Heronian Triangles. I. Systematic relation among the isosceles Heronian triangles.

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Abstract A Heronian triangle (HeT) is defined here to be a triangle whose edges and area Δ are integers. A HeT (*a*, *a*, *b*) with coprime *a* and *b* is called a primitive isosceles Heronian triangle (piHeT). All the piHeT's are shown to be classified into two groups, I and II, depending on the value of 2*a*–*b*, and a pair of natural numbers, (*n*, *k*), called "family register codes", are shown to generate a pair of piHeT each belonging to I and II, together with a limited number of non-primitive triangles. However, by including those non-primitive members the *a*, *b*, and Δ values of all the family members of piHeT are uniquely generated and related with each other through simple recursion formulas.

1. Introduction

By using the semiperimeter, $s=(a+b+c)/2$, for a given triangle, (a, b, c), the area, Δ, of this triangle is expressed by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,, \tag{1.1}$$

which is attributed to Heron of Alexandria who lived circa 2000±100 years ago.

Heronian triangle (abbreviated here as HeT) is a triangle whose edges and area are rational numbers.[1-3] However, here we will be concerned only with HeT with integers, *a*, *b*, *c*, and Δ. Since all the Pythagorean triangles (or triples) which are right triangles composed of integer edges, are Heronian, they are excluded from the HeT except for (3, 4, 5), which is the progenitor of all the family members of both Pythagorean and Heronian triangles. If (*a*, *b*, *c*) have no common divisor, the triangle is called primitive. Here primitive HeT's are the main targets, but some special non-primitive HeT's are also included in the discussion.

Contrary to the Pythagorean triangles the HeT's have not been extensively studied by both professional and amateur mathematicians for a long time, except for Brahmagupta, Euler, Schubert,[4] Carmichael,[5] and a few others. Very recently by the help of computers Buchholz[6] dug out many hidden properties of HeT's and showed us its interesting and deep mathematical structure, but still majority of those problems are open.

One big reason for this situation might be the lack of useful family register code(s) for each member of HeT's contrary to the case of Pythagorean triangles.[7-9] Although Carmichael[5] showed that the edges of all rational number triangles can be expressed in terms of the set of positive integers (*m*, *n*, *h*) by

$$a = n(m^2 + h^2),$$
$$b = m(n^2 + h^2),$$

and $$c=(m+n)(mn-h^2) \quad \text{under } mn>h^2, \tag{1.2}$$
it is very tedious to select the correct set of the HeT family out of the possible combinations of m, n, and h. Thus with this method one can hardly grasp the mathematical structure of the family of HeT's.

The purpose of the present study is to seek some crucial key for finding systematic relations among the edges and area of the family of HeT's. As a first step for this project we have studied the mathematical structure of the primitive isosceles HeT's (abbreviated here as piHeT's) of the type (a, a, b). The main result obtained here is that piHeT's can be classified into two big groups which originate from (5, 5, 6) and (5, 5, 8), and the values of a, b, and Δ for a member in each group can be expressed by polynomials written in terms of a pair of natural numbers, (n, k). Further, the values of a, b, and Δ for all the members in each group are related with each other by a pair of common recursive formulas.

Before introducing our results let us show how the Carmichael's formulation (1.2) does not work well even for piHeT's. In this case one can put $m=n$, and we have

$$a=b=m(m^2+h^2)$$
and $$c=2m(m^2-h^2). \tag{1.3}$$

Since we have chosen $m^2>h^2$, it is straightforward to generate the family of piHeT's by taking possible combinations of m and h $(m>h>0)$ to calculate m^2+h^2 and compare it with $2(m^2-h^2)$ for checking if they are coprime with each other. If they are coprime, we get one piHeT having $a=b=m^2+h^2$ and $c=2(m^2-h^2)$. Otherwise a piHeT can be obtained by dividing them with their cofactor.

However, as will be shown in Section 3, it was found that this is a rather messy work and, further, the set of (m, h) does not function as good family register codes from which useful information on the mathematical structure of isosceles HeT's is revealed.

2. Main results

All the family members of piHeT's are found to be divided into two groups depending on their $2a-b$ values, which are either $(2n)^2$ or $2(2n-1)^2$. Let them be called groups I and II, respectively. Each of these two big groups is further classified into infinitely many subgroups according to the value of natural number, n, and the members in each subgroup are numbered by another natural number code, k, according to their cardinality. Then if a pair of codes, (n, k), are chosen, there exist a pair of piHeT's each of which belongs to either group I or II. In some cases the obtained isosceles HeT is non-primitive, but it is to be included in the family. The edge lengths, a and b, and area, Δ, of these isosceles HeT's are determined as in the following Theorem.

[Main Theorem]

For a given pair codes (n, k) there exist a pair of isosceles HeT's (mostly primitive)。

Group I

$$a_{n,k}=(k+2n-1)^2+k^2, \quad b_{n,k}=2(2n-1)(2k+2n-1), \quad \Delta_{n,k}=2k(2n-1)(k+2n-1)(2k+2n-1) \tag{2.1}$$

Group II

$$a_{n,k}=(k+2n-1)^2+k^2, \quad b_{n,k}=4k(k+2n-1), \quad \Delta_{n,k}=2k(2n-1)(k+2n-1)(2k+2n-1) \tag{2.2}$$

(Remark 1) Besides all the piHeT's, a limited number of non-primitive triangles are also included.

(Remark 2) The pair of triangles with the same (n, k) codes differ only in the base edge, b and have the same area.

(Remark 3) The edge length of the triangles with the same n (or k) code in the same big group obey the following recursion formula:

$$f_j = 3 f_{j-1} - 3 f_{j-2} + f_{j-3} \quad \text{for } f = a, b \text{ and } j = k \text{ (or } n). \tag{2.3}$$

(Remark 4) The area of the triangles with the same n (or k) code in the same big group obey the following recursion formula:

$$f_j = 4 f_{j-1} - 6 f_{j-2} + 4 f_{j-3} - f_{j-4} \quad \text{for } f = \Delta \text{ and } j = k \text{ (or } n). \tag{2.4}$$

In the rest of this section, how these results were obtained will be explained. Now consider an isosceles HeT (a, a, b). Since its semiperimeter is $s=a+b/2$, $s-a=b/2$ and $s-b=a-b/2$. Then its area Δ is expressed by

$$\Delta = \frac{b}{4}\sqrt{4a^2 - b^2}, \tag{2.5}$$

and wẹ can put

$$4a^2 - b^2 = h^2. \tag{2.6}$$

This means that b and h should be even, and we further put

$$b = 2B \quad \text{and} \quad h = 2H, \tag{2.7}$$

leading to

$$a^2 = B^2 + H^2. \tag{2.8}$$

This means that our HeT can be divided into two equivalent Pythagorean triangles (B, H, a) as shown in Fig. 1a.

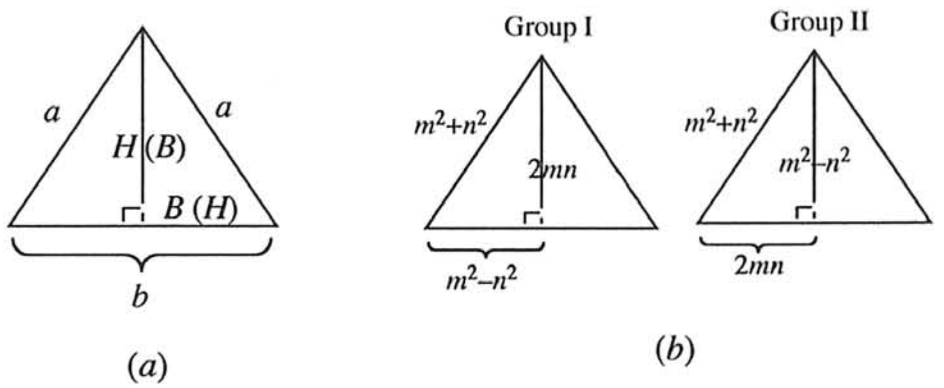


Fig. 1. (a) Isosceles Heronian triangle. (b) Two groups of primitive isosceles Heronian triangles.

It is known that the hypotenuse of a primitive Pythagorean triangle (pPT) is either a prime $p\equiv 1$ (mod 4) or a product of such prime numbers.[10] Then we have

$$a = p \equiv 1 \pmod 4 \quad \text{or a product of those } p\text{'s}. \tag{2.9}$$

Actually the a values of smaller pPT's are known to be as follows:

$$a = 5, 13, 17, 25, 29, 37, 41, 53, 61, 65, 73, 85, 89, 97, \cdots,$$

where $25=5\times 5$, $65=5\times 13$, and $85=5\times 17$, and all others are primes as given by (2.9).

Since a, B, and H are the edges of a pPT, they can be expressed by

$$a = m^2 + n^2, \tag{2.10}$$

$$B \text{ or } H = m^2 - n^2 \text{ or } 2mn, \tag{2.11}$$

where m and n ($m>n$) are i) coprime natural numbers and ii) of different parity.[1,9,10]

Then the value of a–B takes either

$$a - B = 2n^2 \text{ or } (m-n)^2. \tag{2.12}$$

In the latter case m–n should be an odd integer because of condition ii). After doubling (2.12) one can conclude that piHeT's can be classified into two groups according to the following criterion:

$$2a - b = \text{(square of even integer)}, \text{ or } 2\text{(square of odd integer)}. \tag{2.13}$$

Let us call them Groups I and II, respectively.

Very recently the present author proposed an efficient classification scheme for pPT's by taking three kinds of differences among their three edges for assigning triple family register codes for each pPT.[9] Naturally in that case, it was shown that the values of (hypotenuse–odd leg) and (hypotenuse–even leg) take, respectively, double of square number and square of odd integer just as in (2.12).

Thus, as illustrated in Fig. 1*b*, Groups I and II of piHeT's are found to be formed by merging, respectively, the even and odd legs of twin pPT's. This means that if there are l pairs of solutions for a given a value in (2.10), there exist $2l$ piHeT's. Then the possible a values for piHeT's were listed up as in Table 1 by arranging the set of (m, n) belonging to different parities.

Table 1. Rectangular arrangement of a (m, n) for the isosceles HeT family.

5	(2, 1)	17	(4, 1)	37	(6, 1)	**65**	(8, 1)	101	(10, 1)	……
13	(3, 2)	29	(5, 2)	53	(7, 2)	**85**	(9, 2)	**125**	(11, 2)	……
25	(4, 3)	*45*	*(6, 3)*	73	(8, 3)	109	(10, 3)	*153*	*(12, 3)*	……
41	(5, 4)	**65**	(7, 4)	97	(9, 4)	137	(11, 4)	**185**	(13, 4)	……
61	(6, 5)	89	(8, 5)	***125***	*(10, 5)*	169	(12, 5)	221	(14, 5)	……
85	(7, 6)	*117*	*(9, 6)*	157	(11, 6)	**205**	(13, 6)	*261*	*(15, 6)*	……
113	(8, 7)	149	(10, 7)	193	(12, 7)	*245*	*(14, 7)*	**305**	(16, 7)	……
……		……		……		……		……		……

$a = m^2 + n^2$, Italicized a : GCD of (m, n)$\neq 1$, Bold a : More than two (m, n) pairs exist.

It is to be remembered that there exist two piHeT's for each entry of a in Table 1. For example for the case of a=5 (m=2 and n=1) the b value can take either $2(m^2-n^2)$=6 or $4mn$=8, but the area Δ takes the same value of $2mn(m^2-n^2)$=12.

In this table such non-primitive isosceles HeT's are also included in italic which are constructed from non-coprime (m, n) pairs. Those a values are printed in bold which can be expressed by the sum of two squares in more than two different ways. Notice that the square of $p\equiv 1$ (mod 4), such as 25 and 169, has only one pair of (m, n) solution, while their higher powers have more than two solutions.

Now if one assumes

$a_{n,k} = (k+2n-1)^2 + k^2$,

all the a values in Table 1 can be reproduced as in Table 2.

Table 2. $a_{n,k} = (k+2n-1)^2 + k^2$

$k \diagdown n$	1	2	3	…
1	5	17	37	…
2	13	29	53	…
3	25	45	73	…
…	…	…	…	…

Further, one can understand that inclusion of non-coprime (m, n) pairs by neglecting condition i) is essential for systematic preparation of Tables 1 and 2.

It is straightforward to get the $b_{n,k}$ expressions from $a_{n,k}$. Namely, by putting $m=k+2n-1$ and $n=k$ into $2(m^2-n^2)$ and $4mn$ one gets the expressions given in (2.1) and (2.2).

These observations lead us to prepare Table 3, where the a, b, and Δ values of piHeT's in groups I and II are arranged according to the kth row and nth column calculated by the formulas given in (2.1) and (2.2), which were thus inductively derived and include a limited number of non-primitive members. However, the above-mentioned flow of logic is thought to exclude a possibility of other solutions. The most remarkable point is that for a given pair of (n, k) codes there always exist two isosceles HeT's, most of which are primitive. It is very difficult to draw this conclusion from the Carmichael's theory.

Further, it is interesting to note that the a values in each row and also in each column of Table 1 the consecutive members obey the same recursion formula (2.3) as exemplified as

$3\times65 - 3\times37 + 17 = 101$,

$3\times73 - 3\times53 + 37 = 97$, *etc.*

The same recursive properties also apply to the consecutive members of b, while the values of Δ obey a different recursion formula (2.4). These two recursive formulas, (2.3) and (2.4), respectively, come from the characteristic equation, $(x-1)^3$ and $(x-1)^4$.

Table 3. Smaller primitive and non-primitive (italicized) isosceles Heronian triangles.

	n=1				n=2				n=3				n=4			
k	$a_{n,k}$	$b_{n,k}$		$\Delta_{n,k}$	$a_{n,k}$	$b_{n,k}$		$\Delta_{n,k}$	$a_{n,k}$	$b_{n,k}$		$\Delta_{n,k}$	$a_{n,k}$	$b_{n,k}$		$\Delta_{n,k}$
		I	II			I	II			I	II			I	II	
1	5	6	8	12	17	30	16	120	37	70	24	420	65	126	32	1008
2	13	10	24	60	29	42	40	420	53	90	56	1260	85	154	72	2772
3	25	14	48	168	*45*	*54*	*72*	*972*	73	110	96	2640	109	182	120	5460
4	41	18	80	360	65	66	112	1848	97	130	144	4680	137	210	176	9240
5	61	22	120	660	89	78	160	3120	*125*	*150*	*200*	*7500*	169	238	240	14280
6	85	26	168	1092	*117*	*90*	*216*	*4860*	157	170	264	11220	205	266	312	20748
7	113	30	224	1680	149	102	280	7140	193	190	336	15960	*245*	*294*	*392*	*28812*
8	145	34	288	2448	185	114	352	10032	233	210	416	21840	289	322	480	38640
9	181	38	360	3420	*225*	*126*	*432*	*13608*	277	230	504	28980	337	350	576	50400
10	221	42	440	4620	269	138	520	17940	*325*	*250*	*600*	*37500*	389	378	680	64260
11	265	46	528	6072	317	150	616	23100	377	270	704	47520	445	406	792	80388
12	313	50	624	7800	*369*	*162*	*720*	*29160*	433	290	816	59160	505	434	912	98952

	n=5				n=6				n=7				n=8			
k	$a_{n,k}$	$b_{n,k}$		$\Delta_{n,k}$	$a_{n,k}$	$b_{n,k}$		$\Delta_{n,k}$	$a_{n,k}$	$b_{n,k}$		$\Delta_{n,k}$	$a_{n,k}$	$b_{n,k}$		$\Delta_{n,k}$
		I	II			I	II			I	II			I	II	
1	101	198	40	1980	145	286	48	3432	197	390	56	5460	257	510	64	8160
2	125	234	88	5148	173	330	104	8580	229	442	120	13260	293	570	136	19380
3	*153*	*270*	*144*	*9720*	205	374	168	15708	265	494	192	23712	*333*	*630*	*216*	*34020*
4	185	306	208	15912	241	418	240	25080	305	546	272	37128	377	690	304	52440
5	221	342	280	23940	281	462	320	36960	349	598	360	53820	*425*	*750*	*400*	*75000*
6	*261*	*378*	*360*	*34020*	325	506	408	51612	397	650	456	74100	*477*	*810*	*504*	*102060*

$a_{n,k} = (k+2n-1)^2 + k^2$, $\Delta_{n,k} = 2k(2n-1)(k+2n-1)(2k+2n-1)$,

(Group I): $b_{n,k} = 2(2n-1)(2k+2n-1)$, (Group II): $b_{n,k} = 4k(k+2n-1)$.

For a fixed n (or k)

$$f_j = 3f_{j-1} - 3f_{j-2} + f_{j-3} \quad (f = a, b) \qquad j = k \text{ (or } n).$$

$$\Delta_j = 4\Delta_{j-1} - 6\Delta_{j-2} + 4\Delta_{j-3} - \Delta_{j-4} \qquad j = k \text{ (or } n).$$

Next, consider what kind of non-primitive members are included in our algorithm. By enlarging Table 3 we have picked out those non-primitive HeT's whose a and b values are formally derived from (2.1) and (2.2). The results are given in Table 4, where those (n, k) codes are given which yield non-primitive HeT's. Those cofactors are found to be the squares of $2n-1$ and its factors. This can easily be verified by putting $k=2n-1$ into either (2.1) or (2.2). Namely, by this substitution all the a, b, and D values are found to become the multiples of $(2n-1)^2$.

Table 4 (n, k) codes yielding non-primitive HeT's.

$n \backslash k$	1	2	3	4	5	6	cofactors		
1			none					none	
2	3	6	9	12	15	18	3^2		
3	5	10	15	20	25	30	5^2		
4	7	14	21	28	35	42	5^2		
5	3	6	9	12	15	18	3^2	9^2	
6	11	22	33	44	55	66	11^2		
7	13	26	39	52	65	78	13^2		
8	3	5	6	9	10	12	3^2	5^2	15^2
9	17	34	51	68	85	102	17^2		

3. Additional accounts

3.1 Beautiful structure of the family of isosceles HeT's

Advantage of the present (n, k) codes over Carmichael's formulation can be demonstrated visually in the following way. In Fig. 2 the values of edge lengths of isosceles HeT's as given in Table 3 are plotted with a and b, respectively, as abscissa and ordinate, where Groups I and II are connected, respectively, by dashed and solid lines. Non-primitive isosceles HeT's are marked with open circles, and it is clear that inclusion of them is essential for constructing beautiful and stable mathematical structure composed of slant straight lines and a group of non-crossing curves. Further, it is to be noticed that both the abscissas and ordinates of consecutive points situated on all the right-ascending lines and curves obey the same recursive relation (2.3) by including those non-primitive members. Each point on broken (or solid) line can always find another point on solid (or broken) line somewhere along the line passing through it in parallel with the ordinate axis. The arrows at the lower part of the graph indicate that there are more than two pairs of points with this property along the line in parallel with the ordinate axis.

3.2 Failure of Carmichael's scheme

As already mentioned in Introduction, application of Carmichael's formulation to isosceles HeT's does not work well for systematic understanding of their structure. Concrete examples are shown in Table 5, where a and c values of (1.3) are tabulated. More than half of the possible combinations of m and h do not give primitive HeT's but their multiples. Only the members of Group I can be generated as coprime HeT's, while all the Group II members are generated as some multiples in an irregular way. Further, from Table 5 it is quite difficult to select out any twin pairs with the same a and D values, which were systematically derived in Table 3 with the same (n, k) codes.

If some systematic relation is to be sought, sub-groups of Group I with the same n value in Table 3 are found to be lined up along the downward slope originating from the entry with $m=2n$ and $h=1$. On the

Fig. 2 Plot of edges (a, b) of isosceles Heronian triangles (a, a, b) of Groups I (broken) and II (solid).

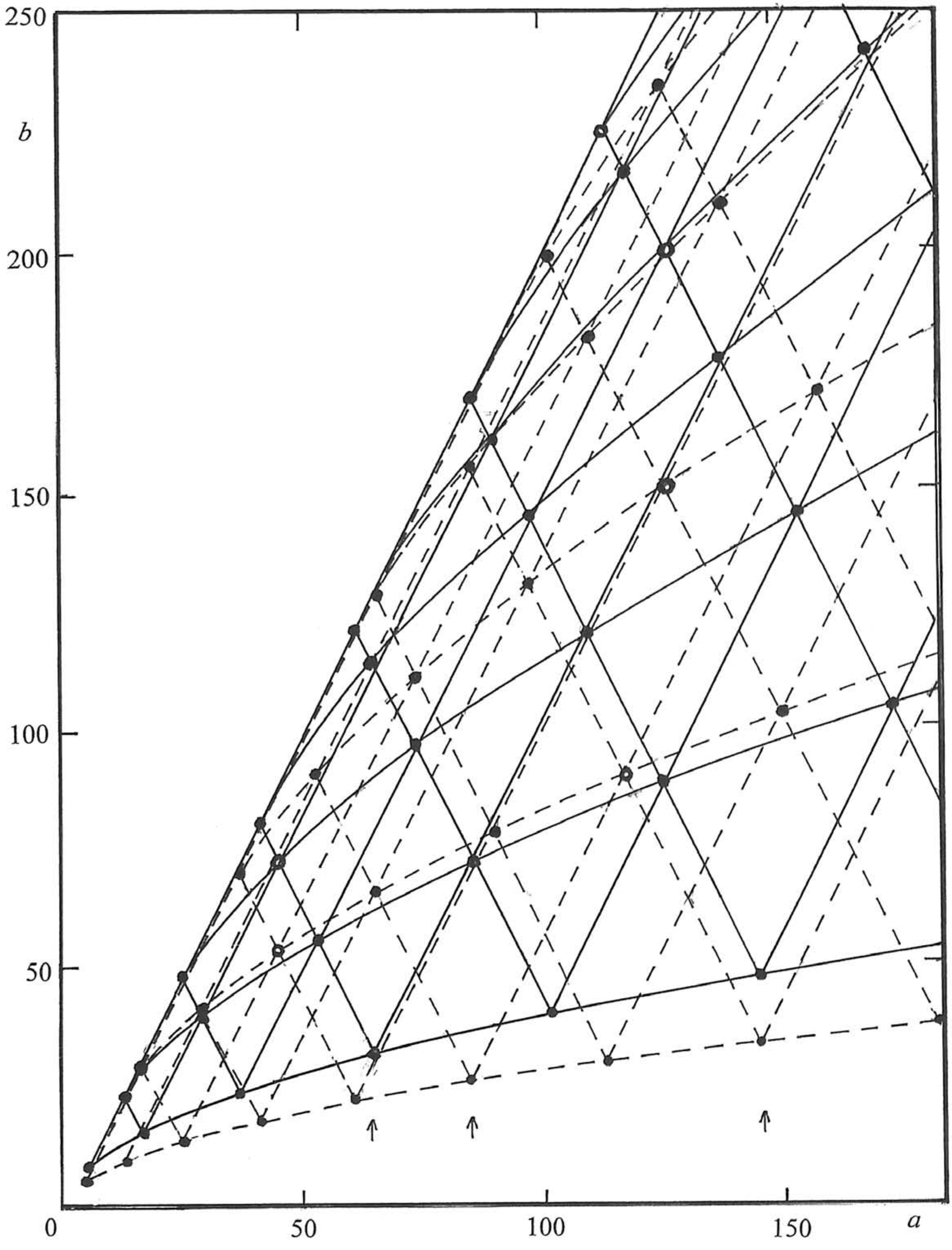


Table 5 Carmichael's scheme for generating piHeT's.

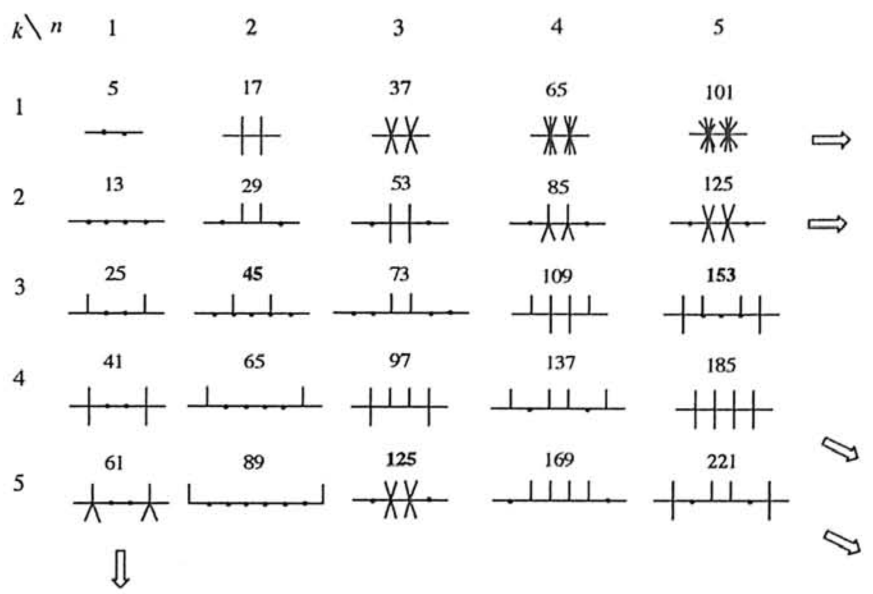
$m \backslash h$	1	2	3	4	5	6	7
2	(5, 6)						
3	2(5, 8)	(13, 10)					
4	(17, 30)	4(5, 6)	(25, 14)				
5	2(13, 24)	(29, 42)	2(17, 16)	(41, 18)			
6	(37, 70)	8(5, 8)	9(5, 6)	4(13, 10)	(61, 22)		
7	2(25, 48)	(53, 90)	2(29, 40)	(65, 56)	2(37, 24)	(85, 26)	
8	(65, 126)	4(17, 30)	(73, 110)	16(5, 6)	(89, 78)	4(25, 14)	(113, 30)

other hand, the double of Group II entries with the same n value in Table 3 can be found vertically but alternately from the entry with m=2n+1 and h=2n–1 in Table 5. However, these rules were obtained from the knowledge of the systematic structure of Table 3.

3.3 Relation of isosceles HeT's with *Z*-index

In 1971 the present author defined the topological index, or TopIx, Z, for characterizing the topological features of a graph,[11,12] and further applied it successfully for elucidating many problems in elementary algebra and geometry.[13-16] The Z-graph is a graph whose TopIx is equal to some prescribed value of a positive integer. Here some observation of the Z-graphs for the edges of isosceles HeT's will be introduced. It has been proved that the hypotenuse of a primitive Pythagorean triangle is expressed by a sum of two squares and then its Z-graph is a caterpillar graph of mirror symmetry, or SymCat.[17] Thus the Z-graph of a edge of a piHeT can be represented by a SymCat. In Table 6 are shown the SymCats for the a edges of smaller members of isosceles HeT's.

Table 6 Z-Graphs of smaller isosceles HeT's revealing interesting features.



At this stage no systematic analysis has been performed, but as the arrows in Table 6 suggest, we can enjoy several series of growing SymCats as the *Z*-graphs of a edges of piHeT's. Namely, the SymCats of $a_{1,k}$ line up along the first column of Table 6, which regularly grow according to a certain rule. Similarly, the SymCats of $a_{n,1}$ and $a_{n,2}$ are regularly growing along the first and second rows, respectively. On the other hand, no regularly growing Symcats can be found corresponding to $a_{n,3}$ series. This phenomenon might be possibly due to the fact that non-primitive members which are marked in bold face are contained in this series. The SymCats of the next series $a_{n,3}$ are found to be divided into two subgroups whose SymCats independently grow regularly.

Further, the SymCats of the series, $a_{2,1}$, $a_{3,2}$, $a_{4,3}$, *etc*., indicated by a slant arrow might form a regularly growing group, and similarly another series, $a_{2,2}$, $a_{3,3}$, $a_{4,4}$, *etc*., seem to form a group. These facts suggest us that TopIx might play an important role in the investigation of other types of members of HeT's. Study along this line is in progress.

References

1) E. W. Weisstein, CRC Concise Encyclopedia of Mathematics, Chapman & Hall/CRC, Boca Raton, FL (2003).
2) http://en.wikipedia.org/wiki/Heronian/triangle
3) http://mathworld.wolfram.com/Heronian Triangle.html
4) H. Schubert, Festgabe 48 Versammlung der Philologen und Schulmänner zu Hamburg, Leibzig (1905), pp. 1.
5) R. D. Carmichael, Diophantine Analysis, Wiley, NY (1915), pp. 11.
6) R. H. Buchholz, PhD Thesis, Univ. of Newcastle (1989).
7) F. J. M. Bahning, *Math. Centrum Amsterdam Afd. Zuivere Wisk*, **ZW-011** (1963) 37.
8) A. Hall, *Math. Gaz.*, **54** (1970) 377.
9) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **59** (2009) (2) 1.
10) A. H. Beiler, Recreations in the Theory of Numbers, Dover, NY (1966).
11) H. Hosoya, *Bull. Chem. Soc. Jpn.*, **44** (1971) 2332-2339.
12) H. Hosoya, *Fibonacci Quart.*, **11** (1973) 255-266.
13) H. Hosoya, *Natural Sci. Rept. Ochanomizu Univ.*, **57** (2006) (2) 35.
14) H. Hosoya, *Croat. Chem. Acta*, **80** (2007) 239.
15) H. Hosoya, *Natural Sci. Rept. Ochanomizu Univ.*, **58** (2007) (1) 15.
16) H. Hosoya, *Natural Sci. Rept. Ochanomizu Univ.*, **59** (2009) (2) 15.
17) H. Hosoya, *Natural Sci. Rept. Ochanomizu Univ.*, **60** (2009) (1) 15.