i]$, then we obtain 6. □

Examples 6.2

1. $\sigma_4(1)\ \Sigma_4(1) = z(P_5)(1)$ $$\begin{pmatrix} 1&0&0&0\\ 1&2&0&0\\ 1&3&3&0\\ 1&4&6&4 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ \frac12&\frac12&0&0\\ \frac16&\frac12&\frac13&0\\ 0&\frac14&\frac12&\frac14 \end{pmatrix}=\begin{pmatrix} 1&0&0&0\\ 2&1&0&0\\ 3&3&1&0\\ 4&6&4&1 \end{pmatrix}$$

2. $\sigma_4(-1)\ \Sigma_4(1) = E_4$ $$\begin{pmatrix} 1&0&0&0\\ -1&2&0&0\\ 1&-3&3&0\\ -1&4&-6&4 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ \frac12&\frac12&0&0\\ \frac16&\frac12&\frac13&0\\ 0&\frac14&\frac12&\frac14 \end{pmatrix}=\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$$

3. $\Sigma_4(1)\ \sigma_4(1) = P_4(1)$ $$\begin{pmatrix} 1&0&0&0\\ \frac12&\frac12&0&0\\ \frac16&\frac12&\frac13&0\\ 0&\frac14&\frac12&\frac14 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ 1&2&0&0\\ 1&3&3&0\\ 1&4&6&4 \end{pmatrix}=\begin{pmatrix} 1&0&0&0\\ 1&1&0&0\\ 1&2&1&0\\ 1&3&3&1 \end{pmatrix}$$

4. $\sigma_4(-1)\ P_4(1) = \sigma_4(1)$ $$\begin{pmatrix} 1&0&0&0\\ -1&2&0&0\\ 1&-3&3&0\\ -1&4&-6&4 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ 1&1&0&0\\ 1&2&1&0\\ 1&3&3&1 \end{pmatrix}=\begin{pmatrix} 1&0&0&0\\ 1&2&0&0\\ 1&3&3&0\\ 1&4&6&4 \end{pmatrix}$$

5. $P_4(1)\ \Sigma_4(-1) = \Sigma_4(1)$ $$\begin{pmatrix} 1&0&0&0\\ 1&1&0&0\\ 1&2&1&0\\ 1&3&3&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ -\frac12&\frac12&0&0\\ \frac16&-\frac12&\frac13&0\\ 0&\frac14&-\frac12&\frac14 \end{pmatrix}=\begin{pmatrix} 1&0&0&0\\ \frac12&\frac12&0&0\\ \frac16&\frac12&\frac13&0\\ 0&\frac14&\frac12&\frac14 \end{pmatrix}$$

6. $\Sigma_4(1) - \Sigma_4(-1) = d_4(1) - E_4 = {}_3^0[\delta_{i-j=1}]^3$

$$\begin{pmatrix} 1&0&0&0\\ \frac12&\frac12&0&0\\ \frac16&\frac12&\frac13&0\\ 0&\frac14&\frac12&\frac14 \end{pmatrix}-\begin{pmatrix} 1&0&0&0\\ -\frac12&\frac12&0&0\\ \frac16&-\frac12&\frac13&0\\ 0&\frac14&-\frac12&\frac14 \end{pmatrix}=\begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix}$$

We use the following equation for the proof of Theorem 6.2.

Lemma 6.4 ${}_n^1[ix^{i;-1}] = x\ {}_n^1\big[x^{i;-1} - (x-1)^{i;-1}\big]$

Examples 6.3 ${}_4^1[ix^{i;-1}] = x\ {}_4^1\big[x^{i;-1} - (x-1)^{i;-1}\big]$

$$\begin{pmatrix} 1x\\ 2x(x-1)\\ 3x(x-1)(x-2)\\ 4x(x-1)(x-2)(x-3) \end{pmatrix} = \begin{pmatrix} (1-x+x)x\\ (2-x+x)x(x-1)\\ (3-x+x)x(x-1)(x-2)\\ (4-x+x)x(x-1)(x-2)(x-3) \end{pmatrix}$$

$$= x\begin{pmatrix} x & -(x-1)\\ x(x-1) & -(x-1)(x-2)\\ x(x-1)(x-2) & -(x-1)(x-2)(x-3)\\ x(x-1)(x-2)(x-3) & -(x-1)(x-2)(x-3)(x-4) \end{pmatrix}$$

For the eigenvalues and eigenvectors of $\Sigma_n(1)$ and $\sigma_n(-1)$, we have the following theorem.

Theorem 6.2 1. $\Sigma_n(1)\ S_n(1) = S_n(1)\ {}_n^1\big[\frac{1}{i}\delta_{i=j}\big]^n$ 2. $\sigma_n(-1)\ S_n(1) = S_n(1)\ {}_n^1[i\delta_{i=j}]^n$

Proof. We will prove $\Sigma_n(1)\ S_n(1)\ {}^1_n[i\delta_{i=j}]^n = S_n(1)$. From Lemma 6.4, we have

${}^1_n[i\delta_{i=j}]^n\ {}^1_n[x^{i;-1}] = {}^1_n[ix^{i;-1}] = x\ {}^1_n[x^{i;-1} - (x-1)^{i;-1}]$. Using (6.1), we have

$\Sigma_n(1)\ S_n(1)\ {}^1_n[i\delta_{i=j}]^n\ {}^1_n[x^{i;-1}] = \Sigma_n(1)\ S_n(1)\ x\ {}^1_n\big[x^{i;-1} - (x-1)^{i;-1}\big] = \Sigma_n(1)\ x\ {}^1_n\big[x^i - (x-1)^i\big]$

$= x\ {}^0_{n-1}\big[W_i(x) - W_i(x-1)\big] = x\ {}^0_{n-1}[x^i]\ = {}^1_n[x^i]\ = S_n(1)\ {}^1_n[x^{i;-1}]$.

Then we obtain 1. 2 follows from 1 by Theorem 6.1, 2. □

Examples 6.4

1. $\Sigma_4(1)\ S_4(1) = S_4(1)\ {}^1_4\big[\frac{1}{i}\delta_{i=j}\big]^4 \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$

2. $\sigma_4(-1)\ S_4(1) = S_4(1)\ {}^1_4[i\delta_{i=j}]^4 \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -3 & 3 & 0 \\ -1 & 4 & -6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

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