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Critical Exponent of Blowup for Heat Equation with Logarithmic Nonlinearity on Paraboloidal Domain

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Abstract

We present explicit critical exponent of blowup for logarithmic version of nonlinear heat equation of Fujita type on generalized paraboloidal domains. This gives a concrete example to our former result on the existence of critical exponent for logarithmic nonlinearity.

1 Introduction

In [1], Fujita showed for the first time the so called Fujita phenomenon for the initial value problem of the nonlinear heat equation on $\boldsymbol{R}^N$

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \qquad u(0,x) = u_0(x).$$

Namely, there exists a critical value $p^* = 1 + 2/N$ such that for $1 < p < p^*$ the solution for any non-negative non-zero initial data blows up in finite time, whereas for $p > p^*$ the solution is time-global for sufficiently small non-negative initial data. This phenomenon was extended to many directions. Among others we mention Levine-Meier [3], who showed that in a conical domain D with spherical profile Ω the Fujita-type phenomenon occurs for the solutions of the homogeneous Dirichlet boundary condition with the critical exponent $p^* = 1+2/(N+\gamma)$, where γ is the positive root of the quadratic equation $\gamma^2 + (N-2)\gamma - \omega = 0$ and ω is the first Dirichlet eigenvalue of the (positive) Laplace-Beltrami operator of the spherical domain Ω. In [2], we introduced logarithmic nonlinearity of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + u(\operatorname{logg} u)^p \quad \text{in } \{t>0\} \times D, \\ &u(0,x) = u_0(x) \quad \text{in } D, \qquad u|_{\partial D} = 0, \end{aligned} \tag{1.1}$$

where the new function symbol $\operatorname{logg} u$ in the nonlinear term was defined as follows:

$$\operatorname{logg} u = \begin{cases} \log u + 1 & \text{for } u \geq 1, \\ \dfrac{1}{1-\log u} & \text{for } 0 < u < 1, \end{cases} \tag{1.2}$$

with the convention $\operatorname{logg} 0 = 0$. We proved there the abstract existence of critical exponent of blowup p^* in this setting without specifying the domain D. Now we consider the above equation on generalized paraboloidal domains of the form

$$x_N > |x'|^q + C, \tag{1.3}$$

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with some $q > 1$, and show that

$$p^* = \frac{2}{q-1}. \tag{1.4}$$

Note that for Fujita's original nonlinearity, these domains as well as cylindrical ones always have the critical exponent $p^* = 1$, namely, the power nonlinearity cannot distinguish them through the Fujita phenomenon. Our approach is an attempt to refine this. Especially, for the true parabola $q = 2$, the value $p^* = 2$ seems fine. But in general, our result is somewhat unexpected, because it means, contrary to our anticipation, that the logarithmic nonlinearity can distinguish only generalized paraboloidal domains of the opening $q < 3$ in our setting, because $p \leq 1$ is meaningless. This is, however, not a true restriction. Simply, we should have chosen another form of nonlinearity on $u > 1$ to assure the blowup even for $0 < p \leq 1$. See Remark at the end of this article in this respect.

In our previous paper, we prepared the following abstract result in order to determine the critical exponent. (See [2], Theorem 2.1; here it is presented in a little modified form.)

Theorem 1.1 1) *Assume that there exists a non-trivial super-solution W of*

$$u_t = \Delta u, \qquad u\big|_{\partial D} = 0, \tag{1.5}$$

which satisfies, for some $\varepsilon > 0$ and $C > 0$,

$$t^{1/(p+1)+\varepsilon} (\mathrm{logg}\, \|W(t,\cdot)\|_\infty) \leq C.$$

Then a global solution of (1.1) *exists.*

2) *Assume that for some non-trivial sub-solution W of* (1.5) *we have*

$$\overline{\lim_{t\to\infty}}\, t^{1/(p+1)} (\mathrm{logg}\, \|W(t,\cdot)\|_\infty) = \infty.$$

Then every solution of (1.1) *blows up in finite time.*

As was remarked in [2], this assures the abstract existence of the critical exponent p^* for any domain (though of course we are interested in the domains for which p^* is strictly between 1 and ∞). It is difficult, however, to apply this characterization to a domain like (1.3), because of the lack of homogeneity (or self-similarity) which played an essential role in the case of the conical domain. We therefore employ a different strategy: To deduce an estimate from above of the critical exponent, we construct a super-solution of the nonlinear equation directly, by means of an elementary calculation based on the coordinate transform bringing our domain to a half-cylinder. For an estimate from below, we employ the following analogy of one from [3] as is just used there:

Lemma 1.2 *Assume that there exists $\lambda > 0$ and a non-negative function $\psi(x)$ on D which satisfies*

$$\Delta\psi + \lambda\psi > 0, \qquad \psi\big|_{\partial D} = 0, \qquad \text{and} \qquad C = \int_D \psi(x)dx < \infty. \tag{1.6}$$

Set

$$G_0 := \frac{1}{C}\int_D u_0(x)\,\psi(x)dx. \tag{1.7}$$

If u is a solution of (1.1) *which exists for $0 < t < T$, then we have*

$$T \leq \int_{G_0}^\infty \frac{ds}{s(\mathrm{logg}\, s)^p - \lambda s}.$$

Proof Set

$$F(t) := \frac{1}{C}\int_D u(t,x)\,\psi(x)dx.$$

Then we have, in view of the boundary condition,

$$\begin{aligned}
F' &= \frac{1}{C}\int_D u_t(t,x)\,\psi(x)dx \\
&= \frac{1}{C}\int_D \{\Delta u + u(\operatorname{logg} u)^p\}\psi(x)dx \\
&= \frac{1}{C}\int_{\partial D} \frac{\partial u}{\partial n}\psi(x)dS - \frac{1}{C}\int_D \nabla u\cdot\nabla\psi(x)dx + \frac{1}{C}\int_D u(\operatorname{logg} u)^p\psi dx \\
&= -\frac{1}{C}\int_{\partial D} u\frac{\partial\psi}{\partial n}dS + \frac{1}{C}\int_D u\,\Delta\psi dx + \frac{1}{C}\int_D u(\operatorname{logg} u)^p\psi dx \\
&\ge -\frac{\lambda}{C}\int_D u\,\psi dx + \frac{1}{C}\int_D u\,\psi dx \,\operatorname{logg}\left\{\frac{1}{C}\int_D u\,\psi dx\right\}^p \\
&= -\lambda F + F(\operatorname{logg} F)^p.
\end{aligned}$$

Here in the last inequality we employed Jensen's inequality which holds in view of $\frac{1}{C}\int_D \psi dx = 1$ and the convexity of $u(\operatorname{logg} u)^p$ proved in [2]. Thus we obtain

$$T \le \int_0^T \frac{F'dt}{F(\operatorname{logg} F)^p - \lambda F} \le \int_{G_0}^\infty \frac{ds}{s(\operatorname{logg} s)^p - \lambda s}.$$

Corollary 1.3 *Let ψ be as in Lemma 1.2. If $p > 1$ and*

$$\frac{\int_D u_0\psi dx}{\int_D \psi dx} > e^{1-\lambda^{-1/p}} \qquad \text{for some } 0 < \lambda < 1, \tag{1.8}$$

then the solution blows up in finite time.

This follows immediately by solving $(\operatorname{logg} s)^p = \dfrac{1}{(1-\log s)^p} = \lambda$ for $s < 1$. (Note that by the comparison theorem (which holds just as for the usual power nonlinearity), we may replace u_0 by a smaller one, thereby a smaller G_0 for which $\operatorname{logg} G_0$ has the above form, to obtain the same conclusion.)

2 Reduction to a half-cylinder

We first discuss a change of variables which brings the paraboloidal domain (1.3) to a cylinder

$$D_y := \{y = (y', y_N)\,;\ |y'| < 1,\ y_N > 0\}. \tag{2.1}$$

It is rather difficult to find a transformation which exactly maps (1.3) to this. Note, however, that our original equation is translation invariant, hence, the critical exponent of blowup is the common value for the family of domains $D_C := \{x_N > |x'|^q + C\}$. Thus by the comparison theorem, any family $\widetilde{D}_C$ cofinal with this has the same critical exponent. Here, cofinal means that for any C there exists C' such that $\widetilde{D}'_C \subset D_C$, and vice versa. In particular, it suffices that our coordinate transform is valid for $D \cap \{x_N > R\}$ for some $R > 0$.

Thus it is enough to employ the following transformation although it is not diffeomorphic at the origin:

$$y' = \frac{x'}{x_N^{1/q}}, \quad y_N = \sqrt{\frac{|x'|^2}{q} + x_N^2}. \tag{2.2}$$

The new coordinate system is so chosen that it is an asymptotically orthogonal curvilinear one, and especially y' and y_N are really orthogonal, as will be seen from the calculation below. With this simplification, it is still difficult to represent x conversely by y explicitly. We therefore describe the transformation formulae for the partial derivatives with x remaining in the coefficients. We have

$$
\begin{aligned}
\frac{\partial}{\partial x_N} &= \frac{x_N}{\sqrt{\frac{|x'|^2}{q}+x_N^2}}\frac{\partial}{\partial y_N} - \frac{1}{qx_N^{(q+1)/q}}\sum_{j=1}^{N-1} x_j\frac{\partial}{\partial y_j},\\
\frac{\partial^2}{\partial x_N^2} &= \frac{x_N^2}{\frac{|x'|^2}{q}+x_N^2}\frac{\partial^2}{\partial y_N^2}\\
&\quad - \frac{2x_N}{qx_N^{(q+1)/q}\sqrt{\frac{|x'|^2}{q}+x_N^2}}\sum_{j=1}^{N-1} x_j\frac{\partial^2}{\partial y_j\partial y_N} + \frac{1}{q^2x_N^{2(q+1)/q}}\sum_{j,k=1}^{N-1} x_jx_k\frac{\partial^2}{\partial y_j\partial y_k}\\
&\quad + \Big(\frac{1}{\sqrt{\frac{|x'|^2}{q}+x_N^2}} - \frac{x_N^2}{\sqrt{\frac{|x'|^2}{q}+x_N^2}^3}\Big)\frac{\partial}{\partial y_N} + \frac{q+1}{q^2x_N^{(2q+1)/q}}\sum_{j=1}^{N-1} x_j\frac{\partial}{\partial y_j}\\
&= \frac{x_N^2}{\frac{|x'|^2}{q}+x_N^2}\frac{\partial^2}{\partial y_N^2} - \frac{2}{qx_N^{1/q}\sqrt{\frac{|x'|^2}{q}+x_N^2}}\sum_{j=1}^{N-1} x_j\frac{\partial^2}{\partial y_j\partial y_N} + \frac{1}{q^2x_N^2}\sum_{j,k=1}^{N-1} y_jy_k\frac{\partial^2}{\partial y_j\partial y_k}\\
&\quad + \frac{|x'|^2}{q\sqrt{\frac{|x'|^2}{q}+x_N^2}^3}\frac{\partial}{\partial y_N} + \frac{q+1}{q^2x_N^2}\sum_{j=1}^{N-1} y_j\frac{\partial}{\partial y_j},\\
\frac{\partial}{\partial x_j} &= \frac{1}{x_N^{1/q}}\frac{\partial}{\partial y_j} + \frac{x_j}{q\sqrt{\frac{|x'|^2}{q}+x_N^2}}\frac{\partial}{\partial y_N}, \quad j=1,\dots,N-1,\\
\frac{\partial^2}{\partial x_j^2} &= \frac{1}{x_N^{2/q}}\frac{\partial^2}{\partial y_j^2} + \frac{2x_j}{qx_N^{1/q}\sqrt{\frac{|x'|^2}{q}+x_N^2}}\frac{\partial^2}{\partial y_j\partial y_N} + \frac{x_j^2}{q^2\big(\frac{|x'|^2}{q}+x_N^2\big)}\frac{\partial^2}{\partial y_N^2}\\
&\quad + \Big(\frac{1}{q\sqrt{\frac{|x'|^2}{q}+x_N^2}} - \frac{x_j^2}{q^2\sqrt{\frac{|x'|^2}{q}+x_N^2}^3}\Big)\frac{\partial}{\partial y_N}, \quad j=1,\dots,N-1.
\end{aligned}
\tag{2.3}
$$

Hence,

$$
\begin{aligned}
\Delta_x = L_y :=& \frac{\frac{|x'|^2}{q^2}+x_N^2}{\frac{|x'|^2}{q}+x_N^2}\frac{\partial^2}{\partial y_N^2} + \frac{1}{x_N^{2/q}}\Delta_{y'} + \frac{1}{q^2x_N^2}\sum_{j,k=1}^{N-1} y_jy_k\frac{\partial^2}{\partial y_j\partial y_k}\\
&+ \Big(\frac{N-1}{q\sqrt{\frac{|x'|^2}{q}+x_N^2}} + \frac{(q-1)|x'|^2}{q^2\sqrt{\frac{|x'|^2}{q}+x_N^2}^3}\Big)\frac{\partial}{\partial y_N} + \frac{q+1}{q^2x_N^2}\sum_{j=1}^{N-1} y_j\frac{\partial}{\partial y_j}.
\end{aligned}
\tag{2.4}
$$

Thus our problem is to find the asymptotic behavior of the solution of the Dirichlet problem for the evolution equation with this asymptotically one dimensional degenerate elliptic right-hand side:

$$
\frac{\partial u}{\partial t} = Lu \quad \text{in } \{t>0\}\times D_y, \qquad u(t,y)\big|_{y_N=0} = u(t,y)\big|_{|y'|=1} = 0. \tag{2.5}
$$

Although x_N cannot be explicitly represented in terms of y, we have the following asymptotic relations for large x_N, in view of the assumption $q>1$:

$$
\begin{aligned}
&y_N = x_N\Big\{1+O\Big(\frac{|x'|^2}{x_N}\Big)\Big\} = x_N + O(x_N^{2/q}) \sim x_N, \quad \text{hence} \quad x_N = y_N + O(y_N^{2/q}),\\
&x' \sim y_N^{1/q}y', \quad dx = y_N^{(N-1)/q}dy.
\end{aligned}
\tag{2.6}
$$

We have also

$$
\frac{1}{q} \le \frac{\frac{|x'|^2}{q^2}+x_N^2}{\frac{|x'|^2}{q}+x_N^2} \le 1, \quad \text{and} \quad \frac{\frac{|x'|^2}{q^2}+x_N^2}{\frac{|x'|^2}{q}+x_N^2} = 1 - \frac{q-1}{q^2}\frac{|y'|^2}{y_N^{2(q-1)/q}} + O\Big(\frac{1}{y_N^{4(q-1)/q}}\Big) \sim 1. \tag{2.7}
$$

Now we seek for a concrete super-solutions of (1.1) described in the new coordinates y, or apply Corollary 1.3 therein. In verifying the necessary inequalities for them, we can replace the region (2.1) by another with $y_N > R$ with any $R > 0$ which is sufficiently large, and simultaneously the original domain D by the inverse image of this, since they are cofinal with the original family of translated generalized paraboloids.

In principle, we seek for a positive super-solution of (2.5) of the form

$$u(t,y) = \varphi(y')v(t,y_N)$$

or a positive solution ψ of $L\psi + \lambda\psi \geq 0$ of the form

$$\psi(y,\lambda) = \varphi(y')v(y_N,\lambda),$$

where $\varphi(y')$ is the positive eigenfunction of $\Delta_{y'}$ for the domain $|y'| < 1$ corresponding to the first eigenvalue $-\omega$ with respect to the homogeneous Dirichlet condition. For such u with separated variables, the essential part of the expression Lu becomes, with the simpler notation $r = y_N$,

$$\frac{\partial^2 u}{\partial r^2} + \frac{N-1}{q}\frac{1}{r}\frac{\partial u}{\partial r} - \frac{\omega}{r^{2/q}}u. \tag{2.8}$$

Thus our essential task is to consider the following spatially one dimensional heat equation with variable coefficients:

$$u_t = \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{q}\frac{1}{r}\frac{\partial u}{\partial r} - \frac{\omega}{r^{2/q}}u, \quad \text{on } r > R, \tag{2.9}$$

(plus the same nonlinearity) with the Dirichlet condition $u(t,R) = 0$. Each term of the precise transformed equation has a corresponding term in (2.9) with which the coefficient is of difference $\leq \varepsilon$ for any prescribed $\varepsilon > 0$ if $R > 0$ is chosen correspondingly large enough.

The extraction of this main part is, however, not fully compatible with the boundary condition. Thus, for example, among neglected terms there are

$$\frac{1}{q^2x_N^2}\sum_{j,k=1}^{N-1} y_j y_k \frac{\partial^2}{\partial y_j \partial y_k}, \qquad \frac{q+1}{q^2x_N^2}\sum_{j=1}^{N-1} y_j \frac{\partial}{\partial y_j}, \tag{2.10}$$

which, applied to $\varphi(y')$, yield functions non-vanishing at $|y'| = 1$, complicating the comparison there. Therefore we need a delicate modification in the argument, which will be explained when it becomes necessary. Recall here the fact that the first eigenfunction $\varphi(y')$ is radially symmetric in $\rho = |y'|$, and its explicit form is known as:

$$\varphi(y') = c\rho^{-(N-3)/2}J_{(N-3)/2}(\sqrt{\omega}\rho), \tag{2.11}$$

where $\sqrt{\omega}$ is the first positive zero of the Bessel function $J_{(N-3)/2}(z)$, and c is a normalizing constant. Writing $\varphi(y') = \varphi(\rho)$ with abuse of notation, we have

$$\sum_{j,k=1}^{N-1} y_j y_k \frac{\partial^2}{\partial y_j \partial y_k}\varphi(y') = \rho^2\frac{d^2}{d\rho^2}\varphi(\rho) \leq 0, \qquad \sum_{j=1}^{N-1} y_j \frac{\partial}{\partial y_j}\varphi(y') = \rho\frac{d}{d\rho}\varphi(\rho) \leq 0, \tag{2.12}$$

for $0 \leq \rho \leq 1$ in consistent with the generally known properties of the first eigenfunction.

3 Non-blowup case

Now we seek for a super-solution. First, we shall show that our rather primitive approach is efficient enough by reproving the estimate from above of the critical exponent for the power nonlinearity given by [3]. For that, recall that the transformation by the polar coordinates brings the equation

$$u_t = \Delta u + u^p \tag{3.1}$$

exactly to

$$u_t = u_{rr} + \frac{N-1}{r}u_r + \frac{\Delta' u}{r^2}$$

on $r > 0$, where Δ' is the (negative) spherical Laplacian in the coordinates y'. (We are using notation compatible with our calculation in the preceding section.)

Proposition 3.1 *Let ω and $\varphi(y')$ be the first eigenvalue and the associated eigenfunction for the Laplacian $-\Delta'$ on $D' = D \cap S^{N-1}$. Set*

$$u(t, y', r) = ct^{-a}r^m \exp(-\frac{r^2}{4t})\varphi(y'). \tag{3.2}$$

Then for a suitable choice of a, m, c this serves as a super-solution of (3.1) provided that $p > 1 + \frac{2}{N+\gamma}$, where γ is the positive root of $\gamma(\gamma-1) + \gamma(N-1) = \omega$. Hence the critical exponent of blowup $p^* \leq 1 + \frac{2}{N+\gamma}$.

Proof We have

$$u_r = \Big(mr^{m-1} - \frac{r^{m+1}}{2t}\Big)ct^{-a}\exp(-\frac{r^2}{4t})\varphi(y'),$$
$$u_{rr} = \Big(m(m-1)r^{m-2} - \frac{2m+1}{2t}r^m + \frac{r^{m+2}}{4t^2}\Big)ct^{-a}\exp(-\frac{r^2}{4t})\varphi(y').$$

Hence,

$$u_t - \Delta u - u^p = u_t - u_{rr} - \frac{N-1}{r}u_r - \frac{1}{r^2}\Delta' u - u^p$$
$$= \Big(-\frac{a}{t} + \frac{r^2}{4t^2}\Big)u - \Big(\frac{m(m-1)}{r^2} - \frac{2m+1}{2t} + \frac{r^2}{4t^2}\Big)u - \Big(\frac{m(N-1)}{r^2} - \frac{N-1}{2t}\Big)u + \frac{\omega}{r^2}u - u^p$$
$$= \Big\{\Big(m + \frac{N}{2} - a\Big)\frac{1}{t} + \Big(\omega - m(m-1) - m(N-1)\Big)\frac{1}{r^2}\Big\}u - u^p.$$

Here we normalize $\varphi(y')$ so that $\varphi(y') \leq 1$. (This is equivalent with choosing c smaller and put a part to φ.) Then in view of $p > 1$, we have $\varphi(y') \geq \varphi(y')^p$. Thus in order that the above quantity be non-negative, it suffices to have the inequality

$$\Big\{\Big(m + \frac{N}{2} - a\Big)\frac{1}{t} + \Big(\omega - m(m-1) - m(N-1)\Big)\frac{1}{r^2}\Big\} \geq c^{p-1}t^{-a(p-1)}r^{m(p-1)}e^{-(p-1)r^2/4t},$$

or equivalently,

$$\Big(m + \frac{N}{2} - a\Big)\frac{r^2}{t} + \Big(\omega - m(m-1) - m(N-1)\Big) \geq c^{p-1}t^{-a(p-1)}r^{m(p-1)+2}e^{-(p-1)r^2/4t}.$$

If we choose here a such that

$$m(p-1) + 2 = 2a(p-1), \qquad \text{that is,} \qquad a = \frac{m}{2} + \frac{1}{p-1},$$

then with a new parameter $s = r^2/t$ the above inequality is reduced to

$$\Big(m + \frac{N}{2} - a\Big)s + \Big(\omega - m(m-1) - m(N-1)\Big) \geq c^{p-1}s^{(m/2)(p-1)+1}e^{-(p-1)s/4}.$$

Since the right-hand side has finite maximum for $s \geq 0$, we can make it as small as we wish by choosing c small. Thus in order that this inequality holds, it is enough that the left-hand side is above a positive constant for all $s \geq 0$. To assure this, we can e.g. choose $m + \frac{N}{2} - a = 0$ and $\omega - m(m-1) - m(N-1) > 0$. The first condition implies

$$m = a - \frac{N}{2} = \frac{m}{2} - \frac{N}{2} + \frac{1}{p-1}, \qquad \text{or} \qquad p = 1 + \frac{2}{N+m},$$

and the second implies $m < \gamma$. Thus we have proved that $p > 1 + \dfrac{2}{N+\gamma}$ implies non-blowup. QED.

Now we move to our logarithmic case. The above proof suggests that when the term $\dfrac{\omega}{r^2}$ is replaced by $\dfrac{\omega}{r^{2/q}}$ with $q > 1$, then a function of the form (3.2) would never serve as a super-solution. Thus we consider a function modified as follows:

$$u(t, y', r) = c\exp\big(-at^\lambda - \frac{r^2}{bt}\big)\varphi(y'), \tag{3.3}$$

where we assume $0 < \lambda < 1$. Note that to construct a global super-solution, the boundary value may be non-negative and need not be strictly zero. Note also that when treating a super-solution, in view of (2.12), the corresponding terms, which appear with additional negative sign below, can be simply removed although they cannot be rebound by u. We have,

$$u_r = -\frac{2r}{bt}u, \qquad u_{rr} = \Big(-\frac{2}{bt} + \frac{4r^2}{b^2t^2}\Big)u.$$

Thus it suffices to see the non-negativity of the following:

$$\begin{aligned}&u_t - u_{rr} - \frac{N-1}{qr}u_r - \frac{1}{r^{2/q}}\Delta' u - u(\operatorname{logg} u)^p\\ &= \Big(-\frac{a\lambda}{t^{1-\lambda}} + \frac{r^2}{bt^2}\Big)u - \Big(-\frac{2}{bt} + \frac{4r^2}{b^2t^2}\Big)u + \frac{2(N-1)}{bqt}u + \frac{\omega}{r^{2/q}}u - u(\operatorname{logg} u)^p.\end{aligned}$$

Recall that the residual terms produced by the coordinate transformation of the preceding section can all be absorbed to either of the terms here with a modification of size $\le \varepsilon$ in the corresponding coefficient. Thus we neglect this in the sequel by implicitly admitting these coefficients changed appropriately. This will be justified in the course of our calculation below. Now choosing c further sufficiently small, we can assume that the function (3.3) is less than 1 everywhere, hence the nonlinear term is $u(\operatorname{logg} u)^p = \frac{u}{(1-\log u)^p}$. Thus what we have to prove now is the following inequality:

$$\begin{aligned}&\frac{\omega}{r^{2/q}} + \frac{2}{b}\Big(\frac{N-1}{q} + 1\Big)\frac{1}{t} + \frac{b-4}{b^2}\frac{r^2}{t^2} - \frac{a\lambda}{t^{1-\lambda}}\\ &\ge \frac{1}{\{1 - \log c + at^\lambda + \frac{r^2}{bt}\}^p}.\end{aligned} \tag{3.4}$$

Unlike [3], we choose $b > 4$ in order that the coefficient of $\frac{r^2}{t^2}$ becomes positive. To simplify the notation, we put

$$\beta := \frac{b-4}{b^2}.$$

First of all we have to dominate the unique negative term $-\frac{a\lambda}{t^{1-\lambda}}$ by another positive term. In the sequel we can assume that $t \ge 1$, by translating the origin of the time. (Alternatively, for $t < 1$ we can dominate the negative term simply by the term $\frac{2}{b}\big(\frac{N-1}{q} + 1\big)\frac{1}{t}$, by choosing the constant a smaller than this coefficient.)

We divide the (t, r)-space according to the criterion which term in the left- resp. right-hand side is dominant. First we consider the left-hand side.

(1) In the region $\frac{\omega}{r^{2/q}} \ge \frac{2a}{t^{1-\lambda}}$, that is, $r \le \big(\frac{\omega}{2a}\big)^{q/2} t^{(1-\lambda)q/2}$, the negative term is dominated by $\frac{\omega}{2r^{2/q}}$ whatever λ may be. In this case the main term in the left-hand side may be either (1a) $\frac{\omega}{r^{2/q}}$ when $\frac{\omega}{r^{2/q}} \ge \beta\frac{r^2}{t^2}$, that is, when $r \le \big(\frac{\omega}{\beta}\big)^{q/2(q+1)} t^{q/(q+1)}$, or (1b) $\beta\frac{r^2}{t^2}$ when $r \ge \big(\frac{\omega}{\beta}\big)^{q/2(q+1)} t^{q/(q+1)}$. We do not distinguish, however, these two subcases, because as will be shown below, the term $\dfrac{\omega}{2r^{2/q}}$ suffices to dominate the right-hand side in this case throughout.

(2) In the complementary region of (1), that is, $r \ge \big(\frac{\omega}{2a}\big)^{q/2} t^{(1-\lambda)q/2}$, we have to dominate the negative term by $\beta\frac{r^2}{t^2}$, namely, $\beta\frac{r^2}{t^2} \ge \frac{2a}{t^{1-\lambda}}$, that is, $r \ge \sqrt{2a/\beta}\, t^{(1+\lambda)/2}$. Hence, in order

to cover all the (t, r)-space we have to choose λ so that the first assumption implies the second, namely,

$$\frac{(1-\lambda)q}{2} \geq \frac{1+\lambda}{2}, \qquad \text{that is,} \qquad \lambda \leq \frac{q-1}{q+1}, \tag{3.5}$$

and that a is small enough such that $\left(\frac{\omega}{2a}\right)^{q/2} > \sqrt{2a/\beta}$.

Next we classify the right-hand side.

(i) In case $at^\lambda \geq \frac{r^2}{bt}$, that is, $r \leq \sqrt{ab}t^{(1+\lambda)/2}$, the main factor of the right-hand side is $\frac{1}{a^p t^{p\lambda}}$.

(ii) In the complementary case $r \geq \sqrt{ab}t^{(1+\lambda)/2}$, the main factor of the right-hand side is $b^p \frac{t^p}{r^{2p}}$.

Now we examine the inequality (3.4) for each combination. In case (1) vs (i), we should have,

$$\frac{\omega}{2r^{2/q}} > \frac{1}{a^p t^{p\lambda}} \qquad \text{that is,} \qquad r < \left(\frac{a^p \omega}{2}\right)^{q/2} t^{pq\lambda/2}.$$

Taking (3.5) into account, in order that (1) and (i) imply this, it suffices that

$$\frac{pq\lambda}{2} > \min\left\{\frac{1+\lambda}{2}, \frac{(1-\lambda)q}{2}\right\} = \frac{1+\lambda}{2}, \qquad \text{that is,} \qquad \lambda > \frac{1}{pq-1}. \tag{3.6}$$

In fact, this will ensure the desired inequality for sufficiently large t. For the remaining region of t, the term $-\log c$ in the denominator of the right-hand side of (3.4) will achieve a necessary estimate for sufficiently small choice of c (whatever the constant a and ω may be).

Next consider the case (1) vs (ii). We have

$$\sqrt{ab}t^{(1+\lambda)/2} \leq r \leq \left(\frac{\omega}{2a}\right)^{q/2} t^{(1-\lambda)q/2}.$$

Under this condition we should have

$$\frac{\omega}{2r^{2/q}} > b^p \frac{t^p}{r^{2p}} \qquad \text{that is,} \qquad r > \left(\frac{2b^p}{\omega}\right)^{q/2(pq-1)} t^{pq/2(pq-1)}.$$

In order that this holds, it suffices that

$$\frac{pq}{2}\frac{1}{pq-1} < \frac{1+\lambda}{2} \qquad \text{that is,} \qquad \lambda > \frac{1}{pq-1}. \tag{3.7}$$

(The same remark as above applies here to handle t in a bounded region by means of c. We will not repeat this any more.)

In view of (3.5), the combination (2) vs (i) need not be considered. Thus the final case is (2) vs (ii), namely, $r \geq \left(\frac{\omega}{2a}\right)^{q/2} t^{(1-\lambda)q/2}$. Then we should have

$$\frac{\beta r^2}{2t^2} > b^p \frac{t^p}{r^{2p}}, \qquad \text{that is,} \qquad r > \left(\frac{2b^p}{\beta}\right)^{1/2(p+1)} t^{(p+2)/2(p+1)}.$$

In order to have this, it suffices that

$$\frac{(1-\lambda)q}{2} > \frac{p+2}{2(p+1)}, \qquad \text{that is,} \qquad \lambda < 1 - \frac{p+2}{q(p+1)}. \tag{3.8}$$

Summing up the restrictions (3.5), (3.6)=(3.7), (3.8) on λ, we conclude that if

$$\frac{1}{pq-1} < \frac{q-1}{q+1}, \quad \text{and} \quad \frac{1}{pq-1} < 1 - \frac{p+2}{q(p+1)},$$

then we will be able to choose λ so that (3.3) will be a desired super-solution of our nonlinear heat equation (provided c is chosen small enough). Curiously, the above two inequalities reduce to the same one

$$p > \frac{2}{q-1}. \tag{3.9}$$

We have thus proved the following:

Theorem 3.2 *The logarithmic critical exponent p^* of the domain (1.3) satisfies $p^* \leq 2/(q-1)$.*

Note that similar calculation shows the following result. This implies that the combination of paraboloidal region and logarithmic nonlinearity is a natural object next to that of conical domain and power nonlinearity from the viewpoint of critical exponent of blowup.

Proposition 3.3 *Let $f(u)$ be any nonlinear term satisfying the monotonicity and $f(u)/u \to 0$ as $u \to 0$. Let $D \subset \boldsymbol{R}^N$ be a finite domain, or a half cylinder $D' \times \{x_N > 0\}$, or a cylinder $D' \times \boldsymbol{R}_{x_N}$, $D' \subset \boldsymbol{R}^{N-1}$ being a bounded domain. Then there always exists an initial value $u_0(x)$ such that the solution of*

$$u_t = \Delta u + f(u), \qquad u(0,x) = u_0(x), \quad u\big|_{\partial D} = 0$$

exists globally in time.

Proof We omit the proof for the bounded domain since it is well known.

Consider thus a half cylinder $D' \times \{x_N > 0\}$. Let $\varphi(x')$ be the positive eigenfunction corresponding to the first eigenvalue ω of $-\Delta_{x'}$ on D' with the homogeneous Dirichlet condition. Write r for x_N for simplicity. Consider the function

$$u(t,x',r) = ce^{-\lambda t} re^{-kr}\varphi(x').$$

Then, in view of

$$u_r = ce^{-\lambda t}e^{-kr}\varphi(x') - kce^{-\lambda t}re^{-kr}\varphi(x'),$$
$$u_{rr} = -2kce^{-\lambda t}e^{-kr}\varphi(x') + k^2ce^{-\lambda t}re^{-kr}\varphi(x').$$

we have

$$u_t - \Delta u - f(u) = -\lambda u - \Delta_{x'}u - u_{rr} - f(u) = \Big(-\lambda + \omega + \frac{2k}{r} - k^2\Big)u - f(u).$$

Hence, if we choose k, λ such that $\lambda + k^2 < \omega$, then, just as in the case of bounded D, for sufficiently small c the above quantity will be non-negative. Namely we have found a global super-solution, hence there always exists a global solution irrespectively of $f(u)$ for this domain.

For a full cylinder $D' \times \boldsymbol{R}_r$, we take

$$u(t,x',r) = ce^{-\lambda t}e^{-k\sqrt{r^2+1}}\varphi(x').$$

We can show as above that this is a global super-solution provided that $\lambda + k^2 < \omega$ and c is small enough. QED

4 Blowup case

Employing similar method we might construct a sub-solution of our nonlinear heat equation directly, adding the homogeneous Dirichlet condition at $r = R$ for some $R > 0$. It is, however, not obvious, though might be true, that every solution can be estimated from below by such a sub-solution at some time. Thus we choose the way of applying Lemma 1.2 or Corollary 1.3, and try to find a solution $\psi(y)$ of the inequalities

$$L\psi + \lambda\psi \geq 0, \qquad \text{and} \qquad \frac{\int_D u_0(y)\psi(y)dx}{\int_D \psi(y)dx} > \exp(1 - \lambda^{-1/p}) \tag{4.1}$$

together with an appropriately chosen constant $\lambda > 0$. Recall that the Jacobian of the transformation $x \mapsto y$ is approximately $y_N^{(N-1)/q}$. Hence,

$$\int_D \psi(y)dx \sim \int_{D_y} \psi(y)y_N^{(N-1)/q}dy < \infty,$$
$$\int_D u_0(y)\psi(y)dx \sim \int_{D_y} u_0(y)\psi(y)y_N^{(N-1)/q}dy.$$

Let us write r instead of y_N again. For ψ we adopt the following

$$\psi(y) = (r-R)^m e^{-k(r-R)}(1-\varepsilon\rho^2)\varphi(\rho), \qquad \text{where} \quad \rho = |y'|. \tag{4.2}$$

This is essentially the same as the one introduced by [3], but it is shifted by R and the factor $(1-\varepsilon\rho^2)$ is added to manage the terms (2.10), now appearing with unfavorable sign. As ε, we can adopt any positive constant less than 1. Let us thus calculate $L\psi$.

$$L\psi = (1+O(R^{-1}))\psi_{rr} + \frac{N-1+O(R^{-2(1-1/q)})}{qr}\psi_r + \frac{1}{r^{2/q}}\Delta_{y'}\psi$$
$$+ \frac{1}{q^2x_N^2}\rho^2\frac{\partial^2}{\partial\rho^2}\psi + \frac{q+1}{q^2x_N^2}\rho\frac{\partial}{\partial\rho}\psi.$$

Here,

$$\frac{1}{x_N^{2/q}}\Delta_{y'}((1-\varepsilon\rho^2)\varphi(\rho)) + \frac{1}{q^2x_N^2}\rho^2\frac{\partial^2}{\partial\rho^2}((1-\varepsilon\rho^2)\varphi(\rho)) + \frac{q+1}{q^2x_N^2}\rho\frac{\partial}{\partial\rho}((1-\varepsilon\rho^2)\varphi(\rho))$$
$$= \frac{1}{x_N^{2/q}}\Big\{\frac{d^2}{d\rho^2} + \frac{N-2}{\rho}\frac{d}{d\rho}\Big\}((1-\varepsilon\rho^2)\varphi) + \frac{1}{q^2x_N^2}\rho^2\frac{d^2}{d\rho^2}((1-\varepsilon\rho^2)\varphi) + \frac{q+1}{q^2x_N^2}\rho\frac{d}{d\rho}((1-\varepsilon\rho^2)\varphi)$$
$$= -\frac{\omega(1-\varepsilon\rho^2)}{x_N^{2/q}}\varphi - \frac{2\varepsilon\rho}{x_N^{2/q}}\varphi' - \frac{2\varepsilon}{x_N^{2/q}}\varphi - \frac{2\varepsilon(N-2)}{x_N^{2/q}}\varphi$$
$$+ \frac{\rho^2(1-\varepsilon\rho^2)}{q^2x_N^2}\varphi'' - \frac{2\varepsilon\rho^3}{q^2x_N^2}\varphi' - \frac{2\varepsilon\rho^2}{q^2x_N^2}\varphi + \frac{(q+1)\rho(1-\varepsilon\rho^2)}{q^2x_N^2}\varphi' - \frac{2\varepsilon(q+1)\rho^2}{q^2x_N^2}\varphi$$
$$= -\Big(\frac{2\varepsilon\rho}{x_N^{2/q}} + \frac{2\varepsilon\rho^3}{q^2x_N^2} - \frac{(q+1)\rho(1-\varepsilon\rho^2)}{q^2x_N^2} + \frac{(N-2)\rho(1-\varepsilon\rho^2)}{q^2x_N^2}\Big)\varphi'$$
$$- \Big(\frac{\omega(1-\varepsilon\rho^2)}{x_N^{2/q}} + \frac{2\varepsilon}{x_N^{2/q}} + \frac{2\varepsilon(N-2)}{x_N^{2/q}} + \frac{2\varepsilon\rho^2}{x_N^2} + \frac{2\varepsilon(q+1)\rho^2}{q^2x_N^2} + \frac{\omega\rho^2(1-\varepsilon\rho^2)}{q^2x_N^2}\Big)\varphi,$$

where we used the equality $\varphi'' + \frac{N-2}{\rho}\varphi' = -\omega\varphi$. From this we see that if $R>0$ is chosen sufficiently large, the bracket before φ' becomes non-negative, and this term may be omitted. The bracket before φ is dominated by $\frac{\omega}{x_N^{2/q}}$ with a slightly larger ω depending on ε. Likewise, the constant before $\frac{\partial^2}{\partial r^2}$ and $\frac{\partial}{\partial r}$ should be replaced by a slightly smaller or larger one, according to the sign. This being unessential in the calculation below, we can safely replace the sub-solution estimate (4.1) by

$$L\psi + \lambda\psi \geq \psi_{rr} + \frac{N-1}{qr}\psi_r - \frac{\omega}{r^{2/q}}\psi + \lambda\psi \geq 0. \tag{4.3}$$

Now we examine the last inequality of (4.3). We have

$$\psi_r = \Big(\frac{m}{r-R} - k\Big)\psi, \quad \psi_{rr} = \Big(\frac{m(m-1)}{(r-R)^2} + k^2\Big)\psi,$$

hence,

$$\psi_{rr} + \frac{N-1}{qr}\psi_r - \frac{\omega}{r^{2/q}}\psi + \lambda\psi = f(r)\psi,$$

where,

$$f(r) := k^2 + \lambda + \frac{m(m-1)}{(r-R)^2} + \frac{m(N-1)}{qr(r-R)} - \frac{(N-1)k}{qr} - \frac{\omega}{r^{2/q}}.$$

In [3]'s original situation, $q=1$, and the positivity of $f(r)$ for all $r>0$ could be characterized by square completion. In our case, the term $\frac{\omega}{r^{2/q}}$ prevents such a simple treatment and also the calculation of the exact minimum of $f(r)$. Therefore we seek for a sufficient condition for positivity. Suggested by [3], we a priori choose

$$k^2 = \lambda \tag{4.4}$$

which are to be very small. Then we absorb the term of order $1/r$ by

$$k^2 - \frac{(N-1)k}{qr} = \Big(k - \frac{N-1}{2qr}\Big)^2 - \frac{(N-1)^2}{4q^2}\frac{1}{r^2}.$$

The last term will be dominated by $\frac{m(m-1)}{(r-R)^2}$ if m is chosen large, but still independent of k. Next we notice that for a constant A, the minimum of

$$g(r) := k^2 - \frac{\omega}{r^{2/q}} + \frac{A}{r^2}$$

can be exactly calculated:

$$g'(r) = \frac{2}{q}\frac{\omega}{r^{2/q+1}} - 2\frac{A}{r^3} = 0, \quad \text{hence} \quad r = \Big(\frac{qA}{\omega}\Big)^{q/2(q-1)},$$

and then

$$g(r) = k^2 - \omega\Big(\frac{\omega}{qA}\Big)^{1/(q-1)} + A\Big(\frac{\omega}{qA}\Big)^{q/(q-1)} = k^2 - \frac{\omega^{q/(q-1)}}{A^{1/(q-1)}}\Big(\frac{1}{q^{1/(q-1)}} - \frac{1}{q^{q/(q-1)}}\Big).$$

Therefore, if we choose A so large as to $A = \mu k^{-2(q-1)}$, where μ is a constant independent of k, then $g(r) \geq 0$ everywhere, hence

$$f(r) \geq \frac{m(m-1)}{(r-R)^2} + \frac{m(N-1)}{qr(r-R)} - \frac{(N-1)^2}{4q^2}\frac{1}{r^2} - \frac{\mu k^{-2(q-1)}}{r^2}.$$

Thus if we finally choose m large enough so as to

$$m = ck^{-(q-1)} \qquad (c \sim 2\sqrt{\mu}), \tag{4.5}$$

then $f(r) \geq 0$, and the first condition of (4.1) is satisfied.

Next we verify the second condition of (4.1). Taking again into account the Jacobian of the transformation, we have

$$\begin{aligned}\int_R^\infty (r-R)^m r^{(N-1)/q} e^{-k(r-R)} dr &= \int_0^\infty r^m (r+R)^{(N-1)/q} e^{-kr} dr \\ &\geq \int_0^\infty r^{m+(N-1)/q} e^{-kr} dr \geq \frac{\Gamma(m+1+(N-1)/q)}{k^{m+1+(N-1)/q}},\end{aligned}$$

and

$$\begin{aligned}&\int_0^\infty r^m (r+R)^{(N-1)/q} e^{-kr} dr \\ &\leq 2^{(N-1)/q} \int_R^\infty r^{m+(N-1)/q} e^{-kr} dr + (2R)^{(N-1)/q} \int_0^R r^m e^{-kr} dr \\ &\leq 2^{(N-1)/q} \int_0^\infty r^{m+(N-1)/q} e^{-kr} dr + (2R)^{(N-1)/q} \int_0^\infty r^m e^{-kr} dr \\ &= 2^{(N-1)/q} \frac{\Gamma(m+1+(N-1)/q)}{k^{m+1+(N-1)/q}} + (2R)^{(N-1)/q} \frac{\Gamma(m+1)}{k^{m+1}}.\end{aligned}$$

This is a very rough estimate but it suffices for our purpose. Note that when $k \to 0$ the first term is dominant in the last side. Thus we are to prove that

$$\begin{aligned}&\frac{\int_D u_0(y)\psi(y)dx}{\int_D \psi(y)dx} \\ &\geq C\frac{k^{m+1+(N-1)/q}}{\Gamma(m+1+(N-1)/q)} \\ &\quad \times \int_R^\infty \Big(\int_{|y'|\leq 1} u_0(y)(1-\varepsilon|y'|^2)\varphi(y')dy'\Big)(r-R)^m r^{(N-1)/q} e^{-k(r-R)} dr \\ &> e^{1-\lambda^{-1/p}},\end{aligned}$$

where C is a constant independent of k, m. Substituting hitherto introduced conditions (4.4), (4.5), the above reduces to

$$C\frac{k^{ck^{-(q-1)}+1+(N-1)/q}}{\Gamma(ck^{-(q-1)}+1+(N-1)/q)}$$
$$\times\int_R^\infty\Big(\int_{|y'|\le 1}u_0(y)(1-\varepsilon|y'|^2)\varphi(y')dy'\Big)(r-R)^m r^{(N-1)/q}e^{-k(r-R)}dr$$
$$> e^{1-k^{-2/p}}.$$

In view of a rough estimate $\Gamma(s)\le s^s$, the Gamma function of the denominator is estimated from above by

$$(ck^{-(q-1)}+1+(N-1)/q)^{ck^{-(q-1)}+1+(N-1)/q}\le c_1k^{-c(q-1)k^{-(q-1)}-c_2},$$

where c_j are constants independent of k. Hence the factor before the integral sign is estimated from below by

$$c_3k^{cqk^{-(q-1)}+c_4}=c_3e^{-(cqk^{-(q-1)}+c_4)|\log k|},$$

with other constants independent of k. As in the case of [3], the remaining integral is estimated from below by a quantity not decaying to zero. (In fact, even if the initial value u_0 has support only in the region $R<r<R+1$, after a slight lapse of time it becomes everywhere positive. Hence, taking into account that $k\to 0$ and $m\to\infty$, the integral is estimated from below at least by

$$\int_{R+1}^{R+2}\varepsilon e^{-(r-R)}dr$$

with a constant $\varepsilon>0$ independent of k.) Thus having $k\to 0$ in mind, in order that the above inequality holds, it suffices that

$$q-1<\frac{2}{p},\qquad \text{that is,}\qquad p<\frac{2}{q-1}. \tag{4.6}$$

Thus we have proved the following:

Theorem 4.1 *If $p>1$ satisfies (4.6), then all the solutions of (1.1) blows up in finite time. Namely, the critical exponent p^* satisfies $p^*\ge 2/(q-1)$.*

Putting together Theorems 3.2 and 4.1, we have established that $p^*=\frac{2}{q-1}$ is the critical exponent of blowup of the domain (1.3) for the nonlinearity $u(\log u)^p$ of (1.1).

Remark Note that if we combine our present result with our previous abstract one, we can know that the solution $W(t,x)$ of the linear heat equation on the domain (1.3) decays as

$$\|W(t,\cdot)\|_\infty\sim Ce^{-ct^{(q-1)/(q+1)}}.$$

(Precisely speaking, what we can really prove is that for every $\varepsilon>0$ and some related positive constants C, c, C', c',

$$\|W(t,\cdot)\|_\infty\le Ce^{-ct^{(q-1)/(q+1)-\varepsilon}},$$

and that there exists a sequence $t_n\to\infty$ such that

$$\|W(t_n,\cdot)\|_\infty\ge C'e^{-c't^{(q-1)/(q+1)+\varepsilon}}.)$$

We can show this only for $q<3$, but the same estimate should be true for all $q>1$. (Actually, at the beginning, we sought such a classical estimate in the literature, but we could not find usable one for our concrete domain.)

From this observation, it seems rather unnatural to have the restriction $q<3$. This came from the condition $p>1$ assuring the existence of a blowup solution. But the critical exponent is determined (at least philosophically) by the behavior of the nonlinear term at $u = 0$ and not at $u = \infty$. Our nonlinearity was chosen from the viewpoint of beauty so that it may formally balance between $u > 1$ and $u < 1$ like the model of power nonlinearity. But the calculation of [2] shows that $u(\log g\, u)^p$ is convex in the region $0 < u < 1$ even for $0 < p \le 1$. Thus we might, for example, choose a fixed convex nonlinearity $f(u)$ with $f(1) = 1$ for $u \ge 1$ satisfying the blowup condition $\int_1^\infty \frac{du}{f(u)} < \infty$, and set $f_p(u)$ to be $u(\log g\, u)^p$ for $u \le 1$, $f(u) + c_p(u-1)$ for $u \ge 1$, where c_p is chosen so that f_p is convex everywhere (simply to equate $f_p'(1)$ from both sides if $f(u)$ is required to be differentiable). Once this is done, our calculation above, thanks to its primitive nature, applies without modification to the case $0 < p \le 1$, yielding a critical exponent p^* such that $0 \le p^* \le \infty$ for any generalized paraboloidal domain. The lack of standard choice of nonlinearity for $u > 1$ is a problem. What we are thinking now is to employ instead of $u(\log g\, u)^p$ with (1.2), the following one which satisfies the minimum requirement of continuity, convexity and monotone property though not differentiable at $u = 1$:

$$f_p(u) = \begin{cases} \dfrac{u}{(1-\log u)^{p-1}}, & \text{for } 0 < u \le 1, \\ (1+\log u)^p, & \text{for } u \ge 1. \end{cases} \tag{4.7}$$

Here we shifted the parameter p by one so that now the critical exponent of blowup becomes

$$p^* = 1 + \frac{2}{q-1} \tag{4.8}$$

hence $1 < p^* < \infty$ for generalized paraboloidal region, better in concordance with Fujita's original exponent.

In this article, we treated only the simplest examples of domains related with the logarithmic nonlinearity. It will be an interesting problem to treat more general domains, e.g.

$$x_N > |x_1|^{q_1} + |x_2|^{q_2} + \cdots + |x_{N-1}|^{q_{N-1}}, \qquad (q_1 \le q_2 \le \cdots \le q_{N-1}).$$

What we know for the moment is that the critical exponent is between the one corresponding to $q = q_1$ for (1.3) and the one to $q = q_{N-1}$, in view of the comparison theorem.

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