Reverse inequality of triangle inequality and a generalization

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Abstract

In this study, we derive a reverse inequality of the triangle inequality under various conditions. It is important that the condition is natural and is related to the meaning of original inequality. We introduce an extension of Pythagorean theorem in order to understand the necessity of the condition. The concept of convex and concave functions are related to our consideration of reverse inequality. These novel reverse inequalities are interesting and provide in-depth understanding of the original inequalities.

1 Introduction

Let a, b, c be lengths of three sides of a triangle. Triangle inequality is represented as follows.

$$a + b \geq c, \qquad a, b, c > 0.$$

We consider the reverse inequality in the following type:

$$a + b \leq kc. \tag{1}$$

where k is a positive constant depending on the triangle in a certain way.

First, we consider elementary triangles. For a regular triangle, we have $a+b = 2c$. Therefore $k = 2$ in the inequality (1). For a right angled triangle with $a^2 + b^2 = c^2$, we have

$$\frac{(a+b)^2}{c^2} = \frac{(a+b)^2}{a^2+b^2} = \frac{a^2+b^2+2ab}{a^2+b^2} \leq \frac{2(a^2+b^2)}{a^2+b^2} = 2.$$

When $a = b$, the equality holds. Therefore, $k = \sqrt{2}$ is the lowest uniform bound for the right angled triangles.

In section 2, we consider reverse inequalities about general triangles. We need a condition to make reverse inequalities. The case of right angled triangle shows us a hint for the necessary condition. We first consider an extension of Pythagorean theorem

$$a^p + b^p = c^p \tag{2}$$

as a condition on the lengths a, b, c, and we show a reverse inequality of triangle inequality using the p.

In section 3, we generalize these results for any number of variables $a_1, a_2, \ldots, a_n$ instead of a, b. These proofs contain those of Theorem 1 and Theorem 2 in section 2.

2 Reverse inequality of triangle inequality

Theorem 1. *Let $a > 0, b > 0, c > 0$.*

(i-a) $a + b \geq c, a < c, b < c$ *if and only if there exists a* $p \in [1, \infty)$ *such that* $a^p + b^p = c^p$.

(i-b) $a > c, b > c$ *if and only if there exists a* $p \in (-\infty, 0)$ *such that* $a^p + b^p = c^p$.

(ii) $a + b < c$ *if and only if there exists a* $p \in (0, 1)$ *such that* $a^p + b^p = c^p$.

(iii) $a \leq c \leq b$ *if and only if there does not exist a* p *such that* $a^p + b^p = c^p$.

Each of Figures 1 to 4 corresponds respectively to the cases (i-a), (i-b), (ii), and (iii) in Theorem 1 when $c = 1$. In Figure 5 triangles are illustrated for various p when $a = b$. Figure 6 shows the relation between p and a when $a = b$ and $c = 1$. Then the Pythagorean equality becomes $2a^p = 1$, that is $a = \exp\left(-\dfrac{\log 2}{p}\right)$. This graph explains the situation of Figure 5. In particular we can understand that regular triangles are the cases that p is $\pm\infty$ in $a^p + b^p = c^p$.

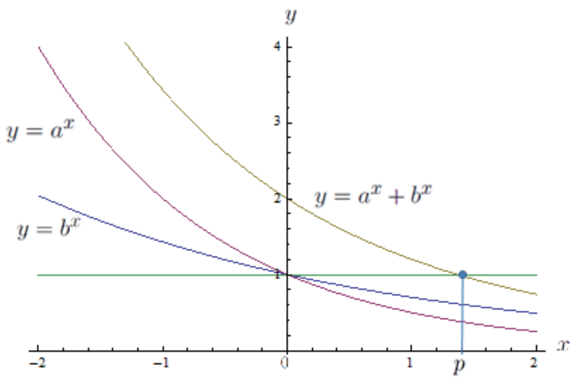

Figure 1. Case (i-a)
The existence of $p \in [1, \infty)$ such that $a^p + b^p = c^p$ when $a < 1, b < 1, c = 1$, $a + b \geq c$.

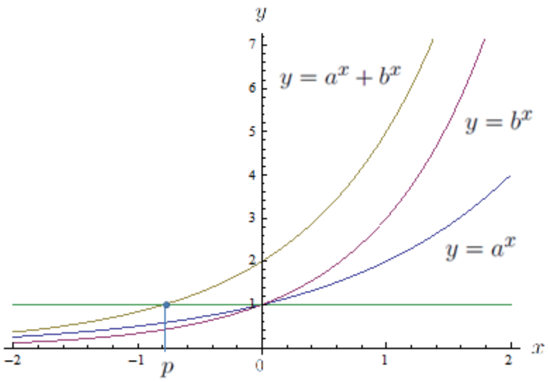

Figure 2. Case (i-b)
The existence of $p \in (-\infty, 0)$ such that $a^p + b^p = c^p$ when $a > 1, b > 1, c = 1$.

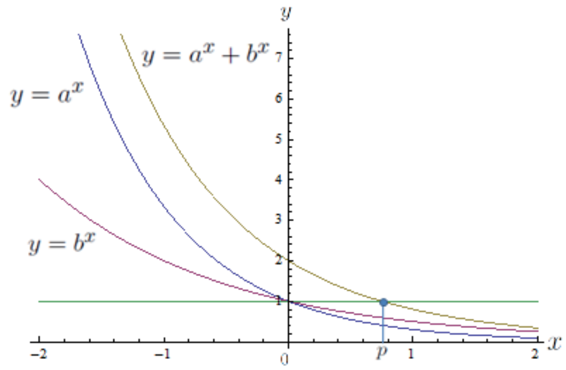

Figure 3. Case (ii)
The existence of $p \in (0, 1)$ such that $a^p + b^p = c^p$ when $c = 1, a + b < c$.

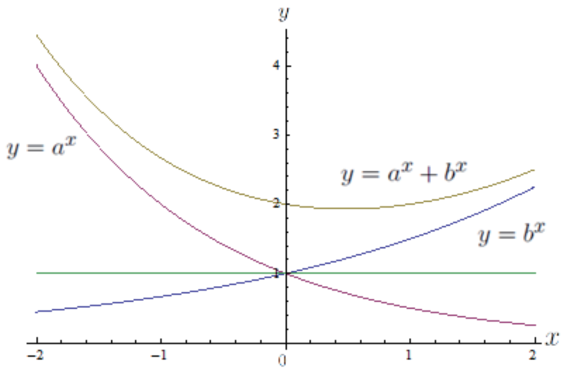

Figure 4. Case (iii)
No p satisfies $a^p + b^p = c^p$ when $c = 1$, $a \leq c \leq b$.

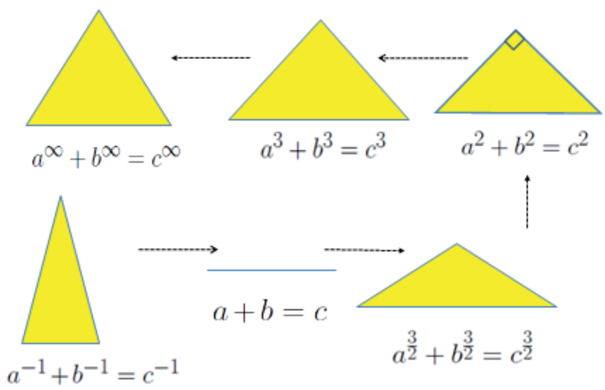

Figure 5.
The equilateral triangles with $a = b$ and $a^p + b^p = c^p$ for the various p.

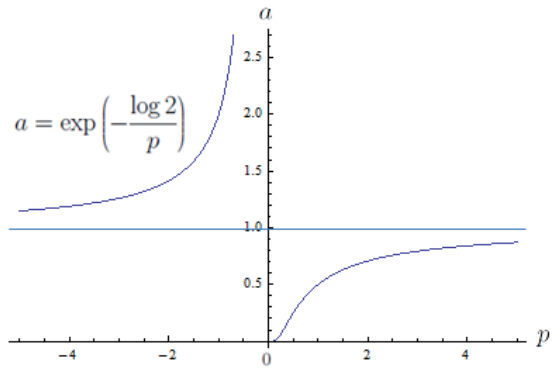

Figure 6.
The relation of p and a : $a^p + b^p = c^p$, $c = 1, a = b$.

In Theorem 1, in the cases (i-a) and (i-b), triangle inequality $a + b \geq c$ holds and we obtain a reverse inequality (3) below. In the case (ii), even though a, b, c do not satisfy triangle inequality because $a + b < c$, we also obtain a reverse inequality (4) below. In the case (iii), triangle inequality $a + b > c$ holds. But we cannot represent a reverse inequality of this case in the same method with cases (i-a), (i-b), and (ii).

Theorem 2. *Let $a > 0, b > 0, c > 0$.*

(i) *In the cases* (i-a) *and* (i-b) *in Theorem* 1*, a reverse inequality*

$$a + b \leq 2^{\frac{p-1}{p}} c \tag{3}$$

holds, where $p \in (-\infty, 0) \cup [1, \infty)$ and $a^p + b^p = c^p$. The equality holds if and only if $a = b$ or $a + b = c$.

(ii) *In the case* (ii) *in Theorem* 1*, a reverse inequality*

$$a + b \geq 2^{\frac{p-1}{p}} c \tag{4}$$

holds, where $p \in (0, 1)$ and $a^p + b^p = c^p$. The equality holds if and only if $a = b$.

3 A generalization and the proofs

The following theorem generalizes Theorem 1.

Theorem 3. *Let $n \geq 2, a_k > 0$ $(k = 1, 2, \ldots, n)$, $b > 0$.*

(i-a) $\displaystyle\sum_{k=1}^{n} a_k \geq b, a_k < b$ $(k = 1, 2, \ldots, n)$ *if and only if there exists a $p \in [1, \infty)$ such that* $\displaystyle\sum_{k=1}^{n} a_k^p = b^p$.

(i-b) $a_k > b$ $(k = 1, 2, \ldots, n)$ *if and only if there exists a $p \in (-\infty, 0)$ such that* $\displaystyle\sum_{k=1}^{n} a_k^p = b^p$.

(ii) $\sum_{k=1}^{n} a_k < b$ *if and only if there exists a* $p \in (0,1)$ *such that* $\sum_{k=1}^{n} a_k^p = b^p$.

(iii) *There exist* i *and* j *such that* $a_i \leq b \leq a_j$ *if and only if there does not exist* p *such that* $\sum_{k=1}^{n} a_k^p = b^p$.

Proof. We may assume that $b = 1$ without loss of generality. Put

$$f_k(x) = a_k^x \ (k = 1, 2, \ldots, n), \ f(x) = \sum_{k=1}^{n} a_k^x.$$

(i-a) Since $0 < a_k < 1$, $f_k(x)$ is a strictly decreasing continuous function and satisfies $\lim_{x\to\infty} f_k(x) = 0$ for each k, so is $f(x)$. $f(1) = \sum_{k=1}^{n} a_k \geq 1$. Then there exists a $p \in [1, \infty)$ such that $\sum_{k=1}^{n} a_k^p = 1$.

(i-b) Since $a_k > 1$, $f_k(x)$ is a strictly increasing continuous function and $\lim_{x\to-\infty} f_k(x) = 0$ for each k, so is $f(x)$. $f(0) = \sum_{k=1}^{n} a_k^0 = n$. Then there exists a $p \in (-\infty, 0)$ such that $\sum_{k=1}^{n} a_k^p = 1$.

(ii) Since $0 < a_k < 1$, $f_k(x)$ is a strictly decreasing continuous function for each k, so is $f(x)$. $f(0) = \sum_{k=1}^{n} a_k^0 = n$ and $f(1) = \sum_{k=1}^{n} a_k < 1$. Then there exists a $p \in (0,1)$ such that $\sum_{k=1}^{n} a_k^p = 1$.

(iii) For $x \in (-\infty, 0]$ there exists i such that $f_i(x) \geq 1$. For $x \in [0, \infty)$ there exists j such that $f_j(x) \geq 1$. Since $f(x) > f_k(x)$ for each k, $f(x) > 1$ for any x. Then there does not exist p such that $\sum_{k=1}^{n} a_k^p = 1$.

Since assumptions in (i) to (iii) exhaust all the cases, respective converses also hold. □

The following theorem generalizes Theorem 2.

Theorem 4. *Let* $n \geq 2, a_k > 0 \ (k = 1, 2, \ldots, n), b > 0$.

(i) *In the cases* (i-a) *and* (i-b) *in Theorem* 3, *a reverse inequality*

$$\sum_{k=1}^{n} a_k \leq n^{\frac{p-1}{p}} b \tag{5}$$

holds, where a $p \in (-\infty, 0) \cup [1, \infty)$ *which satisfies* $\sum_{k=1}^{n} a_k^p = b^p$. *In* (5) *the equalities hold if and only if* $a_1 = a_2 = \cdots = a_n$ *or* $\sum_{k=1}^{n} a_k = b$.

(ii) *In the case* (ii) *in Theorem* 3, *a reverse inequality*

$$\sum_{k=1}^{n} a_k \geq n^{\frac{p-1}{p}} b \tag{6}$$

holds, where a $p \in (0,1)$ *which satisfies* $\sum_{k=1}^{n} a_k^p = b^p$. *In* (6) *the equalities hold if and only if* $a_1 = a_2 = \cdots = a_n$.

Proof. The following facts are well-known as Jensen's inequality [1] for the function x^p : If $p < 0$ or $p > 1$, the function x^p $(x > 0)$ is strictly convex. Then

$$\frac{1}{n}\sum_{k=1}^{n} a_k^p \geq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p. \tag{7}$$

If $0 < p < 1$, the function x^p $(x > 0)$ is strictly concave. Then the opposite sense of inequality of (7) holds. The equalities hold if and only if $a_1 = a_2 = \cdots = a_n$ for both cases. For the case $p = 1$ also, the equality holds in (7).

In Theorem 3 we showed the existence and the uniqueness of p for the respective indicated cases. Therefore we can immediately prove Theorem 4 by using Jensen's inequality. □

References

[1] G. H. Hardy, J. E. Littlewood, G. Pólya : Inequalities. 2nd ed. Cambridge U. P., 1952.

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