Graph-Theoretical Properties of Markoff Numbers. Topological indices of Symmetrical BroComb Graphs and Perfect Matching Numbers of Symmetrical StepOmino Graphs.

Haruo Hosoya

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Abstract Positive integer solutions of $x^2+y^2+z^2=3xyz$ are called Markoff numbers. A number of novel features of Markoff numbers were found from the graph-theoretical standpoint. Namely, for a given Markoff number there exist a pair of graphs, caterpillar and linearly growing polyomino, whose topological index and perfect matching number are, respectively, equal to that number. Efficient stepwise algorithms and recursion formulas are found for enumerating these two characteristic quantities of these special graphs, which have either mirror or rotational symmetry. It is conjectured that any Markoff number can be expressed as the sum of squares of a pair of co-prime integers, From these new findings dramatic advance and application will be expected in the mathematics of Markoff numbers.

1. Introduction

In number theory the Markoff numbers have been attracting the interest of both professional and amateur mathematicians [1]. As will be discussed deeply in this paper all the Markoff numbers can be generated as a member of the family tree, or the genealogy, but it has not yet been rigorously proved that any Markoff number appears only once in that tree. This problem is known as the uniqueness conjecture [2-4]. Geometrical or graph-theoretical features of the Markoff numbers have also not yet been clarified.

Recently the present author discovered [5,6] that certain kinds of caterpillar graphs play an important role in interpreting and connecting various elementary but important concepts of algebra and geometry through the topological index, Z, which has been proposed by the present author for characterizing the topological nature of simple graphs [7-9]. It was also pointed out that the Euler's continuant for facile treatment of continued fractions is equivalent to the Z-value of a specifically defined caterpillar graph and thus efficient algorithms for calculating the continuant and continued fraction were obtained.

Further, very recently it was found that any Markoff number can be expressed by the Z-value of a symmetrically broken comb (BroComb) graph, which belongs to caterpillar graphs, and also by the perfect matching number, K, of a symmetrical step polyomino (StepOmino) graph, which can be constructed by linearly growing square graphs according to a specified code. These two types of graphs and their characteristic numbers were found to be closely related with each other, and a number of interesting and important mathematical properties of Markoff numbers were discovered, The results will be introduced in this paper.

2. Definition and fundamental properties of Markoff numbers

The set of solutions (a, b, c) of positive integers for

$$a^2 + b^2 + c^2 = 3abc \tag{2.1}$$

and their individual members are also called the Markoff numbers. The smallest two trios are (1, 1, 1) and (1, 1, 2), from which all other solutions can be successively obtained by solving the quadratic equation (2.1) with respect to one variable by fixing the other two numbers. Then all the members of the Markoff family can be expressed by the tree, or genealogy as Fig. 1, where each node and open region represent, respectively, one Markoff triple (a, b, c) and one of the Markoff numbers, m's, which run as

m = 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461, 37666, 43261, 51641, 62210, 75025, 96557, 135137, ⋯.

The reason why so many members of the Markoff numbers are given here is to show how uniformly in log scale these numbers distribute. Let us assign the stage numbers both to the Markoff numbers and their trios as in Fig. 1

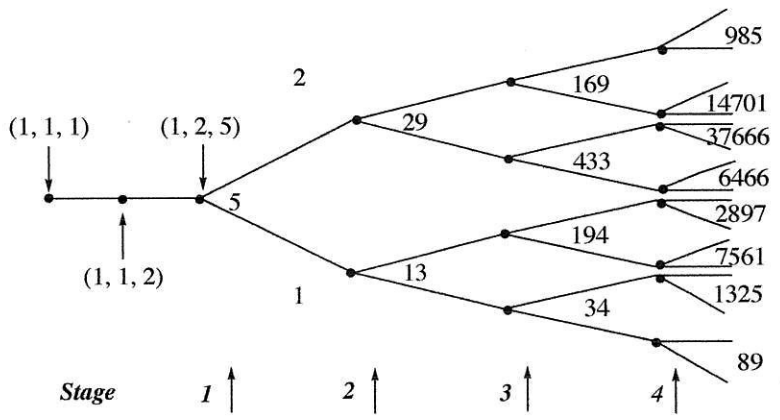


Fig. 1. Genealogy of Markoff numbers.

It has been proved by Markoff that the tree in Fig. 1 generates all the possible Markoff numbers [1], but not yet been rigorously proved that any Markoff number appears only once [2-4].

It is well known that if four adjacent m's are taken from the Markoff tree as shown in Fig. 2 with $d>a,b$ and $d>c$, the largest number d is expressed by the three smaller members as

$$d = 3\,a\,b - c \tag{2.2}$$
$$= (a^2 + b^2)\,/c\,. \tag{2.3}$$

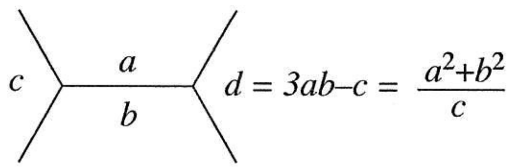


Fig. 2. Relations among four neighbor Markoff numbers.

It is also well known that in the tree in Fig. 1 every other Fibonacci numbers, f_n, and Pell numbers, p_n, appear, respectively, in the upper and lower envelope lines. Note that the initial values of these two famous series of numbers are defined here as

$$f_n = f_{n-1} + f_{n-2}\,, \tag{2.4}$$

with

$$f_0 = f_1 = 1, \tag{2.5}$$

and

$$p_n = 2\,p_{n-1} + p_{n-2}\,, \tag{2.6}$$

with

$$p_0 = 1, \quad \text{and} \quad p_1 = 2, \tag{2.7}$$

which are different from the conventional choices.

On the other hand, it is not well known that any Markoff number can be represented by the sum of squares of a pair of co-prime integers as

$$m = k^2 + l^2, \tag{2.8}$$

at least in one way. This property was found to lead to various important properties as will be explained later in this paper. However, before going into this problem let us explain briefly several graph-theoretical concepts, such as the topological index, perfect matching number, caterpillar graph, and polyomino graph,

3. Fundamental graph-theoretical concepts

In the graph theory a graph is the set of vertices (V) and edges (E). Any edge has two vertices at the both ends, while a single vertex and no vertex (vacant graph, ϕ) can be a member of graphs. Here in this paper only simple tree graphs are treated, which do not have any loop nor ring, while all the component edges are non-directed and single. The simplest series of graphs are path graphs, P_n, composed of n vertices which are connected consecutively by n–1 edges. A star graph, S_n, or $K_{1,n-1}$, is constructed by n–1 vertices and the central vertex, the former of which are joined to the latter. A caterpillar graph, $C_n(\{S_k\})$, is constructed from the path graph P_n with n vertices and the set of n star graphs, $\{S_k\}$, each of which (a_k=K_{1,a_k-1}) is composed of the central vertex and a_k–1 edges of unit length emanating from the central vertex as seen in Fig. 3. A comb graph E_n is C_n (2, 2,⋯, 2) with all a_k being 2, while path graph P_n can be expressed by C_n (1, 1,⋯, 1) with all a_k being unity, A broken comb graph is a caterpillar graph C_n (a_1,⋯, a_n) whose a_k is either 1 or 2.

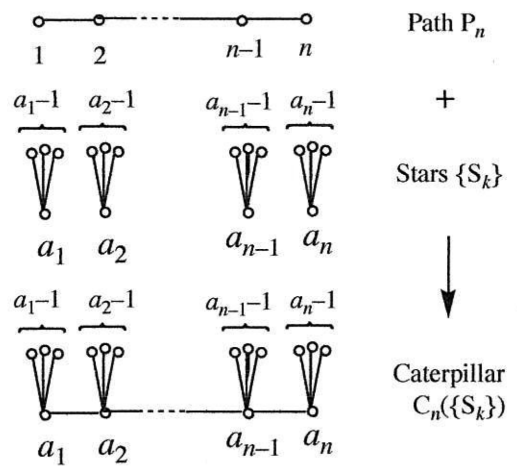


Fig. 3. Construction of a caterpillar graph from a path graph and a set of star graphs.

A polyomino graph is composed of n squares of equal size arranged with coincident edges. In this paper only those step polyominoes, or StepOminoes, are considered (See Fig. 4), which are composed of several steps, s_k's, or linearly joined s_k-ominoes, and a set of junction ominoes between the steps to form a long zigzag shaped strip. In this paper we are concerned only with those StepOminoes which have only one omino at each junction. For those StepOminoes which are composed of l odd-length steps, s_k =2t_k+1, the following shorthand notation will be used as [t_1, t_2,⋯, t_l]. For example, the StepOmino given in Fig. 4 can be denoted as [1, 2, 2], which will be called step-code.

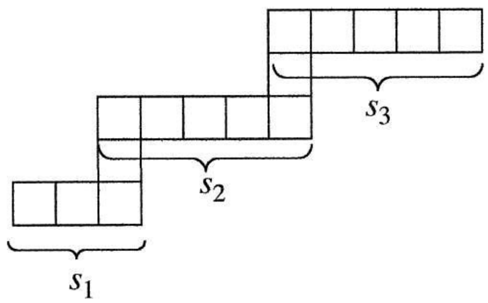


Fig. 4 StepOmino [1, 2, 2]

The topological index Z_G of graph G (with N vertices) is defined as the total sum of non-adjacent number, $p(\mathrm{G},k)$, which is the number of ways for choosing k disjoint edges from G with $p(\mathrm{G},0)=0$ for all the graphs including the vacant graph ϕ [7-9].

$$Z_{\mathrm{G}} = \sum_{k=0}^{\lfloor N/2 \rfloor} p(\mathrm{G},k) \tag{3.1}$$

For graph G with even $N=2m$ the $p(\mathrm{G},m)$ value is called the perfect matching number, which will be called in this paper the K-number and denoted as $K(\mathrm{G})$ or simply as K, taken from the "Kekulé number" in chemistry.

As the calculation of the Z-index for a large polycyclic graph is a time-consuming task, the use of efficient recursive formulas is essential. However, for tree graphs thanks to the following theorem all the $p(\mathrm{G},k)$ numbers can be obtained more easily by expanding the characteristic polynomial $\mathrm{P_G}(x)$ as [7]

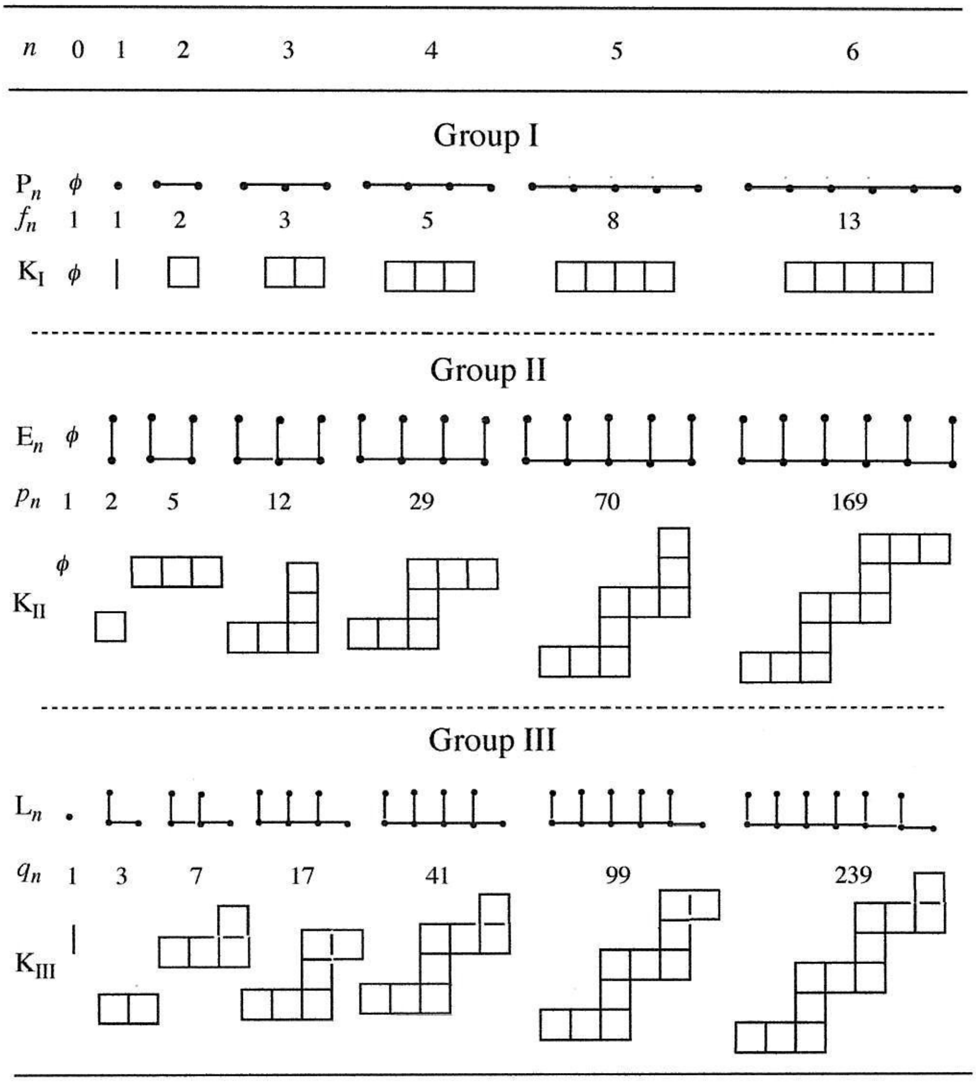
$$\begin{aligned} \mathrm{P_G}(x) &= (-1)^N \det(\mathbf{A} - x\mathbf{E}) \\ &= (-1)^N \begin{vmatrix} -x & 1 & 0 & 0 \\ 1 & -x & 1 & 0 \\ 0 & 1 & -x & 1 \\ 0 & 0 & 1 & -x \end{vmatrix} \\ &= \sum_{k=0}^{\lfloor N/2 \rfloor} (-1)^k p(\mathrm{G},k)\, x^{N-2k} \quad (\mathrm{G} \in \text{tree}). \end{aligned} \tag{3.2}$$

Here $\mathbf{A}$ and $\mathbf{E}$ are, respectively, the adjacency matrix of G and unit matrix of the same size. It has been known that the topological indices of Path graphs P_n's are Fibonacci numbers, f_n, while those of comb graphs E_n's are Pell numbers, p_n, as shown in Table 1, where three groups of the pair of graphs, *i.e.*, caterpillar and polyomino, are given together with their characteristic quantities. That is, the caterpillar and polyomino graphs are paired so that the Z-value of the former and the K-number of the latter take the same value. Groups I and II, respectively, give Fibonacci and Pell numbers whose corresponding graphs have either mirror or rotational symmetry, while the graphs in Group III have no such a symmetry, but play a role in smoothly connecting the adjacent pairs of graphs in Group II.

One can immediately follow from Group I how the Fibonacci numbers are increasing with the linear growth of the corresponding graphs, and from the combined Groups II and III how the Pell-related numbers are increasing with the zigzag growth of the corresponding graphs. In the next section such algorithms will be explained.

From the comparison of Fig. 1 and Table 1 one may be tempted to propose a hypothesis that all the Markoff numbers can be expressed by the Z-values of some hybrids between the path and comb graphs and also by the K-numbers of some hybrids between the linear polyomino and StepOmino graphs, In order to strengthen this hypothesis several recursive algorithms for counting these two kinds of numbers from graphs will be explored.

Table 1. Three groups of graphs whose Z-values and K-numbers give Fibonacci and Pell numbers.



4. Recursive Algorithms for the Topological Index and Perfect Matching Number

In both of these problems a number of knowledge and information have been accumulated in the field of mathematical chemistry, and especially in the research group of the present author. Then instead of giving rigorous mathematical formulations ample examples will be demonstrated here.

First let us explain the algorithm for enumerating the K-numbers of StepOminoes, or 1-dimensionally growing zigzag polyominoes, the direction of whose growth is either linear or making a 90-degree kink. By alternately following the members of Groups II and III in Table 1, one can observe the growth of the zigzag-type "tetragonal animal" in which the $[1,1,\cdots]$-type StepOminoes (See Fig. 4) alternately appear.

Then by following the K-numbers of these growing polyominoes a simple stepwise algorithm for enumerating the n-th K-number (K_n) is derived. That is, for linear growth we have

$$K_n = K_{n-1} + K_{n-2}, \tag{4.1}$$

while for kink-growth

$$K_n = K_{n-1} + K_{n-3}. \tag{4.2}$$

The proof is given in Appendix.

It can easily be proved that this algorithm for the K-numbers can be applied to other types of those StepOminoes whose kink is allowed at least after 3-linear growth.

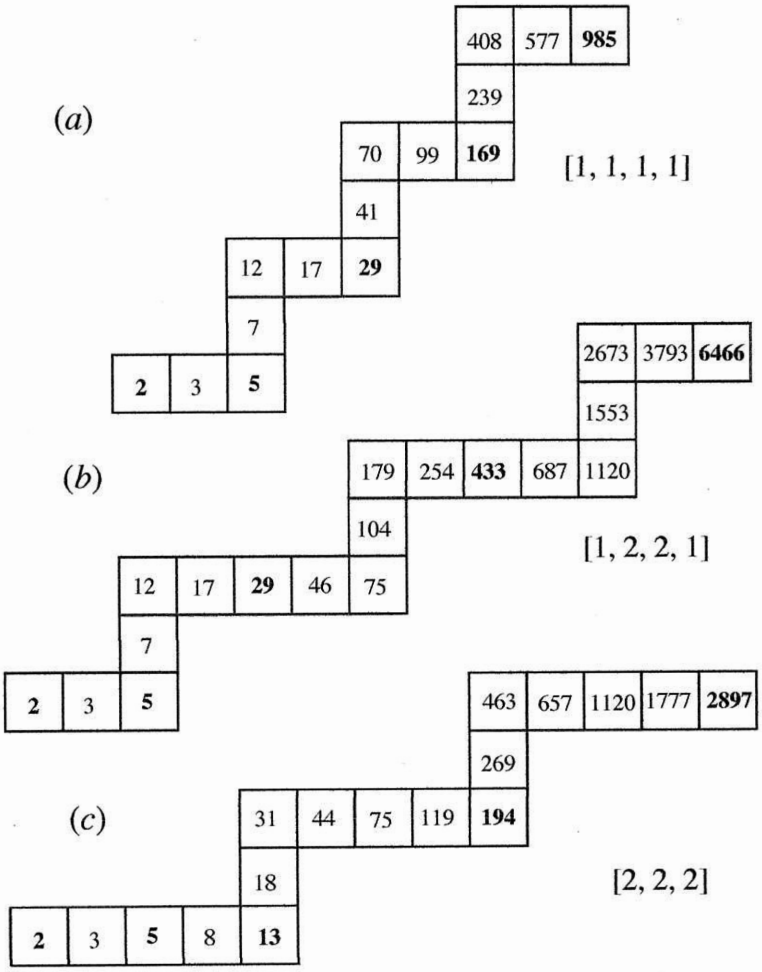


Fig. 5. Stepwise growth of the K-numbers of three kinds of the StepOimnoes giving the Markoff numbers at Stages 3 and 4 in Fig. 1. Note the numbers in bold face and locate them in Fig. 1.

It is striking to observed that the Markoff numbers at Stages 3 and 4 except for 34 could be obtained as the K-numbers of the StepOminoes with rotational symmetry given in Fig. 5. It is easy to be proved that all every other Fibonacci numbers, f_{2m}, including 34 appear as the K-numbers of linearly growing polyominoes.

Enumeration of the K-number of a StepOmino graph is greatly simplified by cutting a pair of parallel edges of an omino in the middle part of the graph followed by using the recursion formulas. As a StepOmino has horizontal and vertical parts, there are two cases for cutting the graph, namely, (a) horizontal and (b) vertical cutting as illustrated in Figs. 6a and b, where the component ominoes are numbered from the left end to the right end. Let the K-number of the StepOmino growing from the left end up to the nth omino be denoted by K_n, and that of the StepOmino growing from the right end up to the nth omino but counted from the left end be K^n.

In the case of horizontal cut (a), one need to have the K-numbers of K_{n-1}, K_{n-2}, K^{n+1}, and K^{n+2}. Then we have

$$K = K_{n-1}K^{n+1} + K_{n-2}\,K^{n+2}. \tag{4.3}$$

The first term is the contribution from the case where the pair of horizontal edges of the nth omino are chosen to be "single," while the second term from the case where they take "double bonds."

On the other hand, in the case of vertical cut (b), we need to have K_{n-1}, K_{n-3}, K^{n+1}, and K^{n+3}, and K is obtained to be

$$K = K_{n-1}\,K^{n+1} + K_{n-3}\,K^{n+3}, \tag{4.4}$$

according to the similar consideration.

Note that our symmetrical StepOminoes are composed of odd number ominoes. Then one can choose a central omino so that a pair of identical (not necessarily symmetric) StepOminoes can be obtained by cutting it. Then in those cases, irrespective of the cases (a) and (b), we have

$$K = K_a^{\,2} + K_b^{\,2}. \tag{4.5}$$

Namely, the Markoff number thus obtained is the sum of two squares. Whether K_a and K_b are co-prime or not will be discussed in section 7.

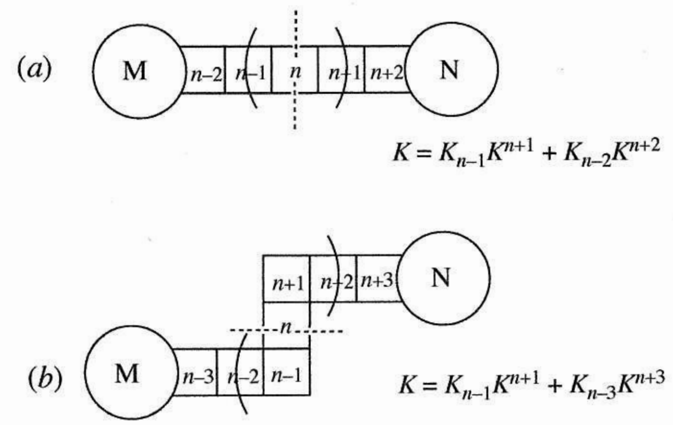


Fig. 6. Recursive formulas for calculating the K-number of StepOmino graphs.

For example, the K-numbers of the three StepOminoes in Fig. 5 can be obtained from (4.5) as follows:

(a): $29^2 + 12^2 = 985$ by vertical cut,

(b): $75^2 + 29^2 = 6466$ by vertical cut,

and (c): $44^2 + 31^2 = 2897$ by horizontal cut,

all of which reproduce the values obtained by the stepwise method.

Now we are going to calculate the Z-values of special kinds of BroComb graphs. By using the algorithms, which will be explained in the next section, for obtaining the BroComb graphs corresponding to the StepOminoes in Fig. 5 we get the three graphs in Fig. 7, where their Z-values obtained stepwise are given. Note that all the K-numbers appearing stepwise in Fig. 5 are reproduced as the Z-values of the corresponding BroComb graphs. The proof of the algorithm will also be given in Appendix.

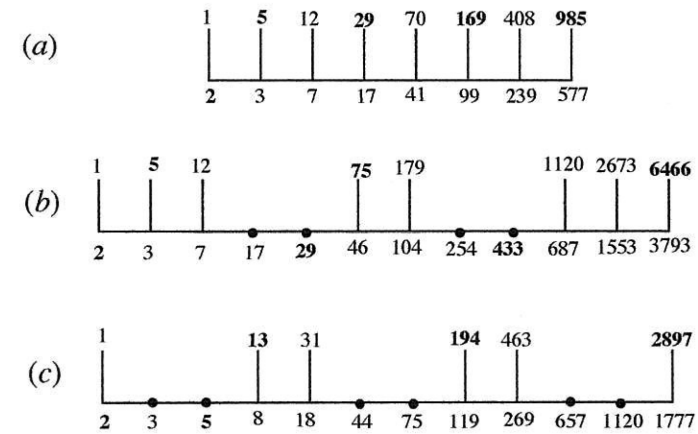


Fig. 7. Stepwise growth of the Z-values of three kinds of BroComb graphs giving the Markoff numbers at Stages 3 and 4. Compare these graphs and Z values with Fig. 5.

It is to be remarked here that in the case of the Z-values of BroComb graphs corresponding to Markoff numbers are expressed by (4.5). See Fig. 8, which explains everything.

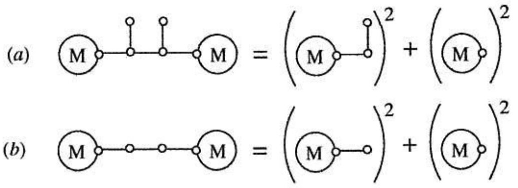


Fig. 8. A BroComb representing a Markoff number is either (a) or (b). In either case The Z-value is the sum of a pair of integers. That they are co-prime cannot be proved, but no counter example has been found.

It is almost certain that all the Markoff numbers can be represented by the BroComb and StepOmino graphs, with which a number of interesting and useful conjectures have come out governing the whole family of Markoff numbers. Although at this stage it is difficult to construct rigorous mathematics to describe all these features, let us try to formulate them as precisely as possible in the form of conjectures and observations.

5. Mathematical structure and relations of the BroComb and StepOmino graphs

[Conjecture 1]

Any Markoff number can be represented by i) the perfect matching number (K-number) of a StepOmino graph, and by ii) the topological index (Z-value) of a BroComb graph.

[Conjecture 2]

The StepOmino representing a Markoff number has the following structure.

i) Each step s_k is composed of linearly joined odd number ominoes ($s_k=2l_k+1$).

ii) Every two consecutive steps are joined by a single omino in head-and-tail manner (See Fig. 4).

iii) The code $[l_1, l_2,\cdots, l_n]$ representing a StepOmino has mirror symmetry, and the geometrical structure of this StepOmino has rotational symmetry (See Fig. 5).

[Conjecture 3]

The BroComb $C_n(a_1, a_2,\cdots, a_n)$ representing a Markoff number has the following structure.

i) n is even.

ii) Mirror symmetry with $a_k = a_{n-k+1}$.

iii) At the both ends $a_1 = a_2 = a_{n-1} = a_n = 1$.

iv) For inner codes the lengths of consecutive 1's and 2's are even.

As already shown in the previous section many Markoff numbers can be expressed by the sum of two squares. However, we can expand this property up to the following conjecture.

[Conjecture 4]

Any Markoff number is represented by the sum of the squares of two co-prime integers.

This is an important property of Markoff numbers, but its discussion will be given after the next section, since the discussion in this section is not relevant to this issue.

In order to discuss the mathematical relation between the BroComb and StepOmino graphs one has to formulate their coding and transformation in more detail as follows by using Figs. 9-11.

[Coding of the BroComb and StepOmino graphs]

First, a BroComb can be expressed by the AB-code as in Fig. 9*a*. An example is given in Fig. 10. The *Z*-value of the BroComb, AABABAA, is obtained to be 2897, which has already been demonstrated in Fig. 7*c*. It is to be remarked here that the AB-codes for all the BroCombs for the Markoff numbers hitherto tested begin with and end at A.

According to the rules shown in Fig. 9b the corresponding StepOmino graph can be obtained as exemplified in Fig. 10 to give the StepOmino [2, 2, 2] as shown in Fig. 5*c*. Then finally one can go back from this StepOmino to the original BroComb by following the rules given in Fig. 9c.

In summary the coding of BroCombs and StepOminoes and interchange between them can uniquely be performed as in the general scheme shown in Fig. 11.

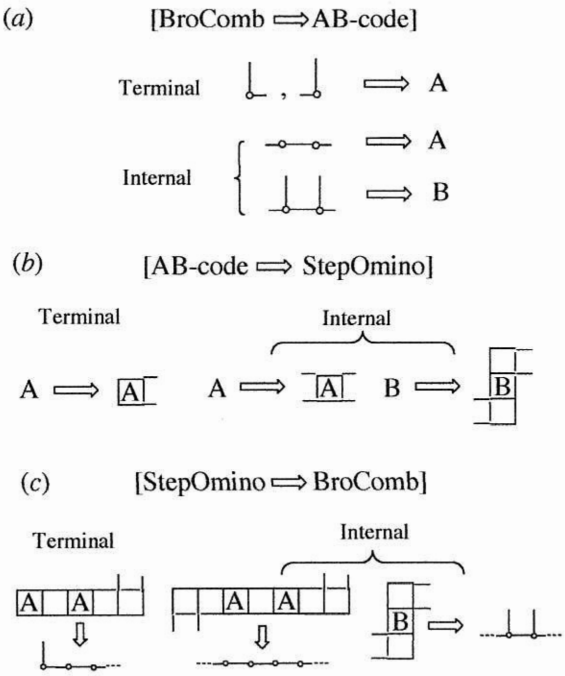


Fig. 9. Coding and transformation of BroComb and StepOmino graphs.

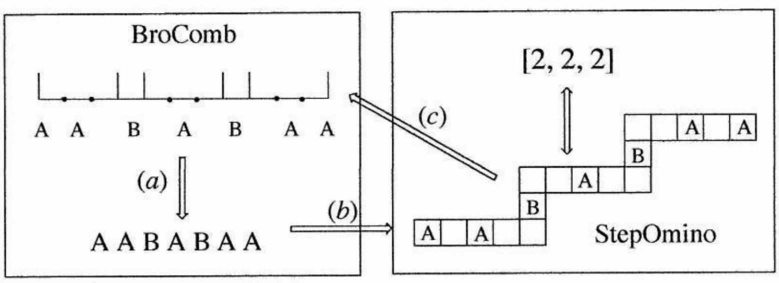


Fig. 10. Examples of coding and transformation of BroCombs and StepOminoes.

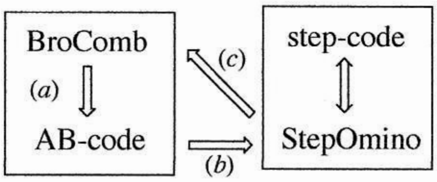


Fig. 11. General schemes of transformation between BroCombs and StepOminoes.

By comparing a number of corresponding BroComb and StepOmino graphs we could get the following Observations.

[Observations]

The number of steps in a StepOmino can be obtained from the AB-code as follows;

i) First cut the AB-code into blocks of either consecutive A_m or B_m ($m \geq 1$).

ii) The terminal A_m constitutes a single step which is composed of linearly joined $2m+1$ ominoes.

iii) An internal A_m (sandwiched by a pair of B's) constitutes a single step which is composed of linearly joined $2m+3$ ominoes.

iv) An isolated B does not constitute a single step.

v) An internal B_m ($m \geq 2$) constitutes $m-1$ step(s) each composed of linearly joined 3 ominoes.

All the above statements including Conjectures, Coding Rules, and Observations were obtained from a number of iterative calculations, and up to now no violation could be detected. Although it is rather cumbersome to write a general proof, the following statement can be given here as a Theorem.

[Theorem 1]

The *K*-numbers and *Z*-values of the two types of graphs, StepOmino and BroComb, are mathematically equivalent and follow the same recursive algorithms.

6. Construction of larger BroComb and StepOmino graphs from smaller ones

In the following discussion let us suppose a set of four Markoff numbers, (a, b, c, d), located as in Fig. 2, and with $d>a>b$ and $d>c$. Numerically d can be calculated by (2.2) from the rest of the three small members as the larger solution of

$$c, d = \frac{3ab \pm \sqrt{9a^2b^2 - 4(a^2 + b^2)}}{2}. \tag{6.1}$$

However, we are going to analyze graph-theoretical features of (2.2) and Fig. 2, which generate the genealogy (Fig. 1) of Markoff numbers.

Let us take the following set of numbers and the corresponding StepOminoes as an example.

$[a] = [1, 2, 1]$ for $a = 433$,

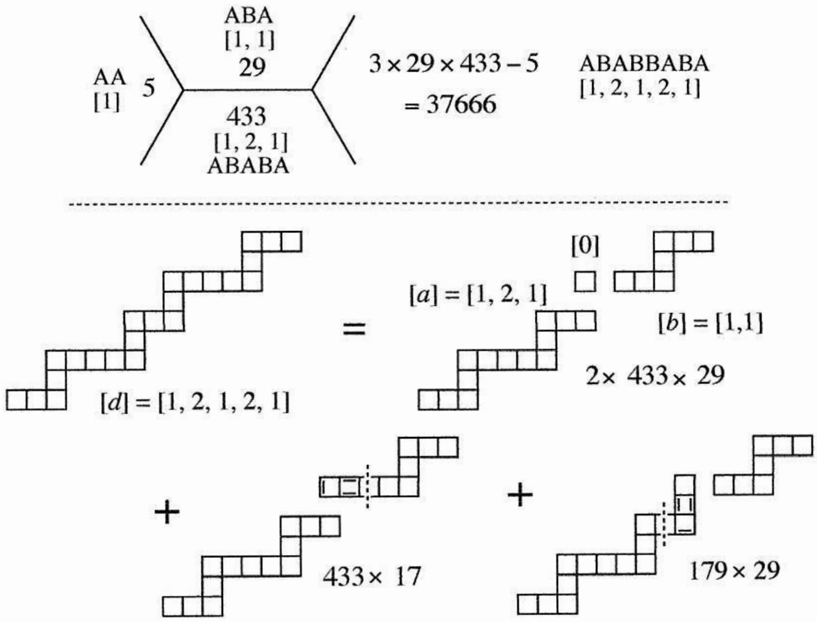
$[b] = [1,1]$ for $b = 29$,

$[c] = [1]$ for $c = 5$,

$[d] = [1, 2, 1, 2, 1]$ for $d = 37666$,

for which the following equality already holds,

$$3 \times 29 \times 433 - 5 = 37666$$



$$= \ 2 \times 433 \times 29 \ + \ 433 \times (29 - 12) \ + \ (433 - 254) \times (12 + 17)$$

$$= \ 3 \times 433 \times 29 \ + \ 433 \times 17 \ - \ 254 \times 29$$

$$= \ 3 \times 433 \times 29 \ - \ 5$$

Fig. 12. Perfect matching number of [1, 2, 1, 2, 1] can be obtained from smaller StepOminoes.

The main frame of $[d]$ can be constructed from $[a]$, $[b]$, and [0], a monomino, as in the middle of Fig. 12. However, the product of the K-numbers of them gives a much smaller value as

$$2 \times 433 \times 29 = 25114,$$

which counts only those contributions in which the four edges joining the three StepOminoes are not involved in the perfect matching patterns.

Then for getting the K-number of $[d]$ to be 37666 one has to add the two contributions as shown in Fig. 12. Namely, we have

$$2 \times 433 \times 29 + 433 \times 17 + 179 \times 29 = 37666.$$

The last two terms in the left-hand-side count those perfect matching patterns where a pair of edges emanating from [0] to the neighbors [b] and [a], respectively, are chosen to be "double bonds." Then by using some tricky manipulation as given in the lower part of Fig. 12, we get the important correction term as

$$433 \times 17 - 254 \times 29 = -5,$$

which can be reduced down to

$$12 \times 1 - 17 \times 1 = -5$$

as in Fig. 13.

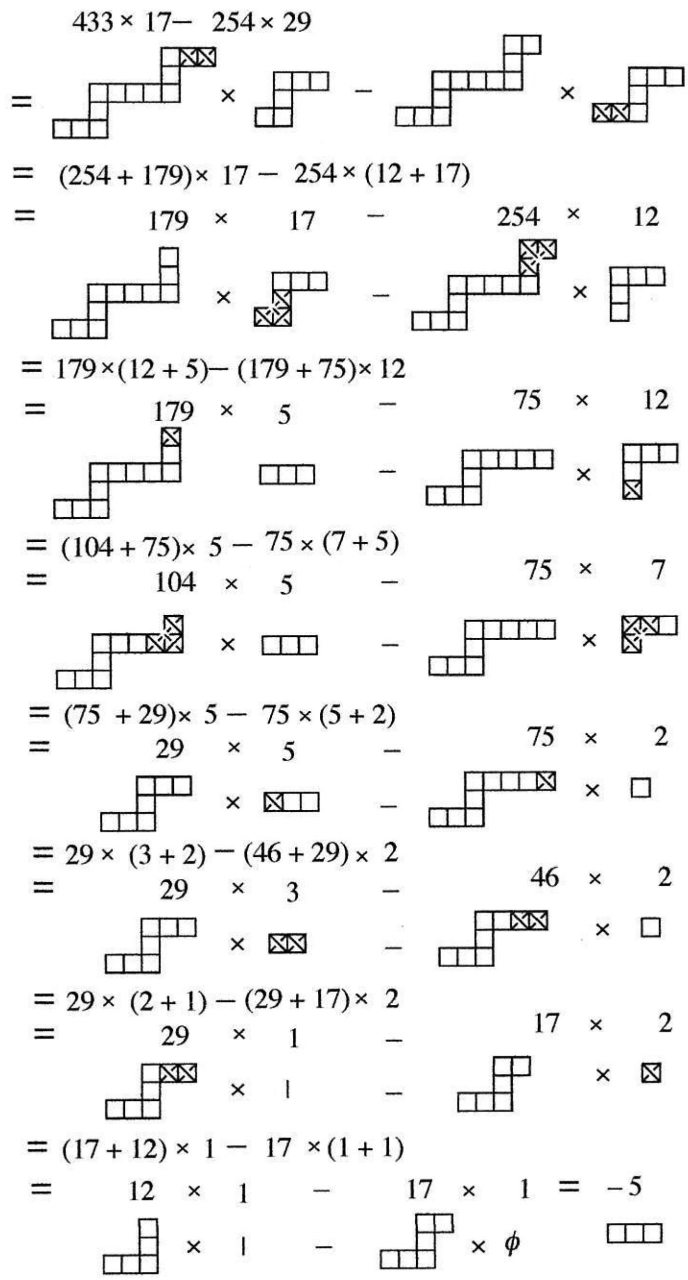


Fig. 13. Degradation of the correction terms in Fig. 11 into smaller StepOmino graphs.

Although it seems to be rather cumbersome to get to the above result, it is important to note that the negative of the resultant value, 5, in the right-hand-side of the last line in Fig. 13 is equal to c. We have calculated a number of similar cases taken from the genealogy of Markoff numbers, but have not yet met any exceptional case.

Then we could reach an important conclusion how the set of four neighboring Markoff numbers, (a, b, c, d), in Fig. 2 are related with each other through their corresponding StepOmino graphs. Before presenting it the following notation $[a]\oplus[b]$ will be introduced. Namely, it denotes the joining of [a] and [b] by inserting an extra monomino [0]. Then the following conjecture is posed.

[Conjecture 5]

Consider four StepOminoes, [a]~[d], corresponding to the four neighboring Markoff numbers with $d>a>b$ and $d>c$. Then StepOminoes [c] and [d] can be obtained from [a] and [b] as

$$[c] = [a] - ([b] \oplus [0]) \tag{6.2}$$

$$[d] = [a] \oplus [b] \oplus [0] \tag{6.3}$$

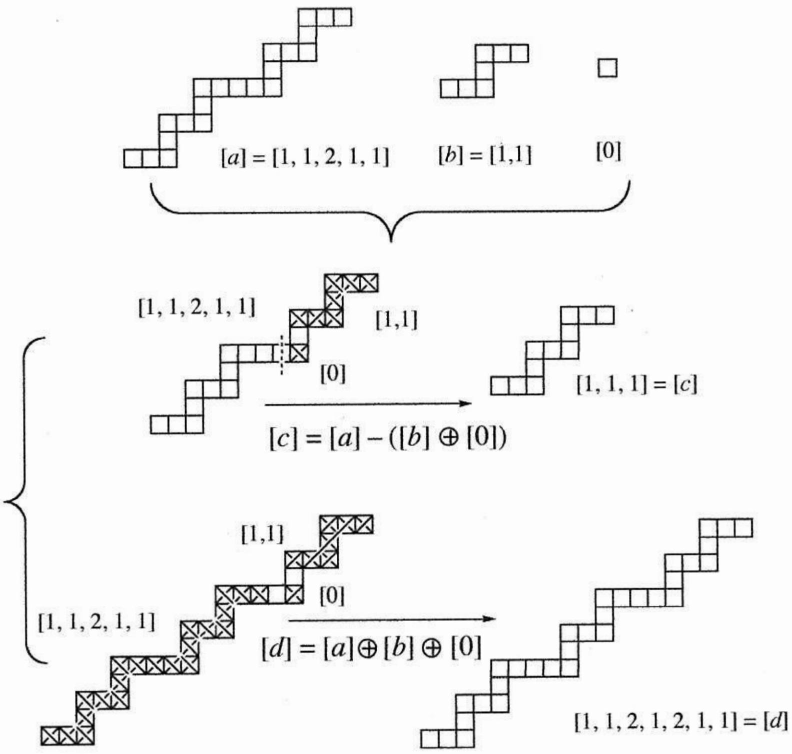


Fig. 14. Example of the algorithm to obtain the StepOminoes [c] and [d] from [a] and [b].

An example is given with [*a*] = [1, 1, 2, 1, 1] and [*b*] = [1, 1] as in Fig. 14. The corresponding Markoff numbers are a=14701 and b=29, and c and d can be calculated from (6.1) as c=169 and d=1278818. Of course, d can be calculated by using (2.2), if a, b, and c are known. On the other hand, Conjecture 5 and Fig. 14 can give rather easily c and d without manipulating the quadratic surd of a large number of more than ten digits, but only with the graph-theoretical information on a and b.

7. Factors determining Markoff numbers

In this section we will discuss what kind of integers is qualified to be a Markoff number. By scrutinizing the results of Figs. 12 and 14 an interesting algorithm for generating [*d*] from the addition of [*a*] and [*b*] was obtained. In Fig. 15 it is illustrated together with two other cases.

$$[1, 2, 1] + [1, 1] \longrightarrow [1, 2, 1, 2, 1] \qquad \text{Fig. 12}$$

$$[1, 2, 1] + [1] \longrightarrow [1, 2, 2, 1]$$

$$[1, 1, 2, 1, 1] + [1, 1] \longrightarrow [1, 1, 2, 1, 2, 1, 1] \qquad \text{Fig. 14}$$

$$[1, 1, 2, 1, 1] + [1, 1, 1] \longrightarrow [1, 1, 2, 1, 1, 2, 1, 1]$$

Fig. 15. Rule for joining of two symmetrical StepOminoes to give a larger StepOmino.

When two symmetrical StepOminoes [*a*] and [*b*] are joined to form a larger symmetrical StepOmino [*d*], the code number of one of the inner terminal steps of one StepOmino is found to increase by one. In Fig. 15, the first and third lines are the cases with Figs. 12 and 14, respectively. Note that the cases given in Fig. 15 are applied only to the StepOminoes whose step-codes at both the terminals are 1. If 1 is added to all the entries in the first line of Fig. 15, one gets

$$[2, 3, 2] + [2, 2] \longrightarrow [2, 3, 2, 3, 2],$$

which can be found in Fig. 16, where the genealogy of Markoff numbers already shown in Fig. 1 is redrawn by using the step-codes. One can realize that whole the genealogy tree except for the upper and lower envelope lines is constructed according to the rule in Fig. 15.

Thus one can understand why many possible step-codes, such as 212, 2112, 12321, etc. do not appear in the genealogy of Markoff numbers. Also it turns out that Conjectures 2 and 3 for the StepOmino and BroComb graphs are only sufficient conditions for a Markoff number. Up to now no fundamental discussion has been had on Conjecture 4, especially on the co-primeness of two integers.

Although the discussion using Figs. 6 and 8 leading to (4.5) ensures that a Markoff number is expressed as the sum of the squares of two integers, m and n, (m, n)=1 has not yet been guaranteed.

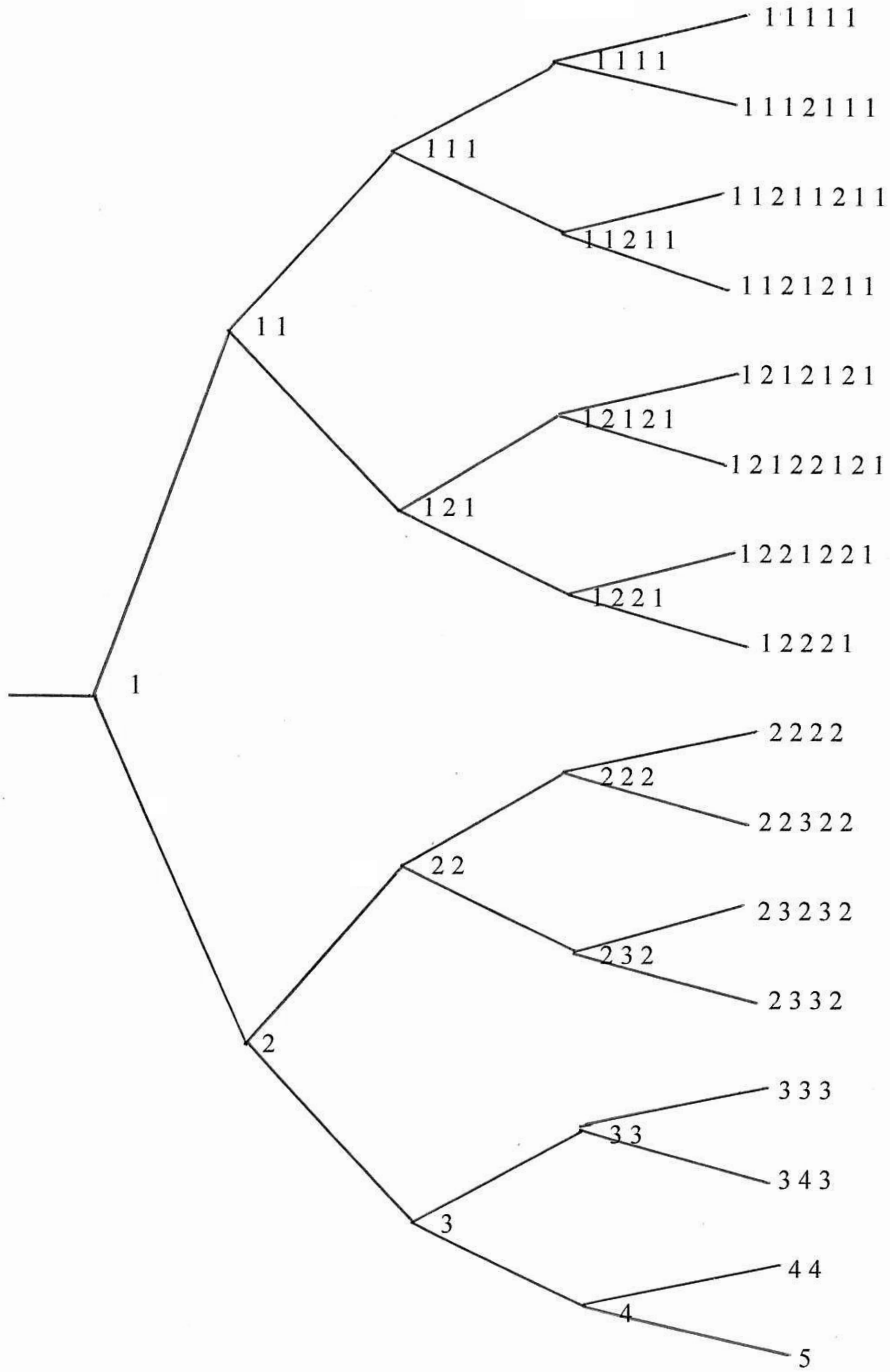


Fig. 16. The genealogy of Markoff numbers drawn by using the step-codes.

The following discussion is not complete, but its direction is believed to be right. Table 2 lists up smaller members of various Fibonacci-type series of numbers obeying the recursion formula, $f_n = f_{n-1}$

$+ f_{n-2}$. From a number of such observations a common divisibility property for those series of numbers can be deduced. Namely, given a prime number p, then multiples of p appear periodically as

$$(f_n, f_{n+mk}) = p \qquad (m=1,2,\cdots). \tag{7.1}$$

Table 2. Divisibility property of Fibonacci-type series of numbers ensuring $(f_n, f_{n+1})=1$ and $(f_n, f_{n+2})=1$.

1	**2**	*3*	5	**8**	13	*21*	**34**	55	89	***144***	233	377	**610**	*987*	1597
1	*3*	**4**	7	11	***18***	29	47	**76**	*123*	199	**322**	521	*843*	**1364**	2207
1	**4**	5	*9*	**14**	23	37	***60***	97	157	**254**	*411*	665	**1076**	1741	*2817*
1	5	***6***	11	17	**28**	*45*	73	**118**	191	*309*	**500**	809	1309	***2118***	3427
2	5	7	*12*	19	31	**50**	*81*	131	**212**	*343*	*555*	**898**	1453	2351	***3804***
3	**8**	11	19	***30***	49	79	**128**	207	335	**542**	877	1419	**2296**	3715	6011
11	13	***24***	37	61	**98**	*159*	257	**416**	673	*1089*	**1762**	2851	4613	***7464***	12077

n $= 2m$, *n* $= 3m$, n $= 5m$, n $= 7m$, n $= 11m$, n$=13m$. $665 = 5\times7\times19$

In Table 3 the k numbers are given for smaller prime numbers.

Table 3. Periodicity k of smaller primes p's in the Fibonacci-type series of numbers

p	2	3	5	7	11	13	17	19	23	29	31	37
k	3	4	5	8	10	7	9	18	24	14	30	19

This means that for Fibonacci-type series of numbers the following divisible properties can be observed,

$$(f_n, f_{n+1}) = 1 \qquad \text{and} \qquad (f_n, f_{n+2}) = 1, \tag{7.2}$$

This conclusion cannot directly be applied to the StepOminoes, because the periodicity k is diminished to some extent at kink positions. However, this effect does not affect the above conclusion.

8. Concluding remarks

Actually there has been a flow of research on the graph-theoretical analysis of Markoff numbers initiated by Conway and Coxeter and developed by Propp et al. [10-14] using the concepts of triangulated polygons, frieze patterns, and perfect matchings in the field of cluster algebra. However, it is difficult to apply these theories to a big family of Markoff numbers, especially in the manipulation of the numbers at higher stages (even >3 in Fig. 1). On the contrary, as already shown in the discussion in this paper, correlation of StepOmino and BroComb graphs with Markoff numbers is more direct and simple, and extension to larger numbers is analytically possible. Further, the most important conclusion in this paper is Conjecture 4 with respect to the number-theoretic property of Markoff numbers.

At the present stage rigorous proof cannot be obtained yet, but the present author believes that all the conjectures including Conjecture 4 posed in this paper are correct. It is expected that in a near future dramatic progress will be made in the field of mathematics relevant to Markoff numbers by taking the new approach proposed in this paper.

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Appendix. Proof of (4.1) and (4.2)

First consider polyomino in Fig. A*a* which is obtained by linearly adding the *n*th omino to the preceding polyomino, whose *K*-number is $K_{n-1}=b$. This polyomino was obtained also by linearly adding the (n–1)th omino to the polyomino, whose *K*-number is $K_{n-2}=a$. We are going to obtain the *K*-number, K_n, Let the right-most vertical edge in the *n*th omino be called *l*. Since *l* is either "double" or "single," according to the inclusion-exclusion principle, K_n is the sum of the cases of "*l*-double" and "*l*-single." The *K* number for the former case is equal to K_{n-1}, while the *K* for the latter is K_{n-2}. Then (4.1) is proved.

Next consider polyomino in Fig. A*b* which is obtained by adding the *n*th omino to K_{n-1} so that a kink is formed. Let the top horizontal edge in the *n*th omino be called *l*. Since *l* is either "double" or "single," K_n is the sum of the cases of "*l*-double" and "*l*-single." The *K* number for the former case is equal to K_{n-1}, while the *K* for the latter is K_{n-3}. Then (4.2) is proved.

Quite similar algorithms for counting the *Z*-values of BroComb graphs can be obtained as illustrated in Figs. A*c* and *d*.

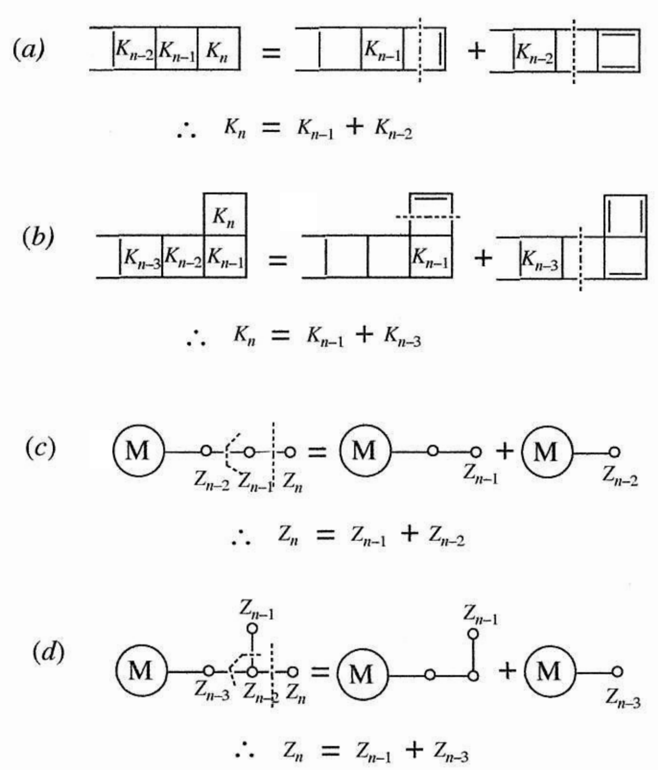

Fig. A. Proof of (4.1) and (4.2), together with the similar recursive relations for the *Z*-values.