A NOTE ON SEMIGROUPS OF LOCALLY LIPSCHITZ OPERATORS ASSOCIATED WITH SEMILINEAR EVOLUTION EQUATIONS

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ABSTRACT. In this note we shall give a simple proof for a part of proof of T. Matsumoto and N. Tanaka [6] Theorem 2.2. This theorem is applied to the global solvability of the mixed problem for the complex Ginzburg-Landau equation by T. Matsumoto and N. Tanaka [5][6].

In this note we shall give a simple proof for a part of proof of T. Matsumoto and N. Tanaka[6] Theorem 2.2.

1. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space with norm $\|\cdot\|$ and D be a closed subset of X.

Definition 1. *A one-parameter family $\{S(t); t \geq 0\}$ of Lipschitz operators from D into itself is called a semigroup of Lipschitz operators on D if the following three conditions are satisfied:*
(S1) $S(0)x = x \quad for \quad x \in D, \qquad S(t+s)x = S(t)S(s)x \quad for \quad s,t \geq 0 \quad and \quad x \in D.$
(S2) *For each $x \in D$, $S(\cdot)x : [0,\infty) \to X$ is continuous.*
(S3) *For each $\tau > 0$, there exists $L_\tau > 0$ such that*

$$\|S(t)x - S(t)y\| \leq \ L_\tau \|x-y\| \quad for \quad x,y \in D \quad and \quad t \in [0,\tau].$$

For semigroups of Lipschitz operators we have the following properties.

Proposition 1. *Let $\{S(t); t \geq 0\}$ be a semigroup of Lipschitz operators on D. Then there exist $M \geq 1$, $\omega \geq 0$ and a nonnegative functional Φ on $X \times X$ satisfying the following three conditions :*
(i) $|\Phi(x_1,y_1) - \Phi(x_2,y_2)| \leq M(\|x_1-x_2\| + \|y_1-y_2\|) \quad for \ \ (x_1,y_1),(x_2,y_2) \in X \times X,$
(ii) $\|x-y\| \leq \Phi(x,y) \leq M\|x-y\| \quad for \ \ (x,y) \in D \times D,$
(iii) $\Phi(S(t)x, S(t)y) \leq e^{\omega t}\Phi(x,y) \quad for \ \ t \geq 0 \ \ and \ \ (x,y) \in D \times D.$

Proof. Cf . Y. Kobayashi, T. Matsumoto and N. Tanaka [3]. □

We consider a semilinear Cauchy problem in X of the form

$$u'(t) = Au(t) + Bu(t) \quad (t>0), \qquad u(0) = u_0 \qquad (SP\,;\,u_0).$$

Here we assume :
(A) A is the infinitesimal generator of an analytic C_0- semigroup $\{T(t); t \geq 0\}$ on X with $\|T(t)\| \leq Const.e^{w_A t}$ for all $t \geq 0$, where $Const. \geq 1$ and $\omega_A < 0$ are some constants.

Remark 1. *We may assume without loss of generality that $Const. = 1$.*

We know that, for any integer $n \in \mathbb{Z}$, the operator A^n is defined. We are then concerned with extending the definition for all real exponents $\alpha \in \mathbb{R}$.

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Definition 2 (Fractional powers). *Let $\alpha > 0$. Define $(-A)^{-\alpha}$ by*

$$(-A)^{-\alpha}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}T(t)xdt \qquad for \quad x \in X \tag{1}$$

where, $\Gamma(\alpha)$ is the Gamma function. An operator $(-A)^{\alpha}$ is defined by $(-A)^{\alpha} = ((-A)^{-\alpha})^{-1}$.

Proposition 2. *$(-A)^{\alpha}$ satisfies the following conditions :*
(i) *For $x \in D((-A)^{\alpha})$*

$$T(t)(-A)^{\alpha}x = (-A)^{\alpha}T(t)x \qquad for \quad t > 0. \tag{2}$$

(ii) *For $\alpha > 0$ there exists $M_\alpha > 0$ such that*

$$\|(-A)^{\alpha}T(t)\| \le M_\alpha t^{-\alpha} \qquad for \quad t > 0. \tag{3}$$

(iii) *For $\gamma \in (\alpha, 1)$ there exists $M_{\alpha,\gamma} > 0$ such that*

$$\|(-A)^{\alpha}(T(t)x - x)\| \le M_{\alpha,\gamma}t^{\gamma-\alpha}\|(-A)^{\gamma}x\| \quad for \quad t \ge 0 \quad and \quad x \in D((-A)^{\gamma}). \tag{4}$$

(iv) *If $0 \le \alpha < \theta < \gamma \le 1$, then there exists $M_{\alpha,\theta,\gamma} > 0$ such that*

$$\|(-A)^{\theta}x\| \le M_{\alpha,\theta,\gamma}\|(-A)^{\alpha}x\|^{\frac{\gamma-\theta}{\gamma-\alpha}}\|(-A)^{\gamma}x\|^{\frac{\theta-\alpha}{\gamma-\alpha}} \qquad for \quad x \in D((-A)^{\gamma}) \tag{5}$$

Proof. Cf . H. Tanabe[7]. □

2. Assumptions and main result

Let $\alpha \in (0,1)$ and $Y = D((-A)^{\alpha})$. Then Y is a Banach space equipped with norm

$$\|v\|_Y := \|(-A)^{\alpha}v\| \qquad for \quad v \in Y \equiv D((-A)^{\alpha}) \tag{6}$$

Obviously $Y \subset X$ and Y is dense in X with X-norm.
Let $\mathcal{C} = D \cap Y$. We assume that $\mathcal{C}$ is dense in D with X-norm. In this case $\mathcal{C}$ is closed in Y.
(B) For the operator B we make the following assumptions:
(B-i) The operator B is continuous from $(\mathcal{C}, \|\cdot\|_Y)$ into $(X, \|\cdot\|)$.
(B-ii) There exists $M_B > 0$ such that $\|Bx\| \le M_B(1 + \|x\|_Y)$ for $x \in \mathcal{C}$.
(Φ) Let Φ be a nonnegative functional on $X \times X$ satisfying the following two conditions:
(Φ-i) There exists $L \ge 0$ such that

$$|\Phi(x_1, y_1) - \Phi(x_2, y_2)| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|) \quad \text{for } (x_1, y_1), (x_2, y_2) \in X \times X.$$

(Φ-ii) There exist $M \ge m > 0$ such that

$$m\|x - y\| \le \Phi(x, y) \le M\|x - y\| \qquad \text{for } (x, y) \in D \times D.$$

(F) Let $\{F_h; h \in (0, h_0]\}(h_0 > 0)$ be a family of nonlinear operators from $\mathcal{C}$ into $\mathcal{C}$ which satisfies the following two conditions:
(F-i) There exists $\omega \ge 0$ such that for any sequence $\{h_n\}_{n=1}^\infty$ with $h_n \downarrow 0$ as $n \to \infty$ and any bounded sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ with respect to Y-norm in $\mathcal{C}$,

$$\limsup_{n\to\infty}\left\{\frac{\Phi(F_{h_n}x_n, F_{h_n}y_n) - \Phi(x_n, y_n)}{h_n} - \omega\Phi(x_n, y_n)\right\} \le 0.$$

(F-ii) There exists $\beta \in (0,1)$ such that for any sequence $\{h_n\}_{n=1}^\infty$ with $h_n \downarrow 0$ as $n \to \infty$ and any convergence sequence $\{x_n\}_{n=1}^\infty$ in $\mathcal{C}$ with respect to Y-norm,

$$\lim_{n\to\infty}\frac{\|F_{h_n}x_n - J(h_n)x_n\|}{h_n} = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{\|F_{h_n}x_n - J(h_n)x_n\|_Y}{h_n^\beta} = 0$$

where

$$J(h)w = T(h)w + \int_0^h T(s)Bwds \qquad for \quad w \in \mathcal{C} \quad and \quad h > 0. \tag{7}$$

Remark 2. *We may assume that in condition* **(F-ii)**, $\beta \in (0\,,\,1-\alpha]$.

Remark 3. *It is easily seen that* **(F-i)** *is equivalent to the following condition :*
(F-i)′ *There exists $\omega \geq 0$ such that for any Y-bounded set $W \subset \mathcal{C}$,*

$$\limsup_{h\downarrow 0}\Big(\sup_{x,y\in W} \Big\{\frac{\Phi(F_h x, F_h y) - \Phi(x,y)}{h} - \omega\Phi(x,y)\Big\}\Big) \leq 0.$$

The main theorem in this note is given by

Theorem 1 ([6] Theorem 2.2.). *Assume that* **(B)**,**(Φ)** *and* **(F)** *hold. Then there exists a semigroup $\{S(t); t \geq 0\}$ of Lipschitz operators on D such that*
(i) $BS(\cdot)x \in C([0,\infty); X)$ *for* $x \in \mathcal{C}$,
(ii) $BS(\cdot)x \in C((0,\infty); X) \cap L^1_{loc}(0,\infty; X)$ *for* $x \in D$,
(iii)

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds \quad for \quad x \in D \quad and \quad t \geq 0. \tag{8}$$

Moreover, the following product formula hold:
(iv)

$$S(t)x = \lim_{h\downarrow 0} F_h^{[\frac{t}{h}]}x \quad for \quad x \in \mathcal{C} \quad and \quad t \geq 0, \tag{9}$$

where the convergence of (9) *is uniform on every compact subset of* $[0,\infty)$. *Here* $[\alpha]$ *is the greatest integer that is less than or equal to* α.

For the proof of the existence of a semigroup $\{S(t); t \geq 0\}$ of Lipschitz operators on D satisfying **(i)**,**(ii)** and **(iii)** T. Matsumoto, and N. Tanaka used [4] Theorem 5.2. But this theorem treated more general case.

3. Mild solutions

We need the followinfg notion of solutions.

Definition 3. *Let $u_0 \in D$ and $\tau > 0$. A function $u \in C([0,\tau]; X) \cap C((0,\tau]; Y)$ is called a mild solution to $(SP; u_0)$ on $[0,\tau]$ if*
(i) $u(t) \in \mathcal{C}$ *for* $t \in (0,\tau]$,
(ii) $Bu \in C((0,\tau]; X) \cap L^1(0,\tau; X)$,
(iii) *u satisfies the integral equation* :

$$u(t) = T(t)u_0 + \int_0^t T(t-s)Bu(s)ds \quad for \quad t \in [0,\tau]. \tag{10}$$

A function $u \in C([0,\infty); X) \cap C((0,\infty); Y)$ is called a global mild solution to $(SP; u_0)$ if for each $\tau > 0$ the restriction u to $[0,\tau]$ is a mild solution to $(SP; u_0)$ on $[0,\tau]$.

The continuous dependence of mild solutions to the Cauchy problem for (SP) on their initial data is given by following Proposition.

Proposition 3. *Let $\tau > 0$ and $x_1, x_2 \in D$. Let $u : [0\,,\,\tau] \to X$ be a mild solution to $(SP; x_1)$ on $[0\,,\,\tau]$ and $v : [0\,,\,\tau] \to X$ be a mild solution to $(SP; x_2)$ on $[0\,,\,\tau]$. Suppose that conditions* **(Φ)** *and* **(F)** *are satisfied. Then there exist $\mathcal{M} > 0$ and $\omega > 0$ such that*

$$\|u(t) - v(t)\| \leq \mathcal{M} e^{\omega t} \|x_1 - x_2\| \qquad for \quad t \in [0\,,\,\tau].$$

Proof. Let $\omega > 0$ be a number appearing in condition **(F-i).** From **(Φ-i),** we have

$$|\Phi(u(s), v(s)) - \Phi(u(t), v(t))| \leq L\Big(\|u(s) - u(t)\| + \|v(s) - v(t)\|\Big) \quad \text{for } s, t \in [0, \tau].$$

The definition of mild solutions shows that $u, v \in C([0, \tau]\ ;\ X)$. Therefore we see that the map $t \mapsto \Phi(u(t), v(t))$ is continuous on $[0, \tau]$. Let $t \in (0, \tau)$ and let $h > 0$ be such that $t + h \leq \tau$. By the semigroup property of $\{T(t)\ ;\ t \geq 0\}$ and (10), we obtain that

$$\begin{aligned}
u(t+h) &= T(t+h)x + \int_0^{t+h} T(t+h-s)Bu(s)ds \\
&= T(h)T(t)x + T(h)\int_0^t T(t-s)Bu(s)ds \\
&\qquad - T(h)\int_0^t T(t-s)Bu(s)ds + \int_0^{t+h} T(t+h-s)Bu(s)ds \\
&= T(h)\left(T(t)x + \int_0^t T(t-s)Bu(s)ds\right) \\
&\qquad - \int_0^t T(t+h-s)Bu(s)ds + \int_0^{t+h} T(t+h-s)Bu(s)ds \\
&= T(h)u(t) + \int_t^{t+h} T(t+h-s)Bu(s)ds \\
&= T(h)u(t) + \int_0^h T(s)Bu(t+h-s)ds.
\end{aligned} \tag{11}$$

With this equation and (7) we have

$$\begin{aligned}
u(t+h) &= T(h)u(t) + \int_0^h T(s)Bu(t)ds \\
&\qquad - \int_0^h T(s)Bu(t)ds + \int_0^h T(s)Bu(t+h-s)ds \\
&= J(h)u(t) + \int_0^h T(s)\Big(Bu(t+h-s) - Bu(t)\Big)ds.
\end{aligned} \tag{12}$$

From the definition of mild solutions we get $Bu \in C((0, \tau]\ ;\ X)$. Then with assumption **(A)** it follows that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h \|T(s)\Big(Bu(t+h-s) - Bu(t)\Big)\| ds = 0. \tag{13}$$

(12) and (13) yields that

$$\lim_{h \downarrow 0} \frac{1}{h} \|u(t+h) - J(h)u(t)\| = 0. \tag{14}$$

Similarly we have

$$\lim_{h \downarrow 0} \frac{1}{h} \|v(t+h) - J(h)v(t)\| = 0. \tag{15}$$

With condition **(Φ-i)**, we have the following estimate:

$$\begin{aligned}
&\frac{1}{h}\Big(\Phi(u(t+h),v(t+h))-\Phi(u(t),v(t))\Big)\\
&\quad\le \frac{1}{h}\Big(\Phi(J(h)u(t),J(h)v(t))-\Phi(u(t),v(t))\Big)\\
&\qquad + L\,\frac{1}{h}\Big(\|u(t+h)-J(h)u(t)\|+\|v(t+h)-J(h)v(t)\|)\Big)\\
&\quad\le \frac{1}{h}\Big(\Phi(F_hu(t),F_hv(t))-\Phi(u(t),v(t))\Big)\\
&\qquad + L\,\frac{1}{h}\Big\{\|u(t+h)-J(h)u(t)\|+\|v(t+h)-J(h)v(t)\|\\
&\qquad\quad +\|J(h)u(t)-F_hu(t)\|+\|J(h)v(t)-F_hv(t)\|)\Big\}.
\end{aligned}\tag{16}$$

From (14),(15) and condition **(F)** we obtain that

$$\begin{aligned}
&\limsup_{h\downarrow 0}\frac{1}{h}\Big(\Phi(u(t+h),v(t+h))-\Phi(u(t),v(t))\Big)\\
&\quad\le \limsup_{h\downarrow 0}\frac{1}{h}\Big(\Phi(F_hu(t),F_hv(t))-\Phi(u(t),v(t))\Big)\le \omega\Phi(u(t),v(t)).
\end{aligned}\tag{17}$$

Therefore we have

$$D^+\Phi(u(t),v(t))\le \omega\Phi(u(t),v(t))\quad\text{for}\quad t\in(0,\tau),\tag{18}$$

where D^+denotes the upper right Dini derivative which defined by

$$D^+f(a)=\limsup_{h\downarrow 0}\frac{f(a+h)-f(a)}{h}.$$

Since $\Phi(u(\cdot),v(\cdot))$ is continuous on $[0,\tau]$ solving the differential inequality (18) yields that

$$\Phi(u(t),v(t))\le e^{\omega t}\Phi(x_1,x_2)\quad\text{for}\quad t\in[0,\tau].$$

An application of condition **(Φ-ii)** shows that

$$\|u(t)-v(t)\|\le \frac{1}{m}\,e^{\omega t}\Phi(x_1,x_2)\le \frac{M}{m}\,e^{\omega t}\|x_1-x_2\|\quad\text{for}\quad t\in[0,\tau].\tag{19}$$

Then we obtain the desired inequality. □

Proposition 4. *Suppose that* **(Φ)** *and* **(F)** *are satisfied. Suppose that for each $x\in\mathcal{C}$ there exist $\tau>0$ and a mild solution u to $(SP;x)$ on $[0\,,\,\tau]$. Then for every $x\in\mathcal{C}$ there exists a global mild solution u to $(SP;x)$.*

Proposition 5. *Suppose that* **(Φ)** *and* **(F)** *are satisfied. Suppose that for each $x\in\mathcal{C}$ there exist a global mild solution u to $(SP;x)$. Then for every $x\in D$ there exists a global mild solution u to $(SP;x)$.*

Proof. From Proposition 2.5 in [4](resp Proposition 2.6 in [4]) with φ defined by $(\varphi)\begin{cases}\varphi(x)=0 & x\in D\\ \varphi(x)=\infty & x\in X\backslash D\end{cases}$, we have Prposition 4 (resp. Proposition 5). □

4. Key estimate

In this section we give a key estimate to showing the convergence of approximate solutions.

Lemma 1. *There exists $K \geq 1$ such that for any $\tau \in (0,1]$ and for any finite sequence $\{s_k\}_{k=0}^N$ satisfying $0 \leq s_0 < s_1 < \cdots < s_N \leq \tau$, the following two assertions hold:*
(i) *Let $M_G > 0$ and let $G : [0,\tau) \to X$ be a measurble function satisfying $\|G(\xi)\| \leq M_G$ for $\xi \in [0,\tau)$. Then*

$$\int_{s_l}^{s_i} \|T(s_i - \xi)G(\xi)\|_Y d\xi \leq KM_G(s_i - s_l)^\beta \qquad for \quad 0 \leq l \leq i \leq N.$$

(ii) *Let $\varepsilon > 0$. Then for any finite sequence $\{\zeta_i\}_{i=1}^N$ in Y satisfying $\|\zeta_i\| \leq \varepsilon(s_i - s_{i-1})$ and $\|\zeta_i\|_Y \leq \varepsilon(s_i - s_{i-1})^\beta$ for $1 \leq i \leq N$, we have*

$$\sum_{l=k+1}^{i} \|T(s_i - s_l)\zeta_l\|_Y \leq K\varepsilon(s_i - s_k)^\beta \qquad for \quad 0 \leq k \leq i \leq N.$$

Here as usual we put $\sum_{l=k+1}^{k} = 0$.

Proof. Cf. T. Matsumoto and N. Tanaka[4] Lemma 3.2. □

In the rest of this paper the symbol K stands for the constant specified in Lemma 1 and we define

$$E_h w = F_h w - J(h)w \qquad \text{for} \quad h \in (0, h_0] \quad \text{and} \quad w \in \mathcal{C}. \tag{20}$$

For $w_0 \in \mathcal{C}, h > 0, \rho > 0, M > 0$ and $\varepsilon > 0$ we introduce the condition

$$\mathbf{V}(w_0; h, \rho, M, \varepsilon) \equiv \left\{ \begin{array}{l} \textbf{(i)}\ \|Bx\| \leq M \quad \text{for} \quad x \in U_Y(w_0, \rho) \cap \mathcal{C}, \\ \textbf{(ii)}\ K(M+\varepsilon)h^\beta + \sup_{s\in[0,h]} \|T(s)w_0 - w_0\|_Y \leq \rho. \end{array} \right\} \tag{21}$$

where $U_Y(\omega_0, \rho)$ denotes the closed ball in Y with center ω_0 and radius ρ and β is a constant appearing in condition **(F-ii)**.

Lemma 2. *Let $w_0 \in \mathcal{C}$. Assume that $0 < h \leq 1$, $\rho > 0$, $M > 0$ and $\varepsilon > 0$, satisfy condition $\mathbf{V}(w_0; h, \rho, M, \varepsilon)$. And take $\sigma > 0$ satisfy $\sigma \leq h$. Assume that there exists a sequence $\{(s_i\, ,\, w_i\, ,\, \zeta_i)\}_{i=1}^N$ in $[0\, ,\, \sigma] \times \mathcal{C} \times Y$ satisfies the following three conditions :*
(i) $0 = s_0 < s_1 < \cdots < s_N \leq \sigma,$
(ii) $w_i = T(s_i - s_{i-1})w_{i-1} + \int_{s_{i-1}}^{s_i} T(s_i - \xi)Bw_{i-1}d\xi + \zeta_i \qquad for \quad 1 \leq i \leq N,$
(iii) $\|\zeta_i\| \leq \varepsilon(s_i - s_{i-1})$ *and* $\|\zeta_i\|_Y \leq \varepsilon(s_i - s_{i-1})^\beta \qquad for \quad 1 \leq i \leq N.$
Then the following assertions **(a)** *and* **(b)** *hold:*
(a) *We have the following estimates with X-norm and Y-norm respectivery* :
  (a-1) $\|T(s_j - s_k)w_k - w_j\| \leq (M+\varepsilon)(s_j - s_k) \qquad for \quad 0 \leq k \leq j \leq N,$
  (a-2) $\|T(s_j - s_k)w_k - w_j\|_Y \leq K(M+\varepsilon)(s_j - s_k)^\beta \qquad for \quad 0 \leq k \leq j \leq N.$
(b) $w_j \in U_Y(w_0\, ,\, \rho)$ *and* $\|Bw_j\| \leq M \qquad for \quad 0 \leq j \leq N.$

Proof. To prove this lemma we use Lemma 1 inductively. □

Given $(t_0, x_0) \in [0, \infty) \times \mathcal{C}$ we set

$$\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{N}) \equiv \left\{ \begin{array}{l} (\mathbf{i})\ 0 = t_0 < t_1 < \ldots < t_N < \tau, \\ (\mathbf{ii})\ t_j - t_{j-1} \le \varepsilon \\ (\mathbf{iii})\ x_j = T(t_j - t_{j-1})x_{j-1} + \int_{t_{j-1}}^{t_j} T(t_j - \xi)Bx_{j-1}d\xi + \zeta_j, \\ (\mathbf{iv})\ \|\zeta_j\| \le \varepsilon(t_j - t_{j-1}) \text{ and } \|\zeta_j\|_Y \le \varepsilon(t_j - t_{j-1})^\beta \\ (\mathbf{v})\ \text{If } x \in \mathcal{C} \text{ satisfies the inequality} \\ \qquad \|x - x_{j-1}\|_Y \\ \ \le K(M_B + 1)(t_j - t_{j-1})^\beta + \sup_{s \in [0, t_j - t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|_Y \\ \qquad \text{then } \|Bx - Bx_{j-1}\| \le \frac{\varepsilon}{4K} \\ (\mathbf{vi})\ (t_j - t_{j-1})(M_B + 1) + \sup_{s \in [0, t_j - t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\| \le \varepsilon \\ \text{where } j = 1, 2, \cdots, N. \end{array} \right\}.$$

(**vii**) $\lim_{j \to \infty} t_j = \tau$.

Proposition 6. *Suppose that condition* (**F**) *is satisfied. Let* $x_0 \in \mathcal{C}$ *and* $\varepsilon \in (0, 1/2]$. *Assume that* $\tau \in (0, 1]$, $\rho_0 > 0$ *and* $M_B > 0$ *satisfy condition* $\mathbf{V}(x_0; \tau, \rho_0, M_B, 1)$. *Then there exists a sequence* $\{(t_j, x_j, \zeta_j)\}_{j=1}^{\infty}$ *in* $[0, \tau) \times \mathcal{C} \times Y$ *satisfying the condition* $\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{\infty})$ *and condition* (**vii**).

Proof. We shall construct inductively a sequence $\{(t_j, x_j, \zeta_j)\}_{j=1}^{\infty}$ in $[0, \tau) \times \mathcal{C} \times Y$ satisfying condition $\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{\infty})$. For this purpose, let $i \in \mathbb{N}$ and assume that a sequence $\{(t_j, x_j, \zeta_j)\}_{j=1}^{i-1} \in [0, \tau) \times \mathcal{C} \times Y$ can be constructed so that it satisfies condition $\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{i-1})$. For $h > 0, t \in [0, \tau), y \in \mathcal{C}$ and $\varepsilon > 0$ we set

$$\boldsymbol{\theta}(h; t, y, \varepsilon) \equiv \left\{ \begin{array}{l} h < \tau - t, \\ h(M_B + 1) + \sup_{s \in [0, h]} \|T(s)y - y\| \le \varepsilon, \\ \|Bx - By\| \le \frac{\varepsilon}{4K} \qquad \text{for} \quad x \in U_Y(y, \rho) \cap \mathcal{C}, \\ \ \text{where } \rho = K(M_B + 1)h^\beta + \sup_{s \in [0, h]} \|T(s)y - y\|_Y \end{array} \right\}. \tag{22}$$

By condition **(B-i)**, the strong continuity of $T(\cdot)$ and **(F-ii)**, there exist $h \in (0, \varepsilon]$ such that

$$\|E_h x_{i-1}\| \le h\varepsilon \quad \text{and} \quad \|E_h x_{i-1}\|_Y \le h^\beta \varepsilon \tag{23}$$

and $(h; t_{i-1}, x_{i-1}, \varepsilon)$ satisfying condition $\boldsymbol{\theta}(h; t_{i-1}, x_{i-1}, \varepsilon)$. We define $\bar{h}_i$ by supremum of such numbers h. Then there exists $h_i \in (0, \varepsilon\,]$ such that $\bar{h}_i/2 < h_i$ which satisfy $\boldsymbol{\theta}(h_i; t_{i-1}, x_{i-1}, \varepsilon)$. We set $t_i = t_{i-1} + h_i$, then condition **(ii)** is satisfied. From (22) we get conditions **(i)**,**(vi)** and **(v)** in $\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{i})$. Next we shall show that there exist $x_i \in \mathcal{C}$ and $\zeta_i \in Y$ satisfying **(iii)** and **(iv)** in $\mathbf{W}(\tau; \varepsilon, \{(t_j, x_j, \zeta_j)\}_{j=1}^{i})$. Here, we define $x_i = F_{h_i} x_{i-1}$ and $\zeta_i = E_{h_i} x_{i-1}$. Obviously $F_{h_i} x_{i-1} \in \mathcal{C}$ and $E_{h_i} x_{i-1} \in Y$ and condition **(iv)** is satisfied by (23). With (7) and (20), we have

$$\begin{aligned} x_i =& F_{h_i} x_{i-1} = J(h_i)x_{i-1} + E_{h_i} x_{i-1} \\ =& T(h_i)x_{i-1} + \int_0^{h_i} T(s)Bx_{i-1}ds + E_{h_i} x_{i-1} \\ =& T(t_i - t_{i-1})x_{i-1} + \int_{t_{i-1}}^{t_i} T(t_i - s)Bx_{i-1}ds + \zeta_i. \end{aligned} \tag{24}$$

It remains to show that condition **(vii)** is satisfied. We can show it in a way similar to that of T. Matsumoto, and N. Tanaka[4.Proposition 3.7]. It is concluded that a sequence

$\{(t_j, x_j, \zeta_j)\}_{j=1}^{\infty}$ in $[0,\tau)\times\mathcal{C}\times Y$ can be constructed so that the condition $\mathbf{W}(\tau;\varepsilon,\{(t_j,x_j,\zeta_j)\}_{j=1}^{\infty})$ and condition **(vii)** are satisfied. □

Proposition 7. *Let $x_0 \in \mathcal{C}$, $0 < \bar\tau \le \min\{\tau , 1\}$, $\rho_0 > 0$, $M_B > 0$ and $0 < \varepsilon$, λ,$\mu \le 1/2$ and suppose condition $\mathbf{V}(x_0;\bar\tau,\rho_0,M_B,1)$ satisfied. For each $\varepsilon = \lambda$ or μ, suppose that there exists a sequence $\{(t_j^\varepsilon, x_j^\varepsilon, \zeta_j^\varepsilon)\}_{j=1}^{\infty}$ in $[0,\bar\tau)\times\mathcal{C}\times Y$ satisfying conditions in $\mathbf{W}(\bar\tau;\varepsilon,\{(t_j^\varepsilon,x_j^\varepsilon,\zeta_j^\varepsilon)\}_{j=1}^{\infty})$ and condition* **(vii)**. *Set $P = \{t_i^\lambda; i = 0,1,\cdots\} \cup \{t_j^\mu; j = 0,1,\cdots\}$, and define $s_0 = 0$ and $s_k = \inf(P \setminus \{s_0, s_1,\cdots, s_{k-1}\})(k \in \mathbb{N})$. Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^{\infty}$ in $\mathcal{C}\times\mathcal{C}$ satisfying the following conditions (where $\varepsilon = \lambda$ or μ):*

(a) *If $s_k = t_i^\varepsilon$, then $z_k^\varepsilon = x_i^\varepsilon$,*

(b) *If $s_k \ne t_i^\varepsilon$, then the element f_k^ε on Y defined by*

$$f_k^\varepsilon = T(s_k - s_{k-1}) z_{k-1}^\varepsilon + \int_{s_{k-1}}^{s_k} T(s_k - \xi) B z_{k-1}^\varepsilon d\xi - z_k^\varepsilon, \tag{25}$$

satisfies $\|f_k^\varepsilon\| \le \varepsilon(s_k - s_{k-1})$ and $\|f_k^\varepsilon\|_Y \le \varepsilon(s_k - s_{k-1})^\beta$.

(c) $\Phi(z_k^\lambda, z_k^\mu) \le e^{\omega\bar\tau}\{L(\lambda+\mu)\bar\tau + \eta_k(\lambda,\mu)\}$ *for $k \ge 0$, where*

$$\eta_k(\lambda,\mu) = 3L\left(\lambda \sum_{t_i^\lambda\in\{s_1,\cdots,s_k\}} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu\in\{s_1,\cdots,s_k\}} (t_j^\mu - t_{j-1}^\mu)\right).$$

Here ω is constants appearing in condition **(F-i)**.

Proof. The proof is assured by Proposition 4.2 in [4] with φ defined by

$$(\varphi)\begin{cases} \varphi(x) = 0 & x \in D \\ \varphi(x) = \infty & x \in X\backslash D \end{cases}.$$

□

5. Characterization of semigroups

We characterize semigroups of locally Lipschitz operators associated with semilinear evolution equations of parabolic type.

Theorem 2. *Assume that condition* **(B)** *is satisfied. Then, the following two statements are equivalent:*

(i) *There exists a semigroup $\{S(t); t \ge 0\}$ of Lipschitz operators on D such that for each $x \in D$, $S(\cdot)x$ is a global mild solution to $(SP;x)$.*

(ii) *There exist a nonnegative functional Φ on $X \times X$ satisfying conditions* **(Φ)** *and a family $\{F_h; h \in (0,h_0]\}$ of nonlinear operators from $\mathcal{C}$ into $\mathcal{C}$ satisfying conditions* **(F)**.

Proof. We begin by showing that **(i)** implies **(ii)**. Applying Proposition 1 with $L = M$ and $m = 1$ there exists a nonnegative functional Φ on $X \times X$ satisfying condition **(Φ)**.
It remains to check the existence of a family $\{F_h; h \in (0,h_0]\}$ of nonlinear operators from $\mathcal{C}$ into $\mathcal{C}$ satisfying conditions **(F)**. Let $h > 0$. From **(iv)** in Proposition 1 we have

$$\Phi(S(h)x\,,\,S(h)y) \le e^{\omega h}\Phi(x\,,\,y) \qquad for \quad (x\,,\,y) \in D\times D. \tag{26}$$

Then from the definition of mild solution we obtain that $S(h)x$ belongs to $\mathcal{C}$.
We define $F_h x = S(h)x$. Now we shall show that $\{F_h; h \in (0,h_0]\}$ satisfies condition **(F)**.

Let W be a bounded subset of $\mathcal{C}$ with respect to Y-norm. By (26), we have

$$\begin{aligned} &\frac{1}{h}\Big(\Phi(F_hx\,,\,F_hy)-\Phi(x\,,\,y)\Big)-\omega\Phi(x\,,\,y)\\ &\quad=\frac{1}{h}\Big(\Phi(S(h)x\,,\,S(h)y)-\Phi(x\,,\,y)\Big)-\omega\Phi(x\,,\,y)\\ &\quad\le\Big(\frac{1}{h}(e^{\omega h}-1)-\omega\Big)\Phi(x\,,\,y)\quad\text{for } h\in(0,h_0] \text{ and } (x\,,\,y)\in W\times W. \end{aligned} \tag{27}$$

Since W is bounded in Y, we have $\sup\{\Phi(x,y);(x,y)\in W\times W\}<\infty$. This and (27) imply that condition **(F-i)**$'$ is satisfied. That is to say, condition **(F-i)** is valid. Next we shall show condition **(F-ii)**. Let take any sequence $\{h_n\}_{n=1}^{\infty}$ such that $h_n\downarrow 0$ as $n\to\infty$ and any convergence sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{C}$.
Note that $S(\cdot)x$ is a mild solution in $(SP;x)$. From (7) and (10) we obtain that

$$\begin{aligned} F_hx-J(h)x&=S(h)x-J(h)x\\ &=\Big(T(h)x+\int_0^h T(h-s)BS(s)xds\Big)-\Big(T(h)x+\int_0^h T(s)Bxds\Big)\\ &=\int_0^h T(h-s)(BS(s)x-Bx)ds. \end{aligned} \tag{28}$$

From (28) we have

$$\begin{aligned} \frac{\|F_{h_n}x_n-J(h_n)x_n\|}{h_n}&\le\frac{1}{h_n}\int_0^{h_n}\|T(h_n-s)(BS(s)x_n-Bx_n)\|ds\\ &\le\max_{0\le s\le h_n}\|BS(s)x_n-Bx_n\|. \end{aligned} \tag{29}$$

With the strong continuity of $S(\cdot)$ and condition **(B-i)**, from (29) it follows that

$$\lim_{n\to\infty}\frac{\|F_{h_n}x_n-J(h_n)x_n\|}{h_n}=0. \tag{30}$$

By (3) and (6), it follows that

$$\begin{aligned} \|F_{h_n}x_n-J(h_n)x_n\|_Y&\le\int_0^{h_n}\|T(h_n-s)(BS(s)x_n-Bx_n)\|_Y ds\\ &=\int_0^{h_n}\|(-A)^{\alpha}T(h_n-s)(BS(s)x_n-Bx_n)\|ds\\ &\le\int_0^{h_n}M_\alpha(h_n-s)^{-\alpha}\|BS(s)x_n-Bx_n\|ds\\ &\le M_\alpha\frac{1}{1-\alpha}h_n^{1-\alpha}\max_{0\le s\le h_n}\|BS(s)x_n-Bx_n\|. \end{aligned} \tag{31}$$

With the strong continuity of $S(\cdot)$ and condition **(B-i)**, from (31) we have that

$$\lim_{n\to\infty}\frac{\|F_{h_n}x_n-J(h_n)x_n\|_Y}{h_n^{1-\alpha}}=0. \tag{32}$$

If $\beta=1-\alpha$, then (32) is the desired estimate. Therefore condition **(F-ii)** is showed .

To prove the converse implication, let $x_0\in\mathcal{C}$. Then, condition **(B-i)** ensures the existence of $\rho_0>0$ and $M_B>0$ satisfying condition $\mathbf{V}(x_0;\tau,\rho_0,M_B,1)$. Therefore, Proposition 6 asserts that for each $\varepsilon\in(0,1/2]$ there exists a sequence $\{(t_j^{\varepsilon},x_j^{\varepsilon},\zeta_j^{\varepsilon})\}_{j=1}^{\infty}$ in $[0,\tau)\times\mathcal{C}\times Y$

satisfying $\mathbf{W}(\tau;\varepsilon,\{(t_j^\varepsilon,x_j^\varepsilon,\zeta_j^\varepsilon)\}_{j=1}^\infty)$ and condition **(vii)**. For each $\varepsilon\in(0,1/2]$, we define a family $\{u^\varepsilon\}$ of step functions by

$$u^\varepsilon(t)=x_i^\varepsilon \qquad \text{for} \quad t\in[t_i^\varepsilon,t_{i+1}^\varepsilon) \quad \text{and} \quad i\in\mathbb{N}.$$

The purpose is to demonstrate that the family $\{u^\varepsilon\}$ converges in the space $C([0,\tau];X)\cap C((0,\tau];Y)$. For this purpose, let λ , $\mu\in(0,1/2]$, and let $\{s_k\}_{k=0}^\infty$ be a sequence constructed as in Proposition 7 . Then, applying Proposition 7 we find a sequence $\{(z_k^\lambda,z_k^\mu)\}_{k=0}^\infty$ in $\mathcal{C}\times\mathcal{C}$ satisfying **(a)**,**(b)** and **(c)** in Proposition 7, which plays an important role in accomplishing the above-mentioned purpose. In the following, ω stands for the constants in **(c)**, which are specified by condition **(F-i)** in Proposition 7.

The first step: We shall show that $\lim_{\varepsilon\downarrow 0}u^\varepsilon(t)=u(t)$ in X. Let $t\in[0\,,\,\tau)$.
We begin by estimating the difference $\|u^\lambda(t)-u^\mu(t)\|$. Take i , j , $k\in\mathbb{N}$ such that:

$$t\in[s_{k-1}\,,\,s_k)\,,\quad t_{i-1}^\lambda\le s_{k-1}<s_k\le t_i^\lambda \quad \text{and} \quad t_{j-1}^\mu\le s_{k-1}<s_k\le t_i^\mu.$$

Then, from the definition of u^ε we have $u^\lambda(t)=x_{i-1}^\lambda$ and $u^\mu(t)=x_{j-1}^\mu$. Take $p\in\mathbb{Z}$ such that $t_{i-1}^\lambda=s_p$. By **(a)** in Lemma 7, we have $z_p^\lambda=x_{i-1}^\lambda$. From Lemma 1 it follows that $\|Bx_{i-1}^\lambda\|\le M_B$. This inequality and condition **(v)** together imply that,

$$\|Bx\|\le M_B+\frac{\lambda}{4K} \qquad \text{for} \quad x\in U_Y(x_{i-1}^\lambda\,,\,\rho_i\lambda)\cap\mathcal{C}.$$

It follows **(b)** in Lemma 7 that

$$z_k^\varepsilon=T(s_k-s_{k-1})z_{k-1}^\varepsilon+\int_{s_{k-1}}^{s_k}T(s_k-\xi)Bz_{k-1}^\varepsilon d\xi-f_k^\varepsilon,$$

satisfies $\|f_k^\varepsilon\|\le\varepsilon(s_k-s_{k-1})$ and $\|f_k^\varepsilon\|_Y\le\varepsilon(s_k-s_{k-1})^\beta$. Since $0=s_p-t_{i-1}^\lambda<s_{p+1}-t_{i-1}^\lambda<\cdots<s_k-t_{i-1}^\lambda<\cdots<t_i^\lambda-t_{i-1}^\lambda$. We apply the sequence $\{(s_{p+k}-t_{i-1}^\lambda\,,\,z_{p+k}^\lambda\,,\,-f_{p+k}^\lambda)\}_{k=1}^\infty$ in $[0\,,\,t_i^\lambda-t_{i-1}^\lambda]\times\mathcal{C}\times Y$ for **(a-1)** in Lemma 2, it follows that

$$\|z_{k-1}^\lambda-T(s_{k-1}-t_{i-1}^\lambda)x_{i-1}^\lambda\|\le(M_B+\frac{\lambda}{4K}+\lambda)(s_{k-1}-t_{i-1}^\lambda).$$

This inequality and **(vi)** in Lemma 6 together imply that $\|z_{k-1}^\lambda-x_{i-1}^\lambda\|\le\lambda$. Similarly we have $\|z_{k-1}^\mu-x_{j-1}^\mu\|\le\mu$. Since it follows from **(Φ-i)** that

$$|\Phi(x_{i-1}^\lambda\,,\,x_{j-1}^\mu)-\Phi(z_{k-1}^\lambda\,,\,z_{k-1}^\mu)|\le L\Big(\|x_{i-1}^\lambda-z_{k-1}^\lambda\|+\|x_{j-1}^\mu-z_{k-1}^\mu\|\Big)\le L(\lambda+\mu). \tag{33}$$

With inequality (33), **(Φ-ii)** and **(c)** in proposition 7, we obtain that

$$\begin{aligned} m\|u^\lambda(t)-u^\mu(t)\| &= m\|x_{i-1}^\lambda-x_{j-1}^\mu\|\le\Phi(x_{i-1}^\lambda\,,\,x_{j-1}^\mu)\\ &\le\Phi(z_{k-1}^\lambda\,,\,z_{k-1}^\mu)+L(\lambda+\mu)\\ &\le e^{\omega\tau}\Big\{L(\lambda+\mu)\tau+\eta_{k-1}(\lambda\,,\,\mu)\Big\}+L(\lambda+\mu)\\ &\le 4Le^{\omega\tau}(\lambda+\mu)\tau+L(\lambda+\mu). \end{aligned}\tag{34}$$

This implies the existence of a measurable function $u:[0\,,\,\tau)\to X$ such that $\lim_{\varepsilon\downarrow 0}u^\varepsilon(t)=u(t)$ uniformly for $t\in[0\,,\,\tau)$.

The second step: We shall show that for any $t\in(0,\tau)$, $\lim_{\varepsilon\downarrow 0}u^\varepsilon(t)=u(t)$ in Y .

The third step: We shall prove that $u\in C([0,\tau);X)\cap C((0,\tau);Y)$.
The proof of *The second step* and *The third step* is obtained in a way similar to that of T.

Matsumoto, and N. Tanaka[4.Theorem 5.2]. Therefore we have proved that to each $x \in \mathcal{C}$ there corresponds $\tau_x > 0$ such that the $(SP; x)$ has a mild solution u on $[0, \tau_x]$. Proposition 4 and Proposition 5 therefore assert that for any $x \in D$ and $t \geq 0$, the $(SP\,;\,x)$ has a global mild solution $u(t; x)$. Next we shall show that the family $\{S(t)x; t \geq 0\}$, defined by $S(t)x = u(t; x)$ for $x \in D$ and $t \geq 0$, is a semigroup of locally Lipschitz operators on D. From the semigroup property of $T(\cdot)$ it follows that

$$\begin{aligned}
S(0)x =& u(0\,;\,x) = x,\\
S(t+s)x =& u(t+s; x) = T(t+s)x + \int_0^{t+s} T(t+s-\xi)Bu(\xi)d\xi\\
=& T(t)T(s)x + \int_0^{t+s} T(t)T(s-\xi)Bu(\xi)d\xi\\
=& T(t)\Big(T(s)x + \int_0^{s} T(s-\xi)Bu(\xi)d\xi\Big) + \int_s^{t+s} T(t+s-\xi)Bu(\xi)d\xi\\
=& T(t)u(s; x) + \int_0^{t} T(t-\xi)Bu(\xi+s)d\xi\\
=& u(t; u(s)) = S(t)u(s\,;\,x) = S(t)S(s)x.
\end{aligned}$$

Therefore we obtain the semigroup property of $\{S(t); t \geq 0\}$. Note that $u(t; x)$ is a global mild solution. For each $\tau > 0$ we have that $S(\cdot)x = u(\cdot) \in C([0\,,\,\tau]; X)$. It proved that $S(\cdot)x : [0\,,\,\infty) \to X$ is continuous. Finally, we check condition **(S3)**. With Lemma 3 we have

$$\|S(t)x - S(t)y\| = \|u(t\,;\,x) - u(t\,;\,y)\| \leq \mathcal{M}e^{\omega\tau}\|x - y\|.$$

If we take $L = \mathcal{M}e^{\omega\tau}$, we obtain the estimate in **(S3)** .
The above argument proves that there exists semigroup $\{S(t)x; t \geq 0\}$ of locally Lipschitz operators on D, which is a global mild solutions to $(SP\,;\,x)$. □

6. Proof of the theorem 1

(ii) and **(iii)** in Theorem 1 is assured by Theorem 2. **(i)** follows from Theorem 2 and condition **(B-i)** too. The proof of **(iv)** follows the one given in T. Matsumoto, and N. Tanaka [6. Chapter 4]. Then the proof is complete.

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