FINDING ROOTS BY A SIMPLE ORIGAMI

ISAO NAKAI, ERI ONO *

(Received September 10, 2014, Revised May 15, 2015)

Abstract. We present a solving method of algebraic equations of one indeterminate of degree up to six by simple paper foldings.

誓いしを思ひかへるの人しれずいとかく物を思ふ頃かな
Bagatelle no.292 from "Kiyosuke Ason shu"

1. Introduction

The history of algebraic equation with one variable goes back to Babylonian mathematics ca. 2000 BC [31], and the solution of the quadratic equation in square root was known to the ancient Egyptians. Through the history of pre-modern and modern mathematics the solution of algebraic equation of degree ≥ 3 in radical form had been long seeked for until it was discovered for degrees 3 and 4 in the first half of the 16th centuries, and negatively settled by Ruffini, Able and Galois by proving the non-existence in general for degrees 5 and higher in the beginning of the 19th centuries.

Origami literally means paper folding in Japanese. Paper crafts in the shapes of Birds and Frogs are, although controversial, alluded in the old texts of the late 12th and 13th centuries [14, 29] (see also page 222 in [5] and page 781 in [22]). The history of Japanese folk art of origami can trace back at least to the late 18th century in the Edo period (see e.g. the books [2]), when origami was popularised to the masses due to a wide distribution of paper.

From the mathematical point of view, paper folding was systematically investigated by an Indian mathematician Sundara Row [28] in the end of the 19th century. His work caught attention of Klein and became widely regarded (see e.g. [21, 16]). In 1936 an Italian female mathematician Beloch [6] showed first a method of solving cubic equations by creases: realizing an enjoyable idea of an Austrian engineer Lill [24] (see Fig.1), and after, its variants were re-discovered by many origami enthusiasts (see [16]). As quartic equations reduce to cubic equations by Ferrari's formula, quartic equations also can be, in principle, solved by paper folding.

Edwards and Shurman [13] showed a method of solving quartic equations by folding lines bi-tangent to a parabola and a circle in 2001, but the procedure given was applicable to a certain partial case (see also the article [25]). The equations of degree higher than or equal to five are known [10] to be solved in the *multi-fold origami* introduced by Alperin and Lang [4], again being inspired by the method of Lill. Although its modification was made by Nishimura [26] in the aspect of "multiplicity" especially for the quintic equations, the multi-fold origami in general is exceedingly difficult to manipulate in practice, and not included in the so-called "Hatori-Huzita-Justin axioms" of origami. In fact, the ancient Greek problem of Angle trisection is nothing but a special case of the 2-fold origami.

In this short note, we present the simplest and concrete procedure of solving quartic equations by compass and paper folding (Proposition 6.1) simplifying a result of Edwards and Shurman and realizing an idea of Ghourabi, Ida, Kaliszyk and Kasem [19, 15], which could, from the mathematical view point, have been discovered more than two milleniums ago if origami on Papyrus had been allowed (see e.g. Figure 4 for a simple example). We present also a nomogram method of solving equations of higher degree using curves B_n introduced in §9, and examine applying it to the various problems such as the quintic equation, the sextic equation etc. in §9, 10.

This work was initiated in the course of graduation study of a former student Eri Ono in Ochanomizu university in the academic year of 2012. The authors thank to Catherine Oikonomides for aid in a literal

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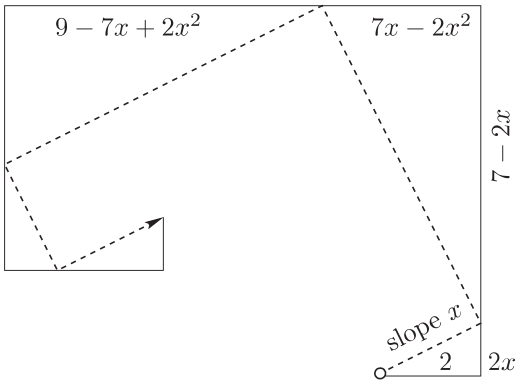
2000 *Mathematics Subject Classification.* MSC 97H20 and 97B60 and 97A30 and 52C99 and 12Y05 and 51M15 .
Key words and phrases. Algebraic equation, Origami, Paper folding, Quartic, Quintic, Sextic, Nomogram.

*Mizuho Bank.

problem in ancient Greek, and also to Ghourabi and the referee for their careful reading of the previous draft of this note.



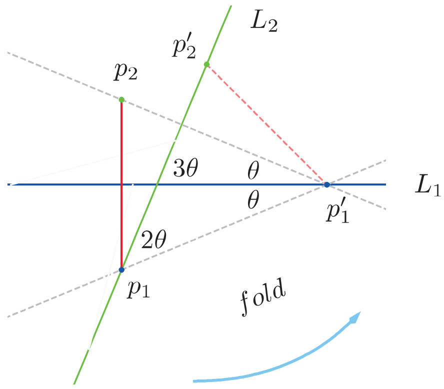


FIGURE 1. Left. A self-explaining diagram due to Lill: a five-fold origami solving $2x^5 - 7x^4 + 9x^3 - 5x^2 + 3x - 1 = 0$. The polygonal diagram in solid line consists of segments of length $2, 7, 9, 5, 3, 1$ successively from the starting point $\circ$. The initial slope of the dashed line is a unique real solution: i.e. $x = \frac{1}{2}$. The dashed trajectory might be found by lapping a transparent graph paper over the polygonal diagram. This method applies to univariate algebraic equations of any degree.

FIGURE 2. Right. Angle trisection by origami due to Abe: Let $p_1 \in L_2$ and assume L_1 is the perpendicular at midpoint of the segment p_1p_2. Fold p_1, p_2 respectively onto some points p_1', p_2' on L_1, L_2. Then the angle of p_1p_1' and L_1 trisects the angle 3θ of L_1 and L_2. There exist two other folds placing p_1, p_2 respectively onto L_1, L_2. Those folds give $\theta/3 + \pi/3$ and $\theta/3 + 2\pi/3$ (cf. the angle trisection in Figure 4).

2. Origami Axioms

A *fold* of a piece of paper (or the Euclidean plane $\mathbb{R}^2$) defines an isometric everting acting on the paper, which fixes every point on a line L: the *folding line*. Huzita [17] and Justin [18] introduced seven allowable operations in paper folding, and called them “axioms” of origami in 1989, which seem now to be widely accepted in the geometry of origami.

The original statement of the origami “axioms” in [17, 18] is pointed to be *non-decidable* as a theory in the sense of computational logic in [19], where a refined set of “axioms” is proposed. We would re-state those operations or “axioms” in slightly modulated forms as follows.

- Given two points $p_1 \neq p_2$, one can make a unique fold that passes p_1 and p_2.
- Given two points $p_1 \neq p_2$, one can make a unique fold that places p_1 onto p_2.
- Given two lines $L_1 \neq L_2$, one can make all (one or two) folds that place L_1 onto L_2.
- Given a line L and a point p off L, one can make a unique fold perpendicular to L that passes p.
- Given two points $p_1 \neq p_2$ and a line L_1, one can make all (0 or 1 or 2) folds that place p_1 onto some point $p_1' \in L_1$ and passes p_2.
- Given two points $p_1 \neq p_2$ and two lines L_1, L_2 such that $p_i \notin L_i$ for $i = 1$ or 2, or $L_1 \cap L_2$ is non-empty, one can make all (0 or 1 or 2 or 3) folds that place p_i onto some point $p_i' \in L_i$ simultaneously for $i = 1, 2$ (see Figure 3, left).
- Given a point p_1 and two lines L_1, L_2 with a non-empty intersection, one can make a unique fold perpendicular to L_2 that places p_1 onto some point $p_1' \in L_1$.

Among these “axioms” the most crucial is the 6th one, which is equivalent to solving a certain cubic equation (see also Remark 1). It seems that most of the other “axioms” are substituted by the above 6th “axiom” placing the initial point p_i on L_i and assuming $L_1 = L_2$ if necessary. For instance, if $p_2 \in L_2$ and $p_2 \notin L_1$ (but not too far from L_1), then the 6th “axiom” provides, in a generic case, two folds placing p_1 onto L_1 and passing p_2, and one fold placing p_1 onto L_1 and also L_2 onto itself. The former folds are

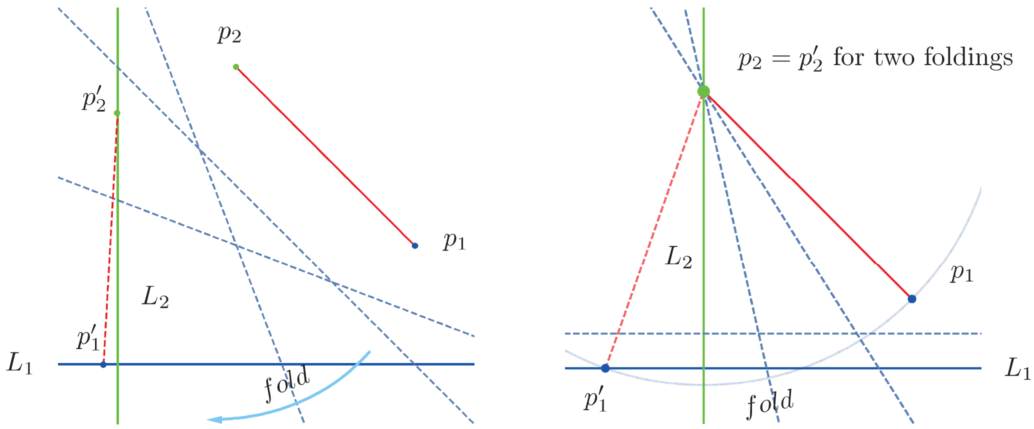


FIGURE 3. Left: the 6th axiom for right angled two lines; Right: the 6th axiom with a singular configuration ($p_2 \in L_2$). The long dashed lines are the folding lines.

those given by the 5th "axiom" and the latter fold is the one given by the 7th "axiom" (see Figure 3, right). This suggests a possibility of reducing the "axioms".

The number of all possible folds in the 6th axiom varies from 0 to 3 depending on the configuration of p_i and L_i. In fact if L_1 and L_2 are parallel and their distance exceeds the distance of p_1 and p_2, there exists no fold as being required. This will be discussed again in §8.

Here let us remind a practical problem of making a fold placing a point p onto a line L. For this sake, one varies and fixes the fold image p' of p when it collides with L. This is carried out in a continuous move of p' with a parameter of dimension one crossing the line L transversely. In order to make a fold that places p onto a prefixed point p' (as in the second Axiom), a parameter of dimension 2 is needed. In general it can be said that probing a point on a plane is much more difficult than a point on a line.

The *neusis* ($\nu\epsilon\tilde{\upsilon}\sigma\iota\varsigma$ in ancient Greek) construction is to place two marked points of a straightedge respectively onto two curves so that it inclines towards a prescribed fixed point (see [11]). This task requires to "slide" the marked points along the curves. This construction was applied by Archimedes, for instance for the angle trisection, to a pair of a straight line and a circle (see Figure 4 and also [11]), but not accepted by most of the other Greek geometers such as Euclid and Pappus, because of the use of a marked straight edge. Interestingly this can be replaced by paper folding construction as in the following figure [19].

The most elementary Angle Trisection by a single-fold with "sliding" along two lines was found by Abe [1] in 1980 before the "axioms" were proclaimed by the others (see Figure 2). Basically, the angle trisection is equivalent to a cubic equation: Chebyshev polynomial of degree 3 is equal to a constant. It is then solved by the sixth origami axiom, which requires to "slide" the extremities of a segment $p_1'p_2'$ ($p_i' \in L_i$) of equal length to p_1p_2 along the lines into a mirror position of p_1p_2. For all above, it seems the sliding construction with a marked straight edge ought to be accepted to explore beyond cubic equations by paper foldings.

In this note the authors would suggest yet another "operation"

- Given two points $p_1 \neq p_2$ and two curves C_1, C_2, one can make all folds that place p_i onto some point $p_i' \in C_i$ simultaneously for $i = 1, 2$.

The adequate choices for the curves C_i would be straight lines, circles and possibly conics, which are already suggested by Ghourabi et al. [19, 15]. Of course the number of such folds varies in general depending on the configuration as before. In this note the authors define the curves $B_n, n = 2, 3, 4, 5, \ldots$ in §9, and present origami solutions to the various equations of degree $2, 3, 4$ and higher with those curves and the above new "operation". We will discuss the various algebraic equations with this "operation" in the following sections.

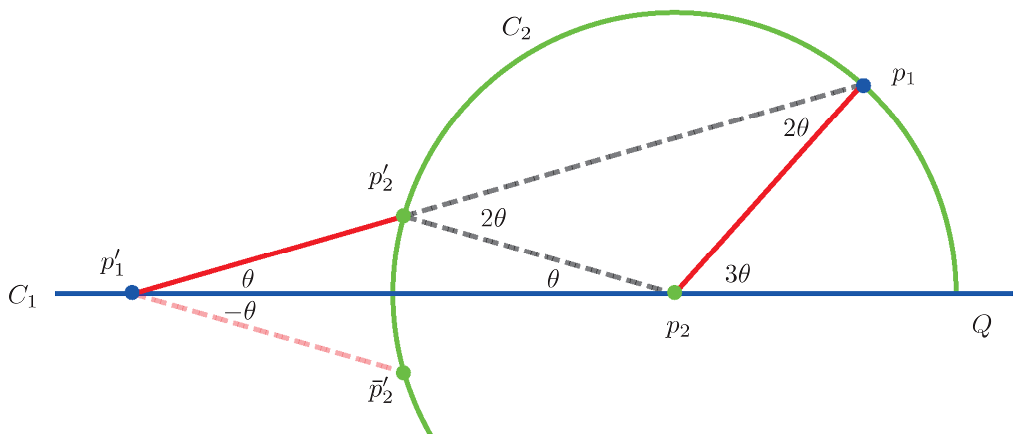


FIGURE 4. The Angle Trisection attributed to Archimedes, and partly to Kasem et al. [19]; Assume the segment $p_1'p_2'$ of length equal to the radius of the circle C_2 inclines towards p_1. Then the angle $\theta = \angle Qp_1'p_2'$ trisects the angle $3\theta = \angle Qp_2p_1$. Kasem et al. pointed this neusis construction can be replaced by an origami placing p_1, p_2 onto $p_1' \in C_1$ and the reflection $\bar{p}_2' \in C_2$ of p_2' by C_1 simultaneously. The dashed segment $p_1'\bar{p}_2'$ in red inclines towards the reflection $\bar{p}_1$ of p_1, and then $\angle Qp_1'\bar{p}_2' = -\theta$ if we measure the angle anti-clockwisely. There exist also three other folds placing p_1, p_2 onto C_1, C_2. In fact, the origami folding placing p_1, p_2 onto C_1, C_2 corresponds to a quartic equation (Remark 1). For those folds, $p_1'\bar{p}_2'$ (or its reverse) inclines towards $\bar{p}_1$ or p_1, and the angle $\angle Qp_1'\bar{p}_2'$ is either $\pi/3-\theta, 2\pi/3-\theta$ or 3θ. The last case holds when p_1, p_2 are transposed by the folding.

3. DUALITY

A *non-vertical line* L (non-parallel to the Y-axis) on a plane with the orthogonal coordinates X, Y is presented by a linear equation $Y = AX + B$ with real coefficients A, B, while (A, B) gives a point on the AB-plane: the *dual plane*. We call (A, B) the *dual point* of L and denote $(A, B) = L^\vee$. We denote also L by $(A, B)^\wedge$ and call the *dual line* of (A, B). Conversely X, Y being fixed, the equation $B = -XA + Y$ defines a (non vertical) *dual line* L' in the AB-plane, which we denote by $(X, Y)^\vee$. We call (X, Y) the *dual* of L' and denote by $L'^\wedge$. The operations $\vee, \wedge$ enjoy the *bi-duality*

$$(X, Y)^{\vee\wedge} = (X, Y) \qquad \text{and} \qquad (A, B)^{\wedge\vee} = (A, B).$$

This argument provides a one-to-one correspondence of the space of non-vertical lines and the space of their defining equations (*dual points*) as follows,

(a) Non-vertical lines in $\mathbb{R}^2_{XY}$ $\overset{\text{Dual}}{\Longleftrightarrow}$ Points in $\mathbb{R}^2_{AB}$,

(b) Points in $\mathbb{R}^2_{XY}$ $\overset{\text{Dual}}{\Longleftrightarrow}$ Non-vertical lines in $\mathbb{R}^2_{AB}$.

This correspondence extends naturally to that of the space of all lines including vertical lines, completing the dual plane $\mathbb{R}^2$ by glueing along the line at infinity to form the projective plane $\mathbb{R}P^2$ (see e.g. [27]). However, we will stay on the above imperfect *projective duality,* respecting Cartesian coordinate structure of the dual plane.

A smooth curve C on a plane is *vertical* at a $p \in C$ if the tangent line T_pC of C at p is vertical. In this note the *dual curve* $C^\vee$ of a smooth plane curve C in the XY-plane is defined to be the set of duals of tangent lines of C at non-vertical points. Let $C : (\alpha(t), \beta(t))$, $t \in \mathbb{R}$, be a C^1-parameterization and assume $\alpha'(t) \neq 0$. Then C is nonsingular, non-vertical, and the tangent line at $C(t)$ is given by

$$Y = \beta'(t)/\alpha'(t)X + \beta(t) - \alpha(t)\beta'(t),$$

which determines a point in the dual AB-plane:

$$C^\vee(t) = \frac{1}{\alpha'(t)}\left(\beta'(t)\ ,\ -\begin{vmatrix} \alpha(t) & \beta(t) \\ \alpha'(t) & \beta'(t) \end{vmatrix}\right).$$

On the one hand, the tangent line of $D : (\alpha(t), \beta(t))$ in the dual AB-plane may be presented as

$$B = \beta'(t)/\alpha'(t)A + \beta(t) - \alpha(t)b'(t) \quad \text{or} \quad \beta(t) - \alpha(t)\beta'(t) = (-\beta'(t)/\alpha'(t))\,A + B.$$

Thus its dual curve denoted $D^\wedge$ in the XY-plane is defined to be

$$D^\wedge(t) = \frac{1}{\alpha'(t)}\left(-\beta'(t)\ ,\ -\begin{vmatrix}\alpha(t) & \beta(t)\\ \alpha'(t) & \beta'(t)\end{vmatrix}\right).$$

If C (respectively D) is C^2-smooth, $C^\vee$ (resp. $D^\wedge$) is C^1-smooth by the above formulas. If $\alpha'(t) \neq 0$ and the Wronskian of $(\alpha'(t), \beta'(t))$ (i.e. $\alpha'\beta'' - \beta'\alpha''$) is nonzero (in other words, the curvature of C does not vanish), the dual curve is smooth, non-vertical and moreover the curvature of $C^\vee$ does not vanish. Thus the repeated dual curve $C^{\vee\wedge}$ is also well defined and smooth. Upon the non-vanishing condition of α' and Wronskian of (α', β'), we have the projective *bi-duality* of plane curves

$$\text{(Bi-duality)} \qquad C^{\vee\wedge}(t) = C(t) \qquad \text{and} \qquad D^{\wedge\vee}(t) = D(t).$$

By this bi-duality, the second dual curve regains the C^2-differentiability. Remark that an isolated vertical point of a C^2-smooth curve C is the dual of a non-vertical asymptotic line of $C^\vee$.

The dual $L^\vee$ of a linearly parameterized line $L : (t, At + B)$ is (A, B) according to the above definition of dual curve. If we regard the lines passing through (A, B) are tangent to (A, B), we obtain $L^{\vee\wedge} = L$ and similarly $L'^{\wedge\vee} = L'$ for lines L' in the AB-plane. Thus the above bi-duality extends the duality in (a), (b).

Let G denote the set (*group*) of affine transformations of the plane in the form $g : (X, Y) \to (pX + q, rX + sY + t)$, $ps \neq 0$, preserving the fibration of the plane by vertical lines. For $g \in G$, define $\phi_\pm(g) \in G$ by

$$\phi_\pm(g) : (X, Y) \to \left(\ \frac{s}{p}X \pm \frac{r}{p}\ ,\ \mp\frac{qs}{p}X + sY - \frac{qr}{p} + t\ \right).$$

These morphisms $\phi_\pm$ are *group isomorphisms* of G: $\phi_\pm(g) = \mathrm{id}$ if and only if $g = \mathrm{id}$, and for $g, h \in G$,

$$\phi_\pm(g \circ h) = \phi_\pm(g) \circ \phi_\pm(h),$$

where $\circ$ denotes the composite. Moreover, for curves C in the XY-plane and D in the AB-plane,

$$g(C)^\vee = \phi_+(g)(C^\vee), \qquad h(D)^\wedge = \phi_-(h)(D^\wedge).$$

To prove these compatibilities it is enough to show each element of G is a product (composite) of three elements (i.e. the generators of G) in the forms,

$$(X, Y) \to (pX, sY), \qquad (X, Y) \to (X, rX + Y), \qquad (X, Y) \to (X + q, Y + t).$$

and the compatibility for these generators.

For an irreducible algebraic curve C of degree ≥ 2 (i.e. defined by an irreducible polynomial equation in X, Y), the dual of the non-vertical smooth part of C defined above is dense in an algebraic curve: the *algebraic dual curve* of C. It is not difficult to show the bi-duality for the algebraic dual curves.

Every smooth quadratic curve (conic) C on the XY-plane is transformed to one of the following normal forms by a suitable element in G,

$$\text{(Conic)} \qquad Y = X^2, \qquad X = Y^2, \qquad XY = 1, \quad Y^2 - X^2 = 1, \quad X^2 - Y^2 = 1, \quad X^2 + Y^2 = 1.$$

The algebraic duals $C^\vee$ of these normal forms C are respectively

$$\text{(Dual)} \qquad B = -\frac{1}{4}A^2, \quad AB = \frac{1}{4}, \qquad A = -\frac{1}{4}B^2, \quad A^2 + B^2 = 1, \quad A^2 - B^2 = 1, \quad B^2 - A^2 = 1.$$

The algebraic duals $C^\wedge$ are given by substituting $-A$ for A in the above list.

4. Bi-tangent lines to a pair of conics

Let C be a smooth plane curve. There is no origami axiom that enables us to fold a tangent line of C at a $q \in C$. If there is given a pair of mirror images (p, p'), $p \neq p'$, with respect to the tangent line of C at q, one can fold the tangent line by placing p onto p'. This fold is allowed by the second origami axiom. Define the *mirror image* of a p by C to be the set of all mirror images of p with respect to tangent lines of C, and denote it by C_p.

Lemma 4.1. *Fold p onto a $p' \in C_p, p' \neq p$. Then the folding line is tangent to C.*

The notion of mirror image of a point may not be of common interest, but the next lemma shows it is a natural generalization of the directrix line of a parabola.

Lemma 4.2. *Let C be a conic, and p one of the foci. If C is an ellipse, C_p is a circle centered at the opposite focus, whose radius is the sum of the distances of a point on C to the foci. If C is a hyperbola, C_p is a two-punctured circle centered at the opposite focus, whose radius is the difference of the distances of a point on C to the foci. If C is a parabola, C_p is the directrix.*

Proof. Let C be an ellipse, f_1, f_2 the foci and $p = f_2$. Let d_i denote the distance of a $q \in C$ to the focus f_i. The bisection of the angle $\angle f_1qf_2$ is perpendicular to the tangent line T_qC. Thus the mirror image f_2' of f_2 by the tangent line is on the line f_1q. The distance of q to f_2' is equal to d_2, and the distance of f_2' to f_1 is equal to $d_1 + d_2$, that is independent of q. As q moves over C, f_2' rotates with the center f_1. Therefore C_p is the circle centered at f_1 with the radius $d_1 + d_2$. If C is a hyperbola, the tangent line TC_q is the bisector of $\angle f_1qf_2$. Thus the mirror image f_2' is on the line f_1q, and the distance of f_2' to f_1 is $|d_1 - d_2|$, which is constant. Therefore C_p is contained in the circle centered at the opposite focus f_1. Since the asymptotic lines of C are not tangent to C, the mirror images of f_2 by those two lines are deleted from the circle by definition. □

Remark that if a focus of an ellipse or a hyperbola is placed onto its mirror image by a fold, then the opposite focus is also placed onto its mirror image.

Let $C_1, C_2 \subset \mathbb{R}^2_{XY}$ be C^2-smooth curves. We will say C_i is *ordinary* at a $p \in C$ if the tangent line of C_i at p has contact of order 2 with C_i at p, or in other words, the curvature is nonzero at p. By the definition of dual curves provided in the previous section, we have the following one-to-one correspondences.

(a bis) Non-vertical lines tangent to C_1 and C_2 in $\mathbb{R}^2_{XY}$ at ordinary points

$\iff$ Points of $C_1^\vee \cap C_2^\vee \subset \mathbb{R}^2_{AB}$ where $C_i^\vee$ are ordinary and non-vertical,

(b bis) Points of $C_1 \cap C_2 \subset \mathbb{R}^2_{XY}$ where C_i are ordinary and non-vertical

$\iff$ Non-vertical lines tangent to $C_1^\vee$ and $C_2^\vee$ in $\mathbb{R}^2_{AB}$ at ordinary points.

In general, if $(X, Y) \in C$ is an isolated vertical point, $(X, Y)^\vee$ is a (non-vertical) asymptotic line of the dual curve $C^\vee$. In particular, if C is either an ellipse or a hyperbola, $C^\vee$ is a hyperbola with two asymptotic lines, one of which is $(X, Y)^\vee$ (see the list (Dual)). By Lemmas 4.1, 4.2, one can fold the tangent lines (possibly vertical) of the dual hyperbola by placing a focus onto its mirror image: a two-punctured circle. One can fold also asymptotic lines by placing a focus onto the punctured holes of the mirror image.

Define the *altered mirror image* of a p by $C \subset \mathbb{R}^2$ by firstly deleting the mirror images of p by the vertical tangent lines of C from C_p, and secondly filling with the mirror images of p by non-vertical asymptotic lines (see e.g. D_1 in the next section and Figure 5 for the case C is the hyperbola $AB = 1/4$ and p is a focus).

From the above dualities (a bis) and (b bis) we obtain

Lemma 4.3. *Let C_i be a non-vertical line or a non-singular conic in the XY-plane and $C_i^\vee$ its (algebraic) dual for $i = 1, 2$. If $C_i^\vee$ is a point, let $f_i = D_i = C_i^\vee$. If $C_i^\vee$ is a conic, let f_i be a focus and D_i its altered mirror image by $C_i^\vee$. Fold the AB-plane placing f_i onto D_i simultaneously for $i = 1, 2$. Then the folding line is not vertical. Write the folding line as $L : Y = AX + B$. Then the dual $L^\wedge = (X, Y)$ is a point of intersection of C_1 and C_2. Conversely if (X, Y) is an intersection point of C_1 and C_2, the fold along the dual line $L = (X, Y)^\vee$ places f_i onto D_i simultaneously for $i = 1, 2$.*

Proof. We prove only the first statement. By Lemma 4.1 and the definition of the altered mirror image, the folding line is either tangent or asymptotic to $C_i^\vee$ for $i = 1, 2$. Write the folding line as $L : Y = AX+B$. Then its dual $L^\wedge = (X, Y)$ is an intersection point of C_1 and C_2 by the above duality (b bis) and the subsequent argument for the case L is asymptotic to $C_i^\vee$. The converse is seen similarly. □

In particular, Y-coordinate of a point of intersection of C_1 and C_2 is read off from the B-intercept of the folding line L in the dual AB-plane placing p_i onto D_i.

In the following sections, we seek intersection points of the various C_1, C_2 in the XY-plane. Due to the above imperfect duality it is sufficient to deal with only non-vertical lines in the dual AB-plane, which are bi-tangent (or possibly asymptotic) to their dual curves.

5. Quadratic equation

Let $C_1 : X = Y^2$ be a parabola in the XY-plane. Then its dual $C_1^\vee$ is a hyperbola with foci f_1, g_1:

$$C_1 : X = Y^2, \quad C_1^\vee : AB = \frac{1}{4}, \quad f_1 : \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad g_1 : \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right).$$

The altered mirror image of f_1 by $C_1^\vee$ is by definition the one-punctured circle

$$D_1 : (A + \frac{1}{\sqrt{2}})^2 + (B + \frac{1}{\sqrt{2}})^2 = 2 \quad \setminus \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

The pair of f_1 and D_1 will be used as a universal template for solving equations of degrees 2,3,4 in §5,6,7,8.

Consider first the quadratic equation

$$\text{(Quad)} \qquad Y^2 + aY + b = 0.$$

This is equivalent to

$$C_1 : X = Y^2 \qquad \text{and} \qquad C_2 : X + aY + b = 0.$$

An intersection point of C_1, C_2 is an $(X, Y) = (Y^2, Y)$, Y being a solution of (Quad). The line C_2 is not vertical if $a \neq 0$, and its dual is $C_2^\vee : (-\frac{1}{a}, -\frac{b}{a})$.

Proposition 5.1. *Fold f_1 onto D_1 fixing $C_2^\vee$. Then the coordinate of the B-intercept of the folding line is a solution of* (Quad)*. This method gives all the real solutions.*

Proof. The statement follows from Lemma 4.3. □

Taking the limit as $a \to 0$, we obtain

Proposition 5.2. *Assume $a = 0$, $b \neq 0$. Let L_1 be the line with slope $-\frac{1}{b}$ passing through the origin. Fold f_1 onto D_1 and L_1 onto itself simultaneously. Then the coordinate of the B-intercept of the folding line is a solution of the equation $Y^2 + b = 0$. This construction gives all the real solutions.*

Proof. As L_1 is folded onto itself, the folding line has slope b. The folding line is tangent to the dual curve $C_1^\vee : AB = \frac{1}{4}$ by Lemmas 4.1, 4.2. The tangent line of $C_1^\vee$ with slope b is $B = bA + \sqrt{-b}$, that has the B-intercept $(0, \sqrt{-b})$. □

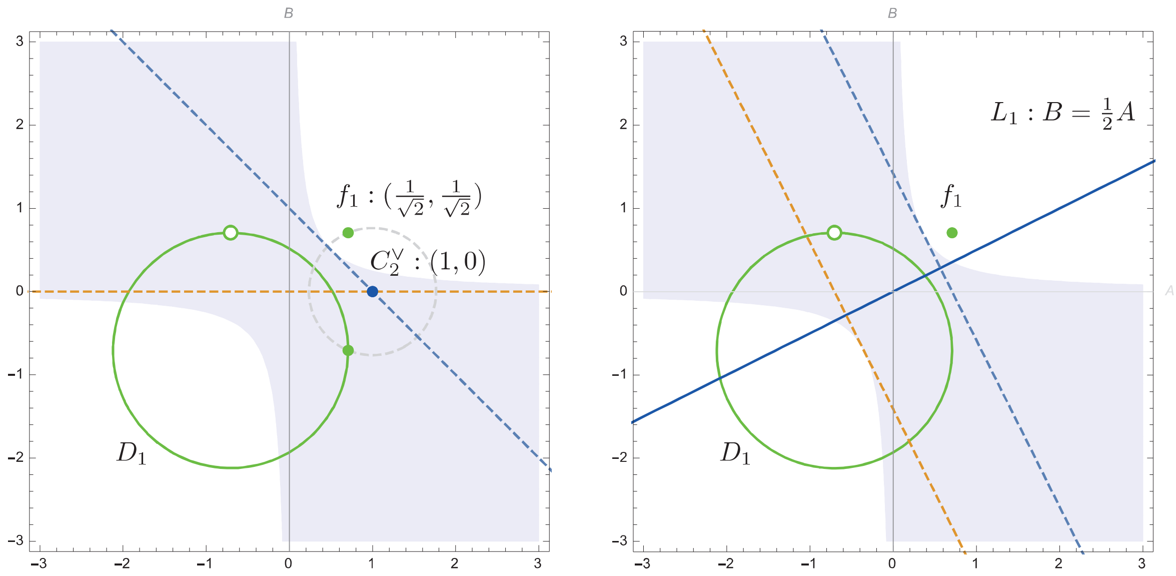


FIGURE 5. Left: $Y^2 - Y = 0$. The region shaded in blue is the region of f_1 for which the equation $Y^2 + aY + b = 0$ has real solutions; Right: $Y^2 - 2 = 0$. The dashed lines are the folding lines. The B-intersepts are the solutions of the equations.

Example 1. *Consider the equation $Y^2 - Y = 0$. Then $C_2^\vee = (1, 0)$. The folding lines passing through $(1, 0)$ and placing f_1 onto D_1 have the B-intercepts 0 and 1, which are the roots of the equation (Fig. 5, left). The horizontal folding line passing through the origin is an asymptotic line of the dual curve $C_1^\vee$, and the fold along it places f_1 onto the point of the altered mirror image D_1, that is appended to fill-in the hole of the mirror image of f_1 by $C_1^\vee$. In order to solve $Y^2 - 2 = 0$, draw the line L_1 with slope $\frac{1}{2}$ passing through the origin. The parallel folding lines given in the proposition have the B-intercepts $\sqrt{2}$ and $-\sqrt{2}$ (Fig. 5, right).*

6. Quartic equation: asymmetric case

Consider the real quartic equation

(Quartic) $$Y^4 + aY^3 + bY^2 + cY + d = 0.$$

This is equivalent to

$$C_1 : X = Y^2, \qquad C_3 : X^2 + aXY + bX + cY + d = 0.$$

By substituting Y with $Y - \frac{1}{4}a$ (i.e. a Tschirnhaus transformation of degree 1 defined in §9), we may assume $a = 0$. Thus our equation will be

(Reduced Quartic) $$Y^4 + bY^2 + cY + d = 0 \quad \text{or} \quad X^2 + bX + cY + d = 0.$$

Assume $c \neq 0$. (The case $b \neq 0, c = 0$ will be dealt in the next section.) Then an intersection point of the parabolas C_1 and

$$C_3 : Y = \frac{-1}{c}(X^2 + bX + d)$$

is $(X, Y) = (Y^2, Y)$, where Y is a solution of the equation (Reduced Quartic). The dual of C_3 is the vertical parabola

$$C_3^\vee : B = \frac{c}{4}(A + \frac{b}{c})^2 - \frac{d}{c} \qquad \text{with focus} \quad f_3 : (-\frac{b}{c}, \frac{1-d}{c}),$$

and its mirror image by $C_3^\vee$ is the directrix

$$D_3 : B = -\frac{1+d}{c}.$$

Proposition 6.1. *Fold f_1, f_3 onto D_1, D_3 simultaneously. Then the folding line is transverse to the B-axis. The coordinate of the B-intercept of the folding line is a solution of* (Reduced Quartic). *This method gives all the real solutions.*

Proof. Since the B-coordinate of f_3 is different from those of points on D_3, the folding line is not vertical. The statement follows from Lemma 4.3. □

Example 2. *Consider the quartic equation with $a = 0, b = -4, c = 1, d = 2$,*

$$Y^4 - 4Y^2 + Y + 2 = 0.$$

This has two real solutions $Y = -2, -0.618\ldots, 1, 1.618\ldots$. These solutions are obtained by folding f_1 onto D_1 and

$$f_3 : (-\frac{b}{c}, \frac{1-d}{c}) = (4, -1)$$

onto

$$D_3 : B = -\frac{1+d}{c} = -3$$

simultaneously. These curves and folding lines are presented in Fig.6, left.

Remark 1. *As mentioned in §2, simultaneous folding (in the 6th axiom) of distinct points $p_1 \neq p_2$ onto non-parallel lines $L_1 \neq L_2$ is equivalent to a cubic equation. More in general let S_i be a circle or a line for $i = 1, 2$ and assume $p_i \notin S_i$ and S_1, S_2 are not simultaneously lines. Then simultaneous folding of p_i onto S_i, $i = 1, 2$, is equivalent to a quartic equation. But the space of configurations of two points and two "circles" is of dimension 10 while the space of reduced quartic equations is of dimension 3. Proposition 6.1 reduces the problem to the special configuration: a point, a horizontal line and a pre-fixed set of a point and a circle; the space of such configurations is of dimension 3 and corresponds in one-to-one to the space of the reduced quartics with $c \neq 0$.*

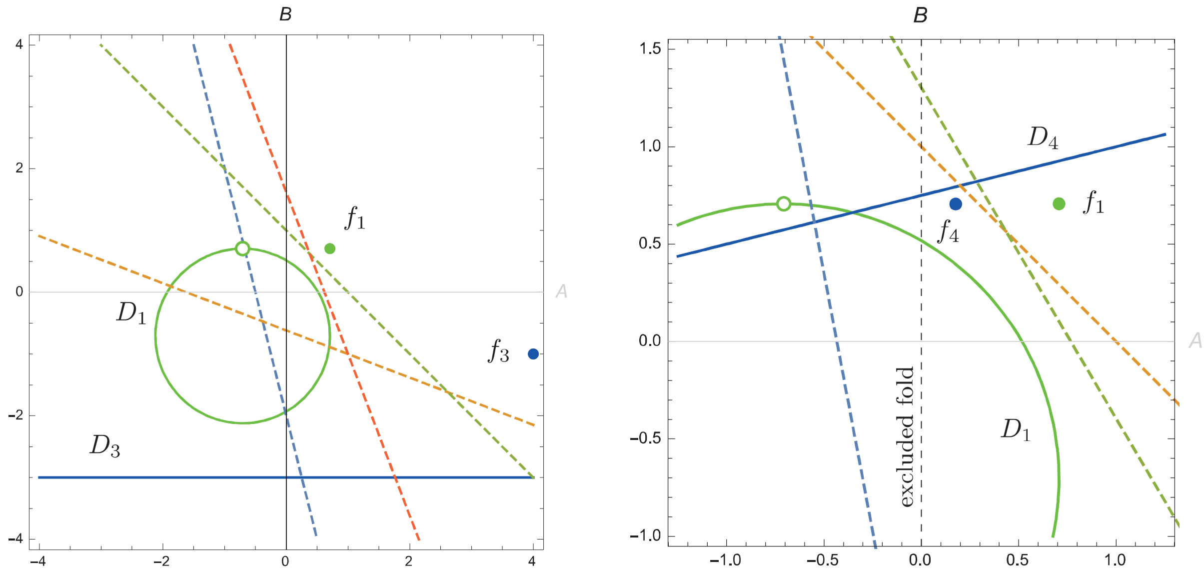


FIGURE 6. Left: Solving $Y^4 - 4Y^2 + Y + 2 = 0$; Right: Solving $Y^3 - 4Y + 3 = 0$: The reflection of f_4 about B-axis is the foot of perpendicular to D_4 from the origin. The folding line in black dashed line is vertical and does not give a solution of the cubic equation.

7. Quartic equation: symmetric case ($c = 0$)

Assume $b \neq 0, c = 0$ and $d \neq 0$. In order to solve $X^2 + bX + d = 0$ first, consider the curves

$$C_1' : Y = X^2, \qquad C_2' : Y + bX + d = 0.$$

The duals of these curves are

$$C_1'^{\vee} : B = -\frac{1}{4}A^2, \qquad C_2'^{\vee} : (-b, -d)$$

and $C_1'^{\vee}$ has

$$\text{focus} \qquad f_2' : (0, -1) \qquad \text{and directrix} \qquad D_2' : B = 1.$$

Fold the plane fixing $C_2'^{\vee}$ and placing $(0, -1)$ onto D_2'. Then the folding line is tangent to $C_1'^{\vee}$ and passing through $C_2'^{\vee}$, hence its dual is a point of intersection of C_1', C_2', that is an (α, α^2), α being a solution of the quadratic equation $X^2 + bX + c = 0$. Thus the slope of the folding line is $-\alpha$. Consider now the $-\pi/2$-rotations of $f_2', D_2', C'{}_2^{\vee}$:

$$f_2'' : (-1, 0), \qquad D_2'' : A = 1, \qquad C''{}_2^{\vee} : (-d, b).$$

Proposition 7.1. (1) *Fold $f_2'' : (-1, 0)$ onto $D_2'' : A = 1$ fixing $C_2''^{\vee} : (-d, b)$. Then the folding line L_2 has a slope $\frac{1}{\alpha}$, where α is a solution of $X^2 + bX + d = 0$.*
(2) *Fold f_1 onto D_1 and the folding line L_2 in (1) with a positive slope onto itself simultaneously. Then the coordinate of the B-intercept of a folding line is a solution of the equation.*
All the real solutions of (Reduced Quartic) *are obtained by this method.*

Proof. The statement (1) is clear by the argument before the proposition. The statement (2) follows from Proposition 5.2. □

Example 3. *Consider a quartic equation with $a = c = 0$ and $b = 2, d = -3$,*

$$Y^4 + 2Y^2 - 3 = 0.$$

Clearly $Y^2 = 1, -3$. Fold $f_2'' : (-1, 0)$ onto $D_2'' : A = 1$ fixing $C_2''^{\vee} : (-d, b) = (3, 2)$. Then the folding lines have slopes 1 and $-\frac{1}{3}$ (see the figure on the left in Fig. 7). Let L_2 be the folding line with positive slope: i.e. 1. Next fold f_1 onto D_1 and $(-d, b) = (3, 2)$ onto L_2 simultaneously. The folding lines have the B-intercepts with coordinates 1 and -1, that are the real solutions of the equation.

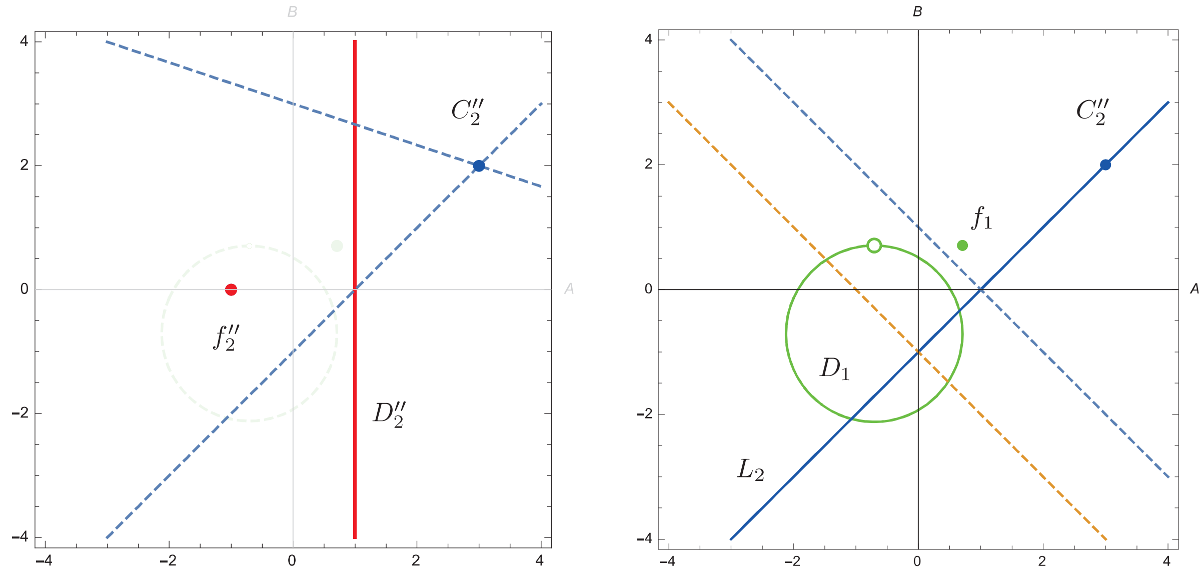


FIGURE 7. Solving $Y^4 + 2Y^2 - 3 = 0$ by two foldings.

8. Cubic equations

We apply the same template to the real cubic equation $Y^3 + aY^2 + bY + c = 0$. This does not provide the simplest solving method, but enables us to discriminate between the cubic and the quartic equations in a common perspective.

Again by a suitable real Tschirnhaus transformation we may assume $a = 0$:

(Reduced Cubic) $$Y^3 + bY + c = 0.$$

Using $X = Y^2$ this equation becomes $XY + bY + c = 0$. Therefore the (X, Y), Y being a solution of the equation (Reduced Cubic), is an intersection point of the parabola $C_1 : X = Y^2$ and the hyperbola

$$C_4 : Y = \frac{-c}{X + b}.$$

The dual of C_4 is, assuming $c \neq 0$,

$$C_4^\vee : A = \frac{(B - bA)^2}{4c},$$

which has the focus and the directrix

$$f_4 : (\frac{c}{1+b^2}, \frac{-bc}{1+b^2}) \quad \text{and} \quad D_4 : B = -\frac{A+c}{b} \quad (A = -c \text{ if } b = 0).$$

Proposition 8.1. *Fold the AB-plane placing f_1, f_4 onto D_1, D_4 simultaneously. Then the folding line is transverse to the B-axis, and the coordinate of its B-intercept is a solution of the cubic equation* (Reduced Cubic). *This method gives all the real solutions of* (Reduced Cubic).

Proof. The statement follows from Lemma 4.3. □

Example 4. *Consider the cubic equation with $a = 0, b = -4, c = 3$:*

$$Y^3 - 4Y + 3 = 0.$$

This has real solutions $Y = -2.302\ldots, 1$ and $1.302\ldots$, which are obtained by folding simultaneously f_1 onto D_1 and

$$f_4 : (\frac{c}{1+b^2}, \frac{-bc}{1+b^2}) = (-\frac{3}{17}, \frac{12}{17}) \quad \textit{onto} \quad D_4 : B = -\frac{A+c}{b} = \frac{A+3}{4}.$$

These curves and folding lines are presented in Fig.6, right. The folding along the B-axis also places f_4 onto D_4 but also f_1 onto the punctured hole of the template D_1.

A "generic" pair of plane conics has four points of intersection counting multiplicity and intersections at infinity by Bézout theorem. For instance the above C_1, C_4 intersect on the line at infinity. This may be better understood if we look at the dual curves $C_1^\vee, C_4^\vee$. Clearly the B-axis is tangent to $C_4^\vee$, but also "tangent" to the hyperbola $C_1^\vee$ at infinity. Thus the B-axis is a common "tangent line", which is excluded from our space of non-vertical lines and not constructed by the scheme of Lemma 4.3. Thus one intersection point of C_1, C_4 is always missing on our XY-plane. This phenomenon is also seen from the configuration of the above f_4, D_4: the mirror image of f_4 by the B-axis is always on the directrix D_4.

Remark 2. *In general simultaneous folding of f_1 and a point f_4 onto D_1 and a line D_4 is equivalent to solving a quartic equation by Lemma 4.3 and Remark 1. If the configuration of f_4, D_4 is symmetric, i.e. D_4 is non-horizontal and the reflection about the B-axis places f_4 onto D_4, the equation reduces to a cubic equation. Proposition 8.1 reduces all cubic equations to special configurations among symmetric configurations, such that the reflection about B-axis places f_4 onto the foot of perpendicular line to D_4 from the origin. Thus the space of such configurations coincides with the space of non-horizontal lines, that is of dimension 2 and corresponds to the space of reduced real cubic equations in one-to-one.*

9. Quintic equations

Consider the real quintic equation

$$Y^5 + aY^4 + bY^3 + cY^2 + dY + e = 0. \tag{Quintic}$$

Tschirnhaus transformation by a polynomial (or possibly an analytic function) $f(Y)$ is an operation on polynomials, respecting degree, defined by

$$(Y - \alpha_1)(Y - \alpha_2)\cdots(Y - \alpha_d) \quad \to \quad (Y - f(\alpha_1))(Y - f(\alpha_2))\cdots(Y - f(\alpha_d)).$$

It is known [7, 20] that a suitable Tschirnhaus transformation of degree four $f(Y) = k_4Y^4 + k_3Y^3 + k_2Y^2 + k_1Y + k_0$ (with complex coefficients) reduces the quintic equation to the (complex) Bring-Jerrard quintic form

$$Y^5 + dY + e = 0. \tag{Bring-Jerrard form}$$

One obtains a root $Y = \alpha_i$ of the original (Quintic) from a root $f(\alpha_i)$ of Bring-Jerrard form, by solving the quartic equation $f(Y) = f(\alpha_i)$ in Y. Although Tschirnhaus transformation has been long investigated since it was introduced in [30], real Tschirnhaus transformation of higher degree seems not to be explored in depth yet from our view point, and would be an interesting research subject. Thus in the current paper, we restrict ourselves to employing rather ad-hoc real transformations to eliminate the various terms of the equations.

If the Bring-Jerrard form is real and $d \neq 0$, one may reduce $d = \pm 1$ by a real linear rescaling of Y. The branch Y of $Y^5 + Y + e = 0$ which is real for real e is known as the ultra-radical or Bring radical of e. Although a real Bring-Jerrard quintic form admits at most three real solutions, we will first give a nomogram solution to this prototypical equation.

Define two curves C_5, C_6 in the XY-plane by

$$C_5 : X = Y^5, \qquad C_6 : X = -dY - e \quad \text{or} \quad Y = -\frac{1}{d}X - \frac{e}{d} \quad \text{assuming} \quad d \neq 0$$

and consider their duals $C_5^\vee : AB^4 = \frac{4^4}{5^5} : (\frac{1}{5t^4}, \frac{4}{5}t)$ and $C_6^\vee : (-\frac{1}{d}, -\frac{e}{d})$. In order to fold a tangent line (possibly a non-vertical asymptotic line) of $C_5^\vee$ it is enough to fold a point p off $C_5^\vee$ onto its altered mirror image by $C_5^\vee$ (Lemma 4.1).

Now let us consider the dual of $C : X = Y^n$ in general for $n \geq 2$. The curve C has a parameterization (t^n, t) and the tangent line $Y = \frac{1}{nt^{n-1}}X + \frac{n-1}{n}t$ at $C(t)$, thus the dual is $C^\vee : AB^{n-1} = \frac{(n-1)^{n-1}}{n^n}$ and its tangent line is $B = -t^nA + t$ at $C^\vee(t)$ by the dualities (b) and (Bi-duality) in §3. We choose the origin as p in general for $n \geq 2$, since there seems to be no other appropriate choice (except for the case $n = 2$). The reflection of the origin by the tangent line at $C^\vee(t)$ is $(\frac{2t^{n+1}}{1+t^{2n}}, \frac{2t}{1+t^{2n}})$, thus the mirror image of the origin by $C^\vee$ is given by

$$B_n : 2^nAB^{n-1} = (A^2 + B^2)^n \qquad \text{or} \qquad 2^n \cos\theta \sin^{n-1}\theta = r^n$$

in the polar coordinates (r, θ). We fill-in the origin and we assume the folding of the origin onto itself is the horizontal fold about A-axis. B_2 is well known as the lemniscate (see Figure 8).

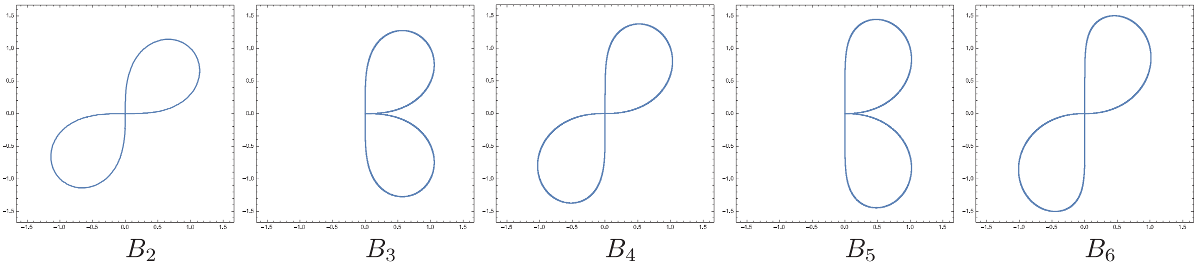


FIGURE 8. B_n is defined by $r^n = 2^n \cos\theta \sin^{n-1}\theta$ in the polar coordinates, in other word, the *inversion* of the curve $AB^{n-1} = 2^{-n}$ by the unit circle centered at the origin. B_2 is the lemniscate.

Similarly to Proposition 5.1, we obtain

Proposition 9.1. *If $d \neq 0$, fold the origin onto B_5 fixing $C_6^\vee : (-\frac{1}{d}, -\frac{e}{d})$. Then the coordinate of the B-intercept of the folding line is a solution of real* (Bring-Jerrard form). *This method gives all the real solutions.*

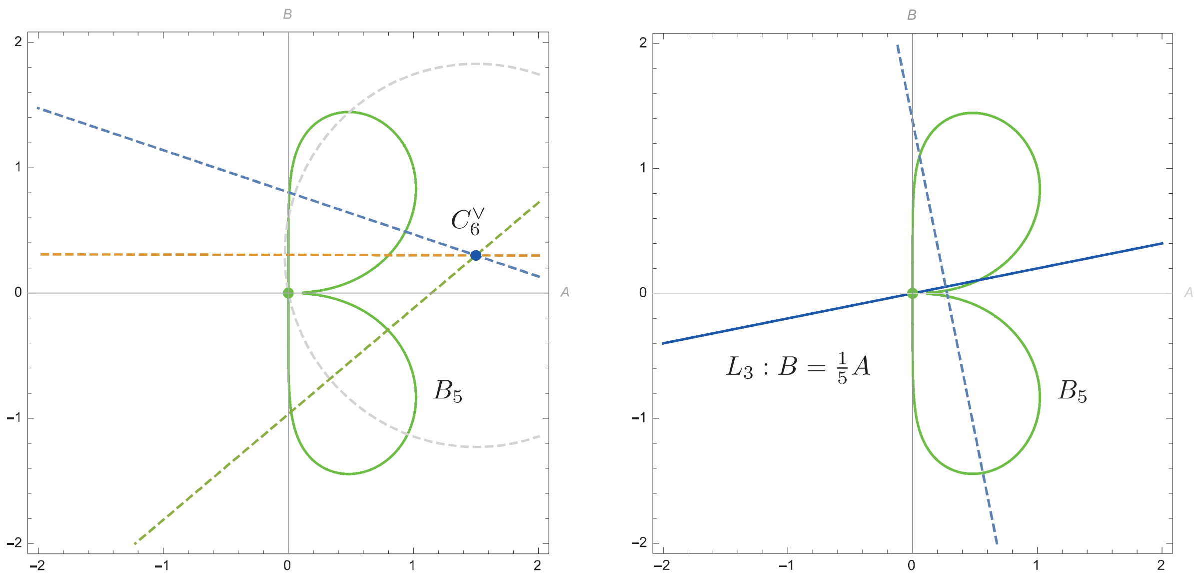


FIGURE 9. Left: solving $y^5 - 2/3y + 1/5 = 0$ by one single fold; Right: the fifth root of 5

Example 5. *Consider the quintic equation with $d = -\frac{2}{3}, e = \frac{1}{5}$,*

$$Y^5 - \frac{2}{3}Y + \frac{1}{5} = 0.$$

This has three real roots $Y = -0.966765\cdots$, $0.303887\cdots$, $0.804032\cdots$. These roots are obtained by folding the origin onto B_5 fixing $C_6^\vee = (\frac{3}{2}, \frac{3}{10})$ (see Fig. 9, left).

For the case $d = 0$, we seek a folding line with slope e that places the origin onto B_5 as in the course of proof of Proposition 5.2.

Proposition 9.2. *Assume $d = 0$, and let L_3 be the line with slope $-\frac{1}{e}$ passing through the origin. Fold the origin onto the point of intersection of B_5 and L_3. Then the coordinate of the B-intercept of the folding line is the fifth root of $-e$.*

Example 6. *Consider the quintic equation with $d = 0, e = -5$:*

$$Y^5 - 5 = 0.$$

Draw the line L_3 with slope $\frac{1}{5}$ passing through the origin. Fold the origin onto the point of intersection of D_5 and L_3. The B-intercept of the folding line is the fifth root of 5*, that is* $1.3797\cdots$ *(see Fig. 9, right).*

Now let us consider the real quintic equation (Quintic). By a suitable real Tschirnhaus transformation in the form $Y \to kY + \ell$, the equation can be reduced to either the canonical form

(Reduced form) $$Y^5 \pm Y^3 + cY^2 + dY + e = 0$$

or

(Canonical form) $$Y^5 + cY^2 + dY + e = 0.$$

Note that if the alternative sign in the left hand side of (Reduced form) is positive, the equation admits at most 3 real solutions like Bring-Jerrard form. Thus the case of negative will be our major interest.

Following the argument for (Bring-Jerrard form) let us prepare the duals of the curves

$$C_{5,\pm} : X = Y^5 \pm Y^3, \qquad C_7 : X = -(cY^2 + dY + e).$$

Assume $c \neq 0$. Then the dual $C_7^\vee$ is a hyperbola such that the B-axis is an asymptotic line. Put

$$\alpha = \frac{d^2 - 4ce}{4c}, \qquad \beta = -\frac{d}{2c}.$$

Then

$$f_7 : \left(\mp\{\frac{1}{2\,|c|}(\sqrt{1+\alpha^2} \mp \alpha)\}^{\frac{1}{2}}, \quad \beta + \{\frac{1}{2\,|c|}(\sqrt{1+\alpha^2} \pm \alpha)\}^{\frac{1}{2}} \right)$$

is a focus of the dual hyperbola, where the ~~sign~~ $\pm$ is determined to be that of c and $\mp$ denotes its opposite. The altered mirror image of f_7 by the dual curve $C_7^\vee$ is a one-punctured circle

$$\begin{aligned} D_7: \quad & \text{radius } \{\frac{2}{|c|}(\sqrt{1+\alpha^2} \pm \alpha)\}^{\frac{1}{2}} \\ & \text{centered at } \left(\pm\{\frac{1}{2\,|c|}(\sqrt{1+\alpha^2} \mp \alpha)\}^{\frac{1}{2}},\ \beta - \{\frac{1}{2\,|c|}(\sqrt{1+\alpha^2} \pm \alpha)\}^{\frac{1}{2}} \right) \\ & \text{punctured at } \left(\pm\{\frac{1}{2\,|c|}(\sqrt{1+\alpha^2} \mp \alpha)\}^{\frac{1}{2}},\ \beta + \{\frac{1}{2\,|c|}(\sqrt{1+\alpha^2} \pm \alpha)\}^{\frac{1}{2}} \right). \end{aligned}$$

The altered mirror image of the origin by the dual curve $C_{5,\pm}^\vee$ is the trajectory of the reflection of the origin by $B = -(t^5 \pm t^3)A + t$, i.e.

$$B_{5,\pm} : 2^5 AB^4 = (A^2 + B^2)^3((A^2 + B^2)^2 \pm 2^2 B^2).$$

Proposition 9.3. *If* $c \neq 0$*, fold the origin onto* $B_{5,\pm}$ *(respectively* B_5*) and* f_7 *onto* D_7 *simultaneously, and if* $c = 0$*, fold the origin onto* $B_{5,\pm}$ *(resp.* B_5*) fixing* $C_6^\vee : (-\frac{1}{d}, -\frac{e}{d})$*. Then the coordinate of the* B*-intercept of the folding line is a solution of the equation* (Reduced form) *(resp.* (Canonical form)*). All the real solutions are obtained in this method.*

Remark 3. *The real* (Canonical form) *is a spacial case of the real* (Alternative reduced form) defined below in this section. Thus the real (Canonical form) *is solvable with* B_4 *by Proposition 9.4,*

Example 7. ($B_{5,+}$) *Consider the equation*

$$Y^5 + Y^3 - 3Y^2 - 3Y + 2 = 0,$$

which is equivalent to

$$C_{5,+} : X = Y^5 + Y^3, \qquad C_7 : X = 3Y^2 + 3Y - 2$$

and has three real solutions $-1,\ 0.480\ldots, 1.396\ldots$*, two other non-real roots, and a negative discriminant* -564223*. Let us employ the method in Proposition 9.3. Similarly to Proposition 9.1, fold the origin onto* $B_{5,+}$ *and* $f_7 : (-0.171\ldots, -1.472\ldots)$ *onto*

D_7*: circle with radius* $1.945\ldots$ *centered at* $(0.171\ldots, 0.472\ldots$ *and punctured at* $(0.171\ldots, -1.472\ldots)$.

Then the coordinate of the B*-intercept of the folding line is is a real root of the equation (see Figure 10, left).*

Example 8. ($B_{5,-}$ as a Quintsectrix) *The Chebyshev polynomial of degree* 5 *of the first kind is*

$$T_5(Y) = 16Y^5 - 20Y^3 + 5Y,$$

which is defined by the relation $T_5(\cos\theta) = \cos 5\theta$. *Solving the equation*

$$16Y^5 - 20Y^3 + 5Y - \cos\theta = 0$$

one obtains $\cos(\theta/5)$ *and also* $\cos(\theta/5+2\pi/5), \cos(\theta/5+4\pi/5), \cos(\theta/5+6\pi/5), \cos(\theta/5+8\pi/5)$. *Although this equation can be transformed to* (Alternative reduced form), *let us employ the method in Proposition 9.3 for simplicity, as Proposition 9.4 does not provide a reasonable procedure in this case. Replacing* Y *with* $\frac{\sqrt{5}}{2}Y$, *the equation is reduced to the following reduced form*

$$Y^5 - Y^3 + \frac{1}{5}Y - \frac{2\cos\theta}{25\sqrt{5}} = 0$$

which is equivalent to

$$C_{5,-} : X = Y^5 - Y^3, \qquad C_8 : X = -\frac{1}{5}Y + \frac{2\cos\theta}{25\sqrt{5}}.$$

Similarly to Proposition 9.1, fold the origin onto $B_{5,-}$ *fixing* $C_8^\vee : (-5, \frac{2\cos\theta}{5\sqrt{5}})$. *Then the coordinate of the* B*-intercept of the folding line is* $\frac{2}{\sqrt{5}}\cos(\theta/5 + 2n\pi/5)$ *for an* $n = 0, 1, 2, 3, 4$.

The real quintic form (Quintic) can be deformed also to the following alternative form by a real Tschirnhaus transformation of degree ≤ 4 under a certain condition.

(Alternative reduced form) $$Y^5 + aY^4 \quad + cY^2 + dY + e = 0.$$

For instance, if $2a^2 - 5b \geq 0$ or particularly the sign in the left hand side of our (Reduced form) is negative, the transformation is found in the form $Y \to Y + k$ with a real k.

The above (Alternative reduced form) is equivalent to

$$C_9 : X = Y^4, \qquad C_{10} : XY + aX + cY^2 + dY + e = 0.$$

Assume C_{10} is non-degenerate, and let f_{10} be one of the foci of the dual curve $C_{10}^\vee$.

Proposition 9.4. (Quintic equation) *Fold the origin onto* B_4 *and* f_{10} *onto its altered mirror image* D_{10} *by* C_{10} *simultaneously. Then the coordinate of the* B*-intercept of the folding line is a solution of* (Alternative reduced form). *All the real solutions of the equation are obtained by this method.*

Example 9. *The Regular Hendecagon (*11*-gon) inscribed in a unit circle is spanned by the vertices on the circle* $\omega^n = \exp 2n\pi/11$, $n = 0, 1, \ldots, 10$. *In order to fold the hendecagon, it suffices to fold* $Z_n = \omega^n + \omega^{-n} = 2\cos 2n\pi/11$, $n = 1, \ldots, 5$. *Clearly* $\omega, \ldots, \omega^{10}$ *fulfill*

$$W^{10} + W^9 + W^8 \cdots + W + 1 = 0,$$

from which, putting $Z = W + W^{-1}$,

$$Z^5 + Z^4 - 4Z^3 - 3Z^2 + 3Z + 1 = 0,$$

and putting $Z = Y - (1 + \sqrt{11})/5$, *we arrive at the alternative reduced form*

$$Y^5 - \sqrt{11}Y^4 + \frac{11}{5^2}(-1 + 4\sqrt{11})Y^2 + \frac{11}{5^3}(-13 + 2\sqrt{11})Y - \frac{11(-34 + 111\sqrt{11})}{5^5} = 0.$$

This equation is known to be solvable using radicals. Here we give a origami procedure of solving this equaltion by Proposition 9.4. Let

$$a = -\sqrt{11}, \quad b = \frac{11}{5^2}(-1 + 4\sqrt{11}), \quad d = \frac{11}{5^3}(-13 + 2\sqrt{11}), \quad e = -\frac{11(-34 + 111\sqrt{11})}{5^5}.$$

Then $f_{10} = (-0.555\ldots, -0.601\ldots)$ *and the circle* D_{10} *has radius* $1.319\cdots$ *and center* $(-0.370\cdots, 0.704\cdots)$. *The five folding lines given by Proposition 9.4 have the* B*-intercepts with coordinates*

$$-1.055\cdots, \quad -0.446\cdots, \quad 0.578\cdots, \quad 1.694\cdots, \quad 2.545\cdots,$$

and the last one is $2\cos 2\pi/11 + \frac{1+\sqrt{11}}{5}$ *(see Figure 10, right).*

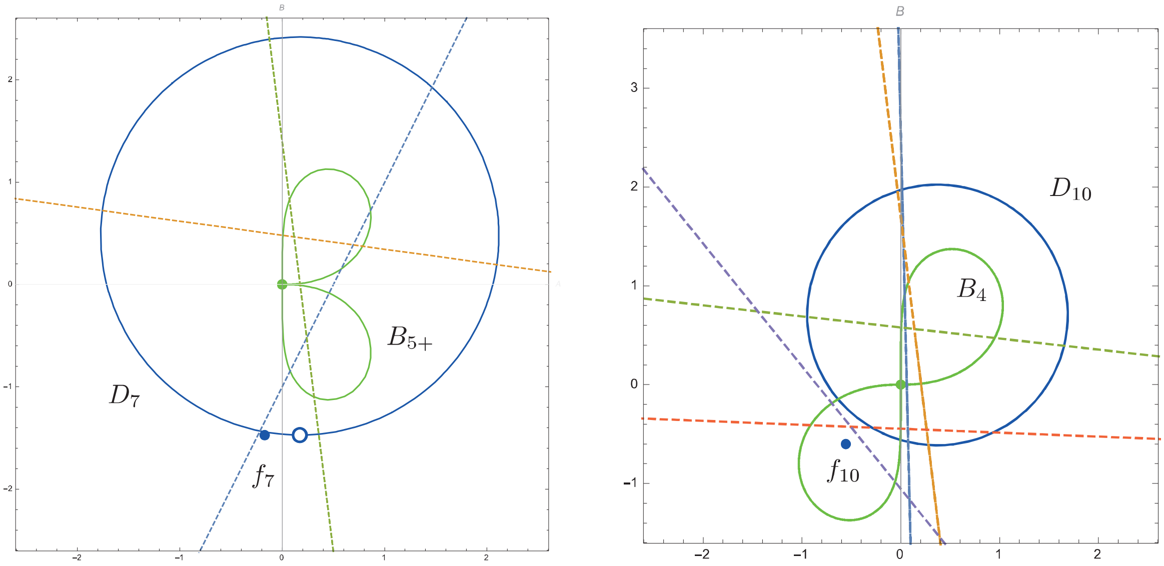


FIGURE 10. Left: Solving $Y^5 + Y^3 - 3Y^2 - 3Y + 2 = 0$ with $B_{5,+}$ (Example 7), Right: Folding $2\cos 2n\pi/11 + \frac{1+\sqrt{11}}{5}$, $n = 1, 2, 3, 4, 5$ (Example 9).

By a result of Cayley [9] in 1861, a real quintic equation with a non-negative discriminant is reduced to a real sextic equation (see [23] for the details). Here the discriminant of a quintic equation is defined to be $\Pi_{i<j}(\alpha_i - \alpha_j)^2$ with the roots $\alpha_1, \ldots, \alpha_5$ of the equation. So if the roots are all real, the discriminant is positive. Surprisingly real sextic equations are solvable by origami with the curve B_3 (Proposition 10.1). Thus we have

Proposition 9.5. *All quintic equations with non-negative discriminants are solvable by origami with B_3.*

10. Sextic equations

Let us consider the real sextic equation

(Sextic) $$Y^6 + aY^5 + bY^4 + cY^3 + dY^2 + eY + f = 0.$$

Again by a suitable real Tschirnhaus transformation one can reduce a to 0. The resulting equation is equivalent to

$$C_{11} : X = Y^3, \qquad C_{12} : X^2 + bXY + cX + dY^2 + eY + f = 0.$$

Assuming C_{12} is non-degenerate: neither a union of two lines nor a double-line, its dual $C_{12}^\vee$ is a non singular conic. Let f_{12} be one of the foci of $C_{12}^\vee$.

Proposition 10.1. (Sextic equation) *Fold the origin onto B_3 and f_{12} onto its altered mirror image by $C_{12}^\vee$ simultaneously. Then the coordinate of the B-intercept of the folding line is a solution of* (Sextic). *All the real solutions of the equation are obtained by this method.*

The degenerated case can be also treated as in the previous sections.

Example 10. *Let us consider*

$$Y^6 - 7Y^4 + +Y^3 + 10Y^2 - 2 = 0,$$

which is equivalent to

$$C_{11} : X = Y^3, \qquad C_{12} : X^2 - 7XY + 10Y^2 + X - 2 = 0.$$

The curve C_{12} is a hyperbola, and its dual is an ellipse with the center at $(7/20, 0)$, the foci at $(0.235\ldots, 0.436\ldots)$, $(0.464\ldots, -0.436\ldots)$ and the major axis $0.900\ldots$. Let $f_{12} = (0.464\ldots, -0.436\ldots)$. Then its altered mirror image by $C_{12}^\vee$ is the circle D_{12} centered at $(0.235\ldots, 0.436\ldots)$ with radius $0.900\ldots$. There exist six folds that place f_{12} onto D_{12} and the origin onto B_3 simultaneously. The coordinates of the B-intercepts of those folding lines are

$$-2.394\ldots, \quad -1.129\ldots, \quad -0.507\ldots, \quad 0.472\ldots, \quad 1.485\ldots, \quad 2.073\ldots,$$

which are the real solutions of the equation (see Figure 11).

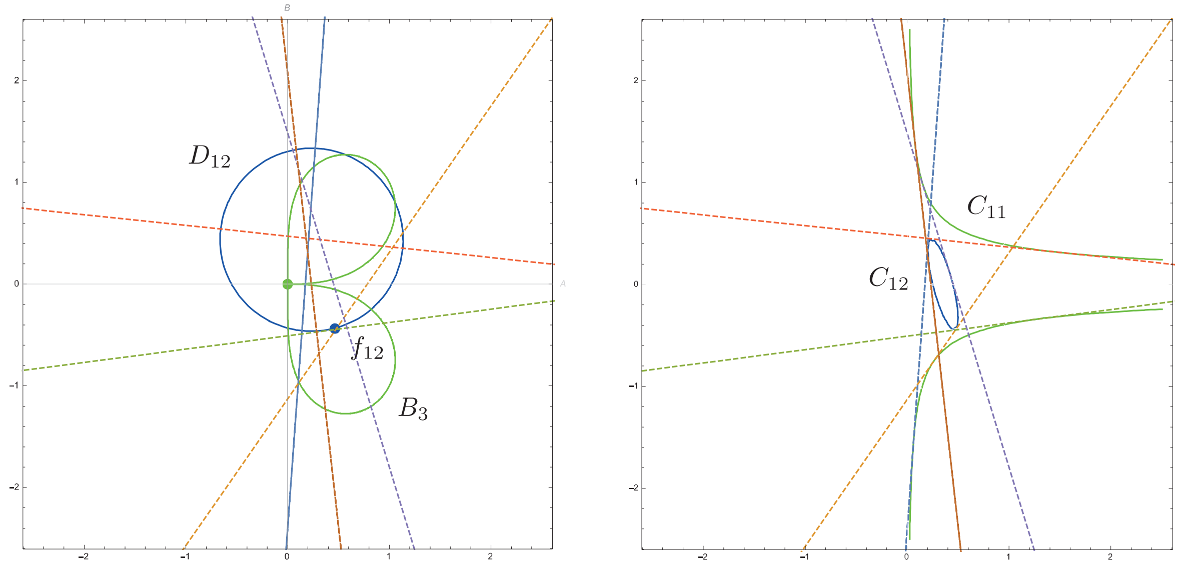


FIGURE 11. Left: Solving $Y^6 - 7Y^4 + +Y^3 + 10Y^2 - 2 = 0$ by one single-fold with B_3. The focus f_{12} is slightly inside the circle D_{12} in blue (see Example 10 for the details), Right: The figure shows the folding lines are tangent to the dual curves of C_{11} and C_{12}.

Our results exhibit importance of the graphical use of the curves B_n. For instance, Propositions 9.4 10.1 allow us the use of real Tschirnhaus transformations of degree 5 and 6, in order to explore the algebraic equations of degree ≥ 7. Then, the drawing of B_n for $n \geq 3$ in an effective method would be our next issue.

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First author: Dept. Math., Faculty of Sci., Ochanomizu Univ., Bunkyo-ku, Ohtsuka 2-1-1, Tokyo, Japan
E-mail address: nakai.isao@ocha.ac.jp