Pythagorean Triples. III. Systematic generation of primitive Pythagorean triples by the topological index and caterpillar graphs.

Haruo Hosoya

(Received April 10, 2009)

Abstract Primitive Pythagorean triples (pPT's) are known to be represented by the topological indices Z's of caterpillar graphs of mirror symmetry. Algorithms for generating these caterpillar graphs are presented with the list of all the pPT's whose hypotenuse is smaller than 1000, together with the relevant caterpillar graphs. Relevant number theoretic discussion including the continuant proposed by Euler is also given.

1. Introduction

In II of the present series of papers[1)] it has been shown that all the family members of the primitive Pythagorean triples (pPT's) can be represented by triplet caterpillar graphs[2,3)] whose topological indices[4,5)] (*Z*, here abbreviated as TopIx) just correspond to the edge values of these PT's. The present author has also recently shown[6)] that the continuant,[7,8)] which was proposed by Euler for facile manipulation of continued fractions, is nothing else but a *Z*-caterpillar, *i.e.*, a TopIx for a caterpillar graph. This means that the concept of the continuant is contained in the TopIx which is defined for all the tree and non-tree graphs.

Although many interesting mathematical properties of the *Z*-caterpillars have been demonstrated in II, no systematic algorithm for deriving these caterpillar graphs has been given. The main purpose of the present paper is to explain the algorithm for obtaining the caterpillar graphs for pPT's. Further, by giving their extensive tabulation several interesting consequences of these problems in the number theory will be discussed.

2. Pythagorean triples and caterpillar graphs

A pPT, $<a, b, c>$, can uniquely be represented by a pair of integers (m, n) as

$$a = m^2 - n^2$$
$$b = 2mn \qquad (2.1)$$
$$c = m^2 + n^2,$$

where a and b are called legs and c hypotenuse, and $m>n>0$ is assumed.[9-11)] The following are the necessary and sufficient conditions for the set of m and n to give a pPT:

i) m and n belong to different parities, and (2.2)

ii) m and n are prime to each other, *i.e.*, $(m, n)=1$. (2.3)

Condition i) ensures that the hypotenuse c of a pPT is 1 modulo 4, but it is not so straightforward how the addition of Condition ii) restricts the values of c. On the other hand, it has long been known that such a prime $p\equiv 1 \pmod 4$ can be expressed by the sum of two squares. Although the proof is not so simple, several important conclusions have been derived from this property.[10,12] Here they will be introduced as theorems without proof.

[Theorem 1]

An integer c appears as the hypotenuse of a pPT if and only if c is a product of primes each of which is congruent to 1 module 4. Namely,

$$c = p_1^{\alpha 1} p_2^{\alpha 2} \cdots p_n^{\alpha n} \quad (p_k \equiv 1 \pmod 4),\ \alpha_1, \alpha_2, \cdots, \alpha_n \geq 0,\ \alpha_1+\alpha_2+\cdots+\alpha_n>0). \tag{2.4}$$

[Theorem 2]

If c has a total of n distinct prime divisors and each is 1 modulo 4, there exist 2^{n-1} pPT's.

[Collorary 1]

Such an integer, either prime or composite, that contains at least one divisor of 3 modulo 4 cannot become a hypotenuse of a pPT.

In graph theory a star graph S_n is defined as a complete bipartite graph $K_{1,n-1}$, which is composed of a central vertex and $n-1$ emanating edges of length 1 thereof, and a path graph P_n as composed of n vertices joined consecutively by $n-1$ edges. Suppose such P_n whose vertices are numbered sequentially either from left or right to another end, prepare a set of n star graphs $X_n(x_1, x_2, \cdots, x_n)$ with each x_k representing Sx_k, and mount X_n one by one on the corresponding vertices of P_n to give a caterpillar graph $C_n(x_1, x_2, \cdots, x_n)$ (See Fig. 1).[2,3]

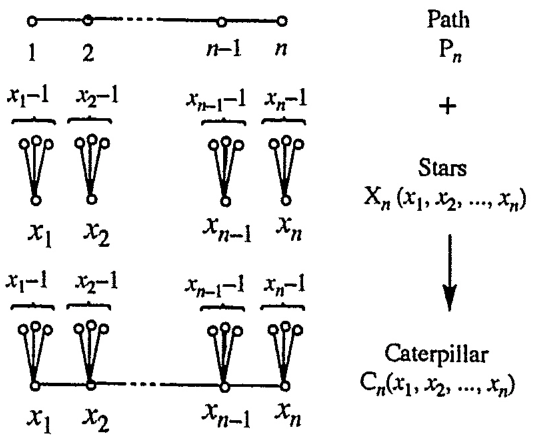


Fig. 1. Path, Stars, and Caterpillar

The topological index TopIx Z was proposed to be defined by the present author in 1971[4,5] for graph G as the total sum of the non-adjacent number $p(G,k)$, which is the number of ways for choosing k disjoint edges from G. Originally TopIx was invented for characterizing the topological nature of the carbon atom skeleton of hydrocarbon molecules and has been used for correlating molecular properties and structures, but later its deep mathematical meaning was clarified and now it is registered as Hosoya index in several data bases in mathematics.[13-15] Fundamental properties and application to number theory are explained elsewhere.[6,16]

Although the continuant was proposed by Euler for facile treatment of continued fractions,[7,8] it was recently found to be nothing else but the TopIx of a caterpillar graph.[6] Thanks to the efficient recursive relations for TopIx, calculation of continuants and continued fractions was dramatically sped up.

Based on these facts and information it was found that the three edges, <a, b, c> of all the family members of pPT's are expressed by the TopIces of triplet caterpillars of mirror symmetry,[1] which will be called here SymCats. By using the algorithm which will be explained later one can obtain the triplet caterpillars {C(a), C(b), C(c)} representing a given PT from the set of (m, n)-codes in (2.1), as long as (m, n)=1 (Condition ii)). Table 1 gives the list of the SymCats of all the pPT's whose edges are shorter than 1000. The abbreviated symbols of the Z-caterpillars, C(a), *etc.*, are explained in the caption of Table 1. The listed pPT's in Table 1 coincide with the ones which were already documented as long as the comparison is possible.[17, 18] One can realize that the selection of the pPT's by (2.4) yields the same result obtained by using Conditions i) and ii) for m and n. It is obvious in Table 1 that the numbers of the vertices of a and b in the main trunk of the caterpillar are odd, while those of c even. Let us call these two groups o- and e-SymCats, respectively.

Although the number of pPT's for a given integer as the length of the hypotenuse is determined by Theorem 2, Table 1 accommodates only the cases of p^2, p^3, p^4, $p_1 p_2$, and $p_1^2 p_2$. Then Table 2 was prepared, where more highly composite c values are selected and the corresponding Z-caterpillars are derived to supplement Table 1. Before analyzing these data, several necessary explanations for the structure of the Z-caterpillars will be given.

3. Fundamental structure and calculation of SymCats

In this study we are concerned only with the e- and o-SymCats, which can be expressed, respectively, as $C_{2s}(x_1, x_2, \cdots, x_s, x_s, \cdots, x_1)$ and $C_{2s-1}(x_1, x_2, \cdots, x_{s-1}, x_s, x_{s-1}, \cdots, x_1)$. As remarked in the caption of Table 1 they are abbreviated, respectively, as [$x_1, x_2, \cdots, x_s$, and ($x_1, x_2, \cdots, x_{s-1}, \underline{x_s}$. Note the underline for the last term of o-SymCat. As has been mentioned in II, the set of triplet caterpillars representing a PT belongs either to T- or W-type with respect to their kernels. See Fig. 2, where the schematic diagrams and the TopIces of the two types of the triplet Z-caterpillars of pPT's are shown. The encircled n represents a moiety of a caterpillar and a indicates the number of edges of a unit length emanating from a certain vertex of the caterpillar.

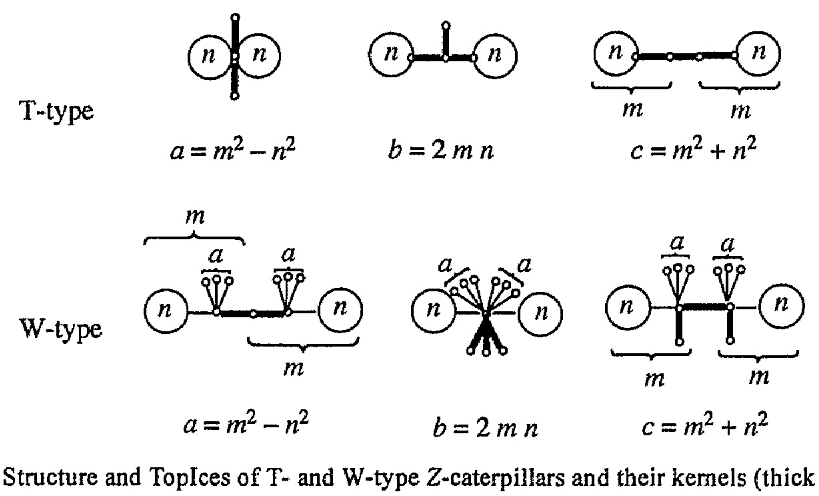


Fig. 2 Structure and TopIces of T- and W-type Z-caterpillars and their kernels (thick edges).

In T-type a larger caterpillar m is constructed by joining an edge to one of the terminal vertices of n, while in W-type the set of (a+1) edges of a unit length (a≥0), or a star graph S_{a+2}, is joined to form m. The most important parts of the Z-caterpillars of pPT's of both the T- and W-types are the kernels drawn in thick edges, whose TopIces form the set of <3, 4, 5>, the progenitor of the whole family tree of pPT's.[19-21]

Although a number of PT's and their triplet caterpillars are calculated as in Tables 1 and 2, no other type than T and W has ever been found. Thus what the present author believes in this problem will be stated as the following conjecture:

[Conjecture 1]

The three edges, a, b, and c, of a pPT can be represented by the TopIces of the triplet caterpillars of either T- or W-type.

This conjecture may be paraphrased into a shorter statement as
"A pPT can be represented by the Z-caterpillars of T- or W-type."

If the restriction of "primitive" is removed from this conjecture, one can prove the following Theorem, which is essentially the same as the union of Theorems 1 and 2 in ref. 1

[Theorem 3]

The TopIces of the set of both the T- and W-type caterpillar graphs, a, b, and c, represent the three edges of PTs.

The proof of Theorem 3 is given in Figs. 3a and b, where the values of TopIces of all the structures given in Figs, 2a and b are calculated by using the recursive formulas of TopIces as explained in Appendix. In following the calculation scheme for the proof it is to be noted that the TopIx of a vertex is unity.

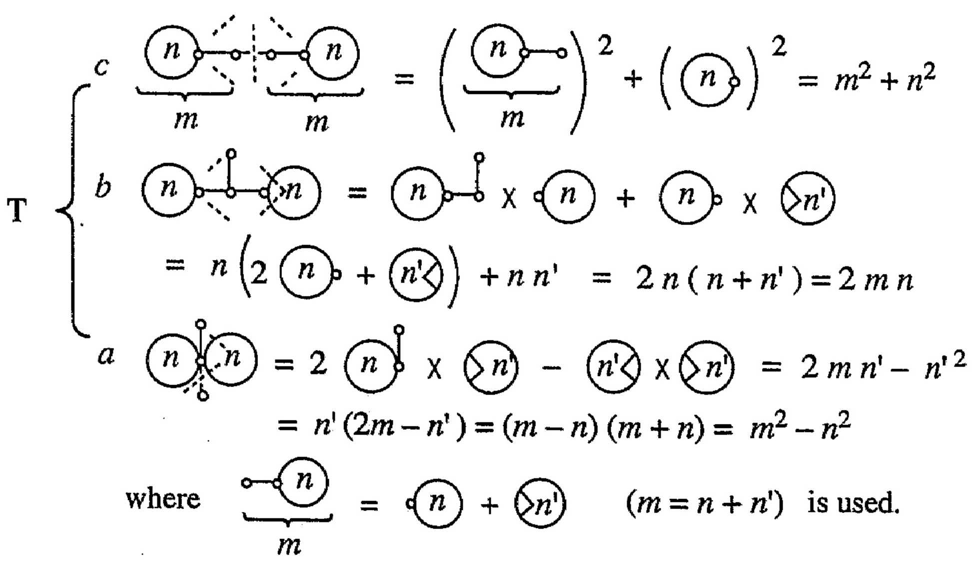


Fig. 3a. T-type caterpillars and their calculation scheme.

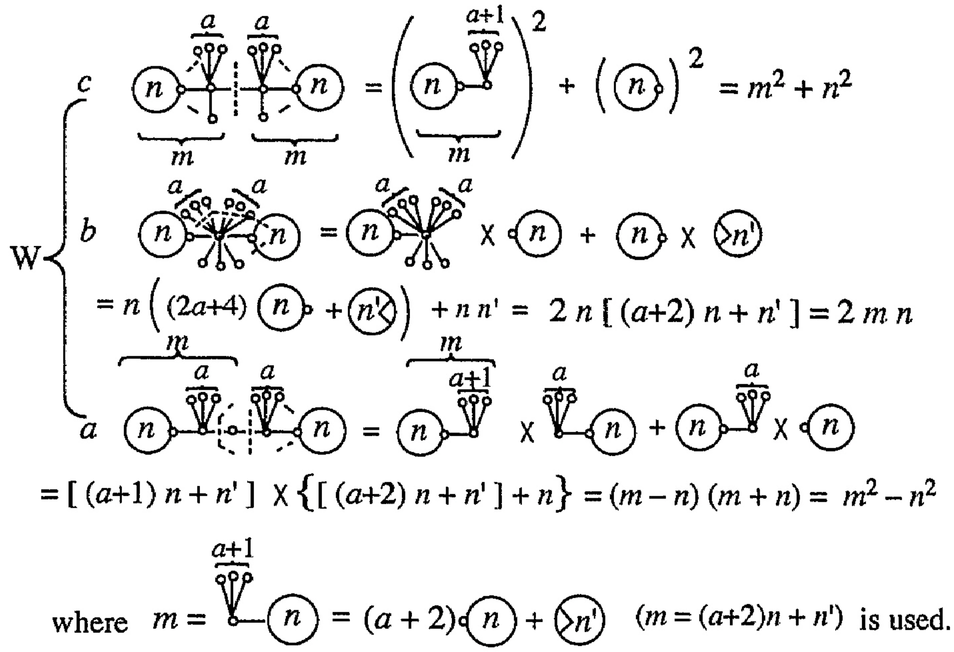


Fig. 3b. W-type caterpillars and their calculation scheme.

Contrary to the hypotenuse c, mathematical conditions for a given integer to appear as one of the legs, *i.e.*, odd a and even b, of pPT's is rather complicated.[10] At this point we have not yet reached a clear-cut conclusion as to this problem. Here the algorithm for finding the Z-caterpillars will be explained in some detail.

4. Algorithms for obtaining the Z-caterpillars of a given pPT

Case i) $N = p^{\alpha}$ ($\alpha \geq 1$) with $p \equiv 1 \pmod 4$

Example 1) $N = 317$

i) There is a unique expression for a natural number N to be the sum of two squares, $N = m^2 + n^2$ ($m > n$). In this case we get $m = 14$ and $n = 11$ as $317 = 14^2 + 11^2$. For this purpose several algorithms, such as the "descent procedure" proposed by Fermat, have already been known.

ii) The targeted pPT is obtained to be $<m^2 - n^2, 2mn, m^2 + n^2> = <75, 308, 317>$.

iii) The corresponding caterpillar graph whose TopIx is $m^2 + n^2$ can be obtained as follows:

From the set of (m, n) one can simultaneously obtain t_s and r_{s-2} under $n > r_{s-2}$

$m = t_s \times n + r_{s-2}$ $\quad 14 = 1 \times 11 + 3 \quad (t_s = 1)$

and similarly we can get the sets of t_s and r_s as follows:

$n = t_{s-1} \times r_{s-2} + r_{s-3}$ $\quad 11 = 3 \times 3 + 2 \quad (t_3 = 3)$

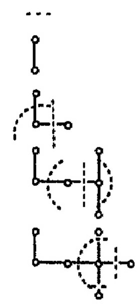
$r_{s-2} = t_{s-2} \times r_{s-3} + r_{s-4}$ $\quad 3 = 1 \times 2 + 1 \quad (t_2 = 1)$

$\cdots$ $\quad 2 = 2 \times 1 \quad (t_1 = 2)$

$r_2 = t_2 \times r_1 + r_0$

$r_1 = t_1 \times r_0$,

where $m = r_s$ and $n = r_{s-1}$. This stepwise calculation quite similar to the GCD algorithm of Euclid terminates after r gets down to 1 ($=r_0$), and the value of s (size of the set of $\{r_s\}$) is determined (4 in this example) as the number of steps in this iteration. It is to be noted that this algorithm is just the trace back of the calculation of the TopIx of the caterpillar graph, $C_s(t_1, t_2, \cdots, t_s)$, where t_k denotes the number of edges of unit length emanating from the kth vertex of the path graph as the main trunk of the caterpillar *plus one*. In Fig. 4 iterative steps for calculating the TopIx of $C_4(2, 1, 3, 1)$ are compared with the calculation scheme of the continuant proposed by Euler, showing the same mathematical structure of both the concepts. However, in the actual calculation of the former efficient recursive relations are available, whereas until recently no such technique has been introduced for the latter.[6]



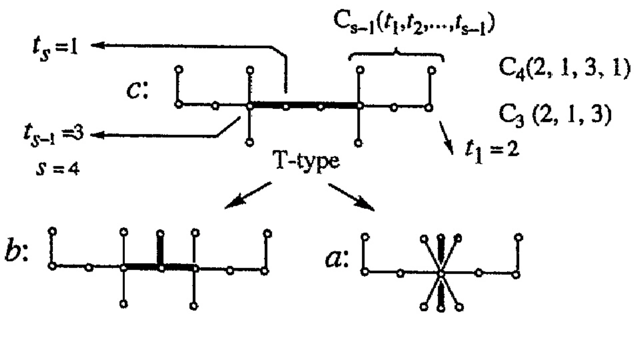
Z-Caterpillar	Continuant
$C_0() = 1$	$K_0() = 1$
$C_1(2) = 2$	$K_1(2) = 2$
$C_2(2, 1) = C_1 + C_0 = 3$	$K_2(2, 1) = K_1 + K_0 = 3$
$C_3(2, 1, 3) = 3\,C_2 + C_1 = 11$	$K_3(2, 1, 3) = 3\,K_2 + K_1 = 11$
$C_4(2, 1, 3, 1) = C_3 + C_2 = 14$	$K_4(2, 1, 3, 1) = K_3 + K_2 = 14$

Fig. 4. Parallelism between the calculation of Z-caterpillar and that of continuant. Dashed straight and curved lines indicate, respectively, the first and second deletions of edges for applying the recursive formulas. Both $C_0()$ and $K_0()$ are defined to be unity.

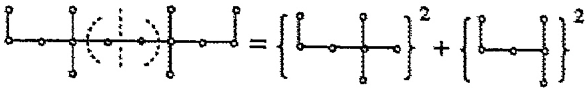
The caterpillar graph $C(c)$ for the hypotenuse c is constructed by joining two C_s graphs as follow:

$$C_{2s}(t_1, t_2, \cdots, t_s, t_s, \cdots, t_1) \qquad C_8(2, 1, 3, 1, 1, 3, 1, 2)$$
$$= [\, t_1, t_2, \cdots, t_s, \qquad = [\, 2, 1, 3, 1,$$

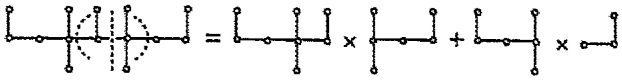
The open square-bracket notations in the second line above are the abbreviated ones for representing e-SymCats as already explained before. In this case, since $t_s = 1$, the triplet caterpillars belong to T-type. Then one can construct the two legs, a and b, of the Pythagorean triple as follows:



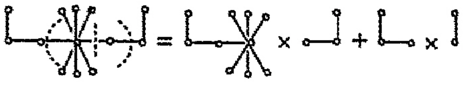
Note that the kernels of the T-type triplets drawn by thick edges are sandwiched by the common pair of caterpillar $C_{s-1}(t_1, t_2, \cdots, t_{s-1})$. The calculation of TopIces of these caterpillars are illustrated as below.



$= 14^2 + 11^2 = 317$



$= 25 \times 11 + 11 \times 3 = 308$



$= 23 \times 3 + 3 \times 2 = 75$

Then we can get the pPT <75, 308, 317>.

Example 2) $N = 349$

ii) In this case we get $m = 18$ and $n = 5$ as $349 = 18^2 + 5^2$.

ii) The targeted pPT is obtained to be <299, 180, 349>.

iv) One can obtain the sets of t_s and r_s as follows:

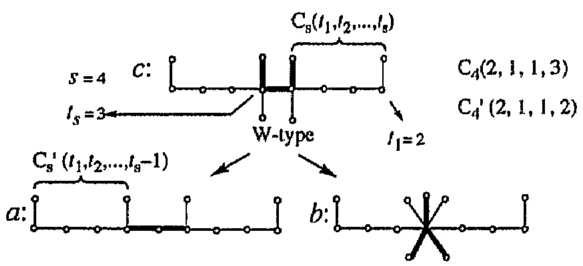
$$18 = 3\times5+3,\quad 5 = 1\times3+2,\quad 3 = 1\times2+1,\quad 2 = 2\times1 \quad (s=4)$$

Thus we get $C_4(2, 1, 1, 3)$

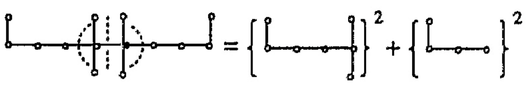
The caterpillar graph for the hypotenuse c is obtained to be $C_8(2, 1, 1, 3, 3, 1, 1, 2)$.

$= [t_1, t_2, \cdots, t_s,$ $\quad = [2, 1, 1, 3,$

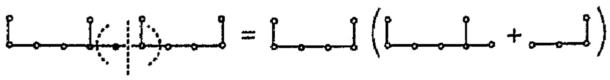
As $t_s \neq 1$ in this case, the triplet caterpillars belong to W-type. The pair of caterpillars sandwiching the kernels are not C_{s-1} but $C_s'(t_1, t_2, \cdots, t_s-1)$. Then one can construct the two legs, a and b, of the Pythagorean triangle as follows:



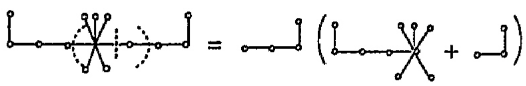
The calculation of these TopIces are illustrated as below.



$= 18^2 + 5^2 = 349$



$= 13(18+5) = 299$



$= 5(33+3) = 180$

Then we can get the pPT <299, 180, 349>.

Case ii) $N = p_1^{\alpha 1} p_2^{\alpha 2} \dots p_n^{\alpha n}$ where $p_k \equiv 1$ (mod 4).

In this case there exist 2^{n-1} distinct Pythagorean triples irrespective of the distribution of $\alpha_1 \sim \alpha_n$.

Example 3) $N = 145 = 5 \times 29$

There are two ways for expressing this number as the sum of two squares. Namely, $145 = 9^2 + 8^2 = 12^2 + 1^2$.

By using the recipe used as above for both the pairs of integers one gets c = [8, 1, and [12, .

Note that the former and latter, respectively, belong to T- and W-types. Then we get the two sets of caterpillar triplets as

(17,	(8, 2,	[8, 1,	for <17, 144, 145>	(Δ1, δ8, d127)
(11, 1,	(24,	[12,	for <143, 24, 145>	(Δ11, δ1, d119).

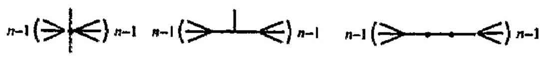
The notations in the parentheses are their classification codes for the PT family proposed by the present author.[21] Here Δ1 and δ1, respectively, mean $c-b=1^2$ and $c-a=2 \cdot 1^2$.

5. Discussion

As has been shown that many series of pPT's can be represented by the special series of Z-caterpillars. Tables 3 and 4 were prepared by picking up the Δ1 and δ1 groups from Table 1. Their general expressions are also given.

Table 3. Z-caterpillars of Δ1 series pPT's.

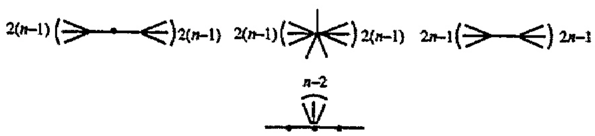
<3, 4, 5>	(3,	(1, 2, [a]	[1, 1, [b]
< 5, 12, 13>	(5,	(2, 2,	[2, 1,
<7, 24, 25>	(7,	(3, 2,	[3, 1,
<9, 40, 41>	(9,	(4, 2,	[4, 1,
<2n+1, 2n(n+1), 2n²+2n+1>	(2n+1,	(n, 2,	[n, 1,



a) Identical to (4, . b) Identical to [2, .

Table 4. Z-caterpillars of δ1 series pPT's.

<3, 4, 5>	(1, 1,	(4, = (2, 0,	[2,
<15, 8, 17>	(3, 1,	(8, = (2, 1,	[4,
<35, 12, 37>	(5, 1,	(12, = (2, 2,	[6,
<63, 16, 65>	(7, 1,	(16, = (2, 3,	[8,
<4n²−1, 4n, 4n²+1>	(2n−1, 1,	(4n, = (2, n−1,	[2n,



The series of pPT's with consecutive legs have also been known to be expressed by a series of caterpillars with some regular structure. However, for other series of graphs, such as Δn and δn (n>1), several different series of caterpillar graphs are involved. Analysis along this line is still in progress, but it goes without saying that the introduction of Z-caterpillar has presented quite a new view in the mathematical structure of PT's.

Refernces

1) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **59** (2) (2009) 15.
2) K. H. Rosen, J. G. Michaels, J. L. Gross, J. W. Grossman, and D. R. Shier (Eds.), Handbook of Discrete and Combinatorial Mathematics, CRC Press, Boca Raton, FL (2000), p. 498.
3) E. W. Weisstein, CRC Concise Encyclopedia of Mathematics, Chapman & Hall/CRC, Boca Raton, FL (2003), p. 351.
4) H. Hosoya, *Bull. Chem. Soc. Jpn.*, **44** (1971) 2332.
5) H. Hosoya, *Fibonacci Quart.*, **11** (1973) 255.
6) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **58** (1) (2007) 15.
7) D. E. Knuth, The Art of Computer Programming, Vol. 1, Addison-Wesley, Reading, MS (1968), Chapter 2.
8) R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematics. A Foundation for Computer Science, Addison-Wesley, Reading, MS (1989), Chapter 6.
9) W. Sierpinski, Pythagorean Triangles, Graduate School of Science, Yeshiva Univ., New York (1962).
10) A. H. Beiler, Recreations in the Theory of Numbers, Dover, New York (1966).
11) E. Maor, The Pythagorean Theorem, Princeton Univ. Press, Princeton (2007).
12) J. H. Silverman, A Friendly Introduction to Number Theory, Prentice-Hall, Upper Saddle River, NJ (2001).
13) http://en.wikipedia.org/wiki/Hosoya_index
14) http://mathworld.wolfram.com/HosoyaIndex.html
15) T. Koshy, Fibonacci and Lucas numbers with applications, Wiley, New York, 2001.
16) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **57** (2) (2006) 35.
17) A020882 (Ordered hypotenuses of primitive Pythagorean triangles) by N. J. A. Sloane, "The On-Line Encyclopedia of Integer Sequences."
18) http://grail.csuohio.edu/~somos/rtritab.txt.
19) A. Hall, *Mathematical Gazette*, **54** (1970) 377.
20) J. Roberts, Elementary Number Theory, MIT Press, Cambridge, MS (1977).
21) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **59** (2) (2009) 1.

Table 1. List of primitive Pythagorean triples and the corresponding caterpillar graphs whose edges are shorter than 1000

p	m^2+n^2	<a, b, c>	C(a)	C(b)	C(c)	Type
5	2^2+1^2	<3, 4, 5>	(3,	(4,	[2,	--
13	3^2+2^2	< 5, 12, 13>	(5,	(2, 2,	[2, 1,	T
17	4^2+1^2	<15, 8, 17>	(3, 1,	(8,	[4,	W
25 $=5^2$	4^2+3^2	<7, 24, 25>	(7,	(3, 2,	[3, 1,	T
29	5^2+2^2	<21, 20, 29>	(2, 1, 1,	(2, 4,	[2, 2,	W
37	6^2+1^2	<35, 12, 37>	(5, 1,	(12,	[6,	W
41	5^2+4^2	<9, 40, 41>	(9,	(4, 2,	[4, 1,	T
53	7^2+2^2	<45, 28, 53>	(2, 2, 1,	(2, 6,	[2, 3,	W
61	6^2+5^2	<11, 60, 61>	(11,	(5, 2,	[5, 1,	T
65	7^2+4^2	<33, 56, 65>	(3, 3,	(3, 1, 2,	[3, 1, 1,	T
$=5\times13$	8^2+1^2	<63, 16, 65>	(7, 1,	(16,	[8,	W
73	8^2+3^2	<55, 48, 73>	(2, 1, 1, 1,	(2, 1, 4,	[2, 1, 2,	W
85	7^2+6^2	<13, 84, 85>	(13,	(6, 2,	[6, 1	T
$=5\times17$	9^2+2^2	<77, 36, 85>	(2, 3, 1,	(2, 8,	[2, 4,	W
89	8^2+5^2	<39, 80, 89>	(2, 1, 3,	(2, 1, 1, 2,	[2, 1, 1, 1,	T
97	9^2+4^2	<65, 72, 97>	(4, 1, 1,	(4, 4,	[4, 2,	W
101	10^2+1^2	<99, 20, 101>	(9, 1,	(20,	[10,	W
109	10^2+3^2	<91, 60, 109>	(3, 2, 1,	(3, 6,	[3, 3,	W
113	8^2+7^2	<15, 112, 113>	(15,	(7, 2,	[7, 1,	T
$125=5^3$	11^2+2^2	<117, 44, 125>	(2, 4, 1,	(2, 10,	[2, 5,	W
	(10^2+5^2)	5^2<3, 4, 5>				
137	11^2+4^2	<105, 88, 137>	(3, 1, 1, 1,	(3, 1, 4,	[3, 1, 2,	W
145	9^2+8^2	<17, 144, 145>	(17,	(8, 2,	[8, 1,	T
$=5\times29$	12^2+1^2	<143, 24, 145>	(11, 1,	(24,	[12,	W
149	10^2+7^2	<51, 140, 149>	(3, 5,	(3, 2, 2,	[3, 2, 1,	T
157	11^2+6^2	<85, 132, 157>	(5, 3,	(5, 1, 2,	[5, 1, 1,	T
$169=13^2$	12^2+5^2	<119, 120, 169>	(2, 2, 1, 1,	(2, 2, 4,	[2, 2, 2,	W
173	13^2+2^2	<165, 52, 173>	(2, 5, 1,	(2, 12,	[2, 6,	W
181	10^2+9^2	<19, 180, 181>	(19,	(9, 2,	[9, 1,	T
185	11^2+8^2	<57, 176, 185>	(2, 1, 5,	(2, 1, 2, 2,	[2, 1, 2, 1,	T
$=5\times37$	13^2+4^2	<153, 104, 185>	(4, 2, 1,	(4, 6,	[4, 3,	W
193	12^2+7^2	<95, 168, 193>	(2, 2, 3,	(2, 2, 1, 2,	[2, 2, 1, 1,	T
197	14^2+1^2	<195, 28, 197>	(13, 1,	(28,	[14,	W

205	13^2+6^2	<133, 156, 205>	(6, 1, 1,	(6, 4,	[6, 2,	W
=5×41	14^2+3^2	<187, 84, 205>	(2, 1, 3, 1,	(2, 1, 8,	[2, 1, 4,	W
221	11^2+10^2	<21, 220, 221>	(21,	(10, 2,	[10, 1,	T
=13×17	14^2+5^2	<171, 140, 221>	(4, 1, 1, 1,	(4, 1, 4,	[4, 1, 2,	W
229	15^2+2^2	<221, 60, 229>	(2, 6, 1,	(2, 14,	[2, 7,	W
233	13^2+8^2	<105, 208, 233>	(2, 1, 1, 3,	(2, 1, 1, 1, 2,	[2, 1, 1, 1, 1,	T
241	15^2+4^2	<209, 120, 241>	(3, 1, 2, 1,	(3, 1, 6,	[3, 1, 3,	W
257	16^2+1^2	<255, 32, 257>	(15, 1,	(32,	[16,	W
265	12^2+11^2	<23, 264, 265>	(23,	(11, 2,	[11, 1,	T
=5×53	16^2+3^2	<247, 96, 265>	(3, 4, 1,	(3, 10,	[3, 5,	W
269	13^2+10^2	<69, 260, 269>	(3, 7,	(3, 3, 2,	[3, 3, 1,	T
277	14^2+9^2	<115, 252, 277>	(4, 1, 3,	(4, 1, 1, 2,	[4, 1, 1, 1,	T
281	16^2+5^2	<231, 160, 281>	(5, 2, 1,	(5, 6,	[5, 3,	W
$289=17^2$	15^2+8^2	<161, 240, 289>	(7, 3,	(7, 1, 2,	[7, 1, 1,	T
293	17^2+2^2	<285, 68, 293>	(2, 7, 1,	(2, 16,	[2, 8,	W
305	6^2+7^2	<207, 224, 305>	(2, 3, 1, 1,	(2, 3, 4,	[2, 3, 2,	W
=5×61	17^2+4^2	<273, 136, 305>	(4, 3, 1,	(4, 8,	[4, 4,	W
313	13^2+12^2	<25, 312, 313>	(25,	(12, 2,	[12, 1,	T
317	14^2+11^2	<75, 308, 317>	(2, 1, 7,	(2, 1, 3, 2,	[2, 1, 3, 1,	T
325	17^2+6^2	<253, 204, 325>	(5, 1, 1, 1,	(5, 1, 4,	[5, 1, 2,	W
$=5^2\times13$	18^2+1^2	<323, 36, 325>	(17, 1,	(36,	[18,	W
	(15^2+10^2)	5^2<5, 12, 13>				
337	16^2+9^2	<175, 288, 337>	(2, 3, 3,	(2, 3, 1, 2,	[2, 3, 1, 1,	T
349	18^2+5^2	<299, 180, 349>	(2, 1, 1, 2, 1,	(2, 1, 1, 6,	[2, 1, 1, 3,	W
353	17^2+8^2	<225, 272, 353>	(8, 1, 1,	(8, 4,	[8, 2,	W
365	14^2+13^2	<27, 364, 365>	(27,	(13, 2,	[13, 1,	T
=5×73	19^2+2^2	<357, 76, 365>	(2, 8, 1,	(2, 18,	[2, 9,	W
373	18^2+7^2	<275, 252, 373>	(3, 1, 1, 1, 1,	(3, 1, 1, 4,	[3, 1, 1, 2,	W
377	16^2+11^2	<135, 352, 377>	(5, 5,	(5, 2, 2,	[5, 2, 1,	T
=13×29	19^2+4^2	<345, 152, 377>	(3, 1, 3, 1,	(3, 1, 8,	[3, 1, 4,	W
389	17^2+10^2	<189, 340, 389>	(3, 2, 3,	(3, 2, 1, 2,	[3, 2, 1, 1,	T
397	19^2+6^2	<325, 228, 397>	(6, 2, 1,	(6, 6,	[6, 3,	W
401	20^2+1^2	<399, 40, 401>	(19, 1,	(40,	[20,	W
409	20^2+3^2	<391, 120, 409>	(2, 1, 5, 1,	(2, 1, 12,	[2, 1, 6,	W
421	15^2+14^2	<29, 420, 421>	(29,	(14, 2,	[14, 1,	T
425	16^2+13^2	<87, 416, 425>	(3, 9,	(3, 4, 2,	[3, 4, 1,	T
$=5^2\times17$	19^2+8^2	<297, 304, 425>	(2, 1, 2, 1, 1,	(2, 1, 2, 4,	[2, 1, 2, 2,	W
	(20^2+5^2)	5^2<15, 8, 17>				
433	17^2+12^2	<145, 408, 433>	(2, 2, 5,	(2, 2, 2, 2,	[2, 2, 2, 1,	T

445	18^2+11^2	<203, 396, 445>	(3, 1, 1, 3,	(3, 1, 1, 1, 2,	[3, 1, 1, 1, 1,	T
$=5\times89$	21^2+2^2	<437, 84, 445>	(2, 9, 1,	(2, 20,	[2, 10,	W
449	20^2+7^2	<351, 280, 449>	(6, 1, 1, 1,	(6, 1, 4,	[6, 1, 2,	W
457	21^2+4^2	<425, 168, 457>	(4, 4, 1,	(4, 10,	[4, 5,	W
461	19^2+10^2	<261, 380, 461>	(9, 3,	(9, 1, 2,	[9, 1, 1,	T
481	16^2+15^2	<31, 480, 481>	(31,	(15, 2,	[15, 1,	T
$=13\times37$	20^2+9^2	<319, 360, 481>	(2, 4, 1, 1,	(2, 4, 4,	[2, 4, 2,	W
485	17^2+14^2	<93, 476, 485>	(2, 1, 9,	(2, 1, 4, 2,	[2, 1, 4, 1,	T
$=5\times97$	22^2+1^2	<483, 44, 485>	(21, 1,	(44,	[22,	W
493	18^2+13^2	<155, 468, 493>	(2, 1, 1, 5,	(2, 1, 1, 2, 2,	[2, 1, 1, 2, 1,	T
$=17\times29$	22^2+3^2	<475, 132, 493>	(3, 6, 1,	(3, 14,	[3, 7,	W
505	19^2+12^2	<217, 456, 505>	(2, 2, 1, 3,	(2, 2, 1, 1, 2,	[2, 2, 1, 1, 1,	T
$=5\times101$	21^2+8^2	<377, 336, 505>	(2, 1, 1, 1, 1, 1,	(2, 1, 1, 1, 4,	[2, 1, 1, 1, 2,	W
509	22^2+5^2	<459, 220, 509>	(2, 2, 3, 1,	(2, 2, 8,	[2, 2, 4,	W
521	20^2+11^2	<279, 440, 521>	(2, 4, 3,	(2, 4, 1, 2,	[2, 4, 1, 1,	T
533	22^2+7^2	<435, 308, 533>	(7, 2, 1,	(7, 6,	[7, 3,	W
$=13\times41$	23^2+2^2	<525, 92, 533>	(2, 10, 1,	(2, 22,	[2, 11,	W
541	21^2+10^2	<341, 420, 541>	(10, 1, 1,	(10, 4,	[10, 2,	W
545	17^2+16^2	<33, 544, 545>	(33,	(16, 2,	[16, 1,	T
$=5\times109$	23^2+4^2	<513, 184, 545>	(3, 1, 4, 1,	(3, 1, 10,	[3, 1, 5,	W
557	19^2+14^2	<165, 532, 557>	(4, 1, 5,	(4, 1, 2, 2,	[4, 1, 2, 1,	T
565	22^2+9^2	<403, 396, 565>	(4, 2, 1, 1,	(4, 2, 4,	[4, 2, 2,	W
$=5\times113$	23^2+6^2	<493, 276, 565>	(5, 1, 2, 1,	(5, 1, 6,	[5, 1, 3,	W
569	20^2+13^2	<231, 520, 569>	(6, 1, 3,	(6, 1, 1, 2,	[6, 1, 1. 1,	T
577	24^2+1^2	<575, 48, 577>	(23, 1,	(48,	[24,	W
593	23^2+8^2	<465, 368, 593>	(7, 1, 1, 1,	(7, 1, 4,	[7, 1, 2,	W
601	24^2+5^2	<551, 240, 601>	(4, 1, 3, 1,	(4, 1, 8,	[4, 1, 4,	W
613	18^2+17^2	<35, 612, 613>	(35,	(17, 2,	[17, 1,	T
617	19^2+16^2	<105, 608, 617>	(3, 11,	(3, 5, 2,	[3, 5, 1,	T
$625=5^4$	24^2+7^2	<527, 336, 625>	(3, 2, 2, 1,	(3, 2, 6,	[3, 2, 3,	W
	(20^2+15^2)	5^2<7, 24, 25>				
629	23^2+10^2	<429, 460, 629>	(3, 3, 1, 1,	(3, 3, 4,	[3, 3, 2,	W
$=17\times37$	25^2+2^2	<621, 100, 629>	(2, 11, 1,	(2, 24,	[2, 12,	W
641	25^2+4^2	<609, 200, 641>	(4, 5, 1,	(4, 12,	[4, 6,	W
653	22^2+13^2	<315, 572, 653>	(4, 2, 3,	(4, 2, 1, 2,	[4, 2, 1, 1,	T
661	25^2+6^2	<589, 300, 661>	(6, 3, 1,	(6, 8,	[6, 4,	W
673	23^2+12^2	<385, 552, 673>	(11, 3,	(11, 1, 2,	[11, 1, 1,	T
677	26^2+1^2	<675, 52, 677>	(25, 1,	(52,	[26,	W

685	19^2+18^2	<37, 684, 685>	(37,	(18, 2,	[18, 1,	T
= 5×137	26^2+3^2	<667, 156, 685>	(2, 1, 7, 1,	(2, 1, 16,	[2, 1, 8,	W
689	20^2+17^2	<111, 680, 689>	(2, 1, 11,	(2, 1, 5, 2,	[2, 1, 5, 1,	T
= 13×53	25^2+8^2	<561, 400, 689>	(8, 2, 1,	(8, 6,	[8, 3,	W
697	21^2+16^2	<185, 672, 697>	(5, 7,	(5, 3, 2,	[5, 3, 1,	T
= 17×41	24^2+11^2	<455, 528, 697>	(2, 5, 1, 1,	(2, 5, 4,	[2, 5, 2,	W
701	26^2+5^2	<651, 260, 701>	(5, 4, 1,	(5, 10,	[5, 5,	W
709	22^2+15^2	<259, 660, 709>	(7, 5,	(7, 2, 2,	[7, 2, 1,	T
725	23^2+14^2	<333, 644, 725>	(4, 1, 1, 3,	(4, 1, 1, 1, 2,	[4, 1, 1, 1, 1,	T
= 5^2×29	26^2+7^2	<627, 364, 725>	(2, 2, 1, 2, 1,	(2, 2, 1, 6,	[2, 2, 1, 3,	W
	(25^2+10^2)	13^2<21, 20, 29>				
733	27^2+2^2	<725, 108, 733>	(2, 12, 1,	(2, 26,	[2, 13,	W
745	24^2+13^2	<407, 624, 745>	(2, 5, 3,	(2, 5, 1, 2,	[2, 5, 1, 1,	T
= 5×149	27^2+4^2	<713, 216, 745>	(3, 1, 5, 1,	(3, 1, 12,	[3, 1, 6,	W
757	26^2+9^2	<595, 468, 757>	(8, 1, 1, 1,	(8, 1, 4,	[8, 1, 2,	W
761	20^2+19^2	<39, 760, 761>	(39,	(19, 2,	[19, 1,	T
769	25^2+12^2	<481, 600, 769>	(12, 1, 1,	(12, 4,	[12, 2,	W
773	22^2+17^2	<195, 748, 773>	(2, 2, 7,	(2, 2, 3, 2,	[2, 2, 3, 1,	T
785	23^2+16^2	<273, 736, 785>	(2, 3, 5,	(2, 3, 2, 2,	[2, 3, 2, 1,	T
= 5×157	28^2+1^2	<783, 56, 785>	(27, 1,	(56,	[28,	W
793	27^2+8^2	<665, 432, 793>	(2, 1, 2, 2, 1,	(2, 1, 2, 6,	[2, 1, 2, 3,	W
= 13×61	28^2+3^2	<775, 168, 793>	(3, 8, 1,	(3, 18,	[3, 9,	W
797	26^2+11^2	<555, 572, 797>	(3, 1, 2, 1, 1,	(3, 1, 2, 4,	[3, 1, 2, 2,	W
809	28^2+5^2	<759, 280, 809>	(2, 1, 1, 4, 1,	(2, 1, 1, 10,	[2, 1, 1, 5,	W
821	25^2+14^2	<429, 700, 821>	(2, 1, 3, 3,	(2, 1, 3, 1, 2,	[2, 1, 3, 1, 1,	T
829	27^2+10^2	<629, 540, 829>	(3, 2, 1, 1, 1,	(3, 2, 1, 4,	[3, 2, 1, 2,	W
841 = 29^2	21^2+20^2	<41, 840, 841>	(41,	(20, 2,	[20, 1,	T
845	22^2+19^2	<123, 836, 845>	(3, 13,	(3, 6, 2,	[3, 6, 1,	T
= 5×13^2	29^2+2^2	<837, 116, 845>	(2, 13, 1,	(2, 28,	[2, 14,	W
	(26^2+13^2)	13^2<3, 4, 5>				
853	23^2+18^2	<205, 828, 853>	(2, 1, 1, 7,	(2, 1, 1, 3, 2,	[2, 1, 1, 3, 1,	T
857	29^2+4^2	<825, 232, 857>	(4, 6, 1,	(4, 14,	[4, 7,	W
865	24^2+17^2	<287, 816, 865>	(3, 2, 5,	(3, 2, 2, 2,	[3, 2, 2, 1,	T
= 5×173	28^2+9^2	<703, 504, 865>	(9, 2, 1,	(9, 6,	[9, 3,	W
877	29^2+6^2	<805, 348, 877>	(5, 1, 3, 1,	(5, 1, 8,	[5, 1, 4,	W
881	25^2+16^2	<369, 800, 881>	(2, 3, 1, 3,	(2, 3, 1, 1, 2,	[2, 3, 1, 1, 1,	T
901	26^2+15^2	<451, 780, 901>	(3, 1, 2, 3,	(3, 1, 2, 1, 2,	[3, 1, 2, 1, 1,	T
= 17×53	30^2+1^2	<899, 60, 901>	(29, 1,	(60,	[30,	W

905	28^2+11^2	<663, 616, 905>	(5, 1, 1, 1, 1,	(5, 1, 1, 4,	[5, 1, 1, 2,	W
= 5 × **181**	29^2+8^2	<777, 464, 905>	(2, 1, 1, 1, 2, 1,	(2, 1, 1, 1, 6,	[2, 1, 1, 1, 3,	W
925	22^2+21^2	<43, 924, 925>	(43,	(21, 2,	[21, 1,	T
= 5^2 × 37	27^2+14^2	<533, 756, 925>	(13, 3,	(13, 1, 2,	[13, 1, 1,	T
	(30^2+5^2)	5^2<35, 12, 37>				
929	23^2+20^2	<129, 920, **929**>	(2, 1, 13,	(2, 1, 6, 2,	[2, 1, 6, 1,	T
937	24^2+19^2	<215, 912, **937**>	(4, 1, 7,	(4, 1, 3, 2,	[4, 1, 3, 1,	T
941	29^2+10^2	<741, 580, **941**>	(9, 1, 1, 1,	(9, 1, 4,	[9, 1, 2,	W
949	30^2+7^2	<851, 420, 949>	(2, 3, 3, 1,	(2, 3, 8,	[2, 3, 4,	W
= **13** × **73**	25^2+18^2	<301, 900, 949>	(3, 1, 1, 5,	(3, 1, 1, 2, 2,	[3, 1, 1, 2, 1,	T
953	28^2+13^2	<615, 728, **953**>	(2, 6, 1, 1,	(2, 6, 4,	[2, 6, 2,	W
965	26^2+17^2	<387, 884, 965>	(8, 1, 3,	(8, 1, 1, 2,	[8, 1, 1, 1,	T
= 5 × **193**	31^2+2^2	<957, 124, 965>	(2, 14, 1,	(2, 30,	[2, 15,	W
977	31^2+4^2	<945, 248, **977**>	(3, 1, 6, 1,	(3, 1, 14,	[3, 1, 7,	W
985	27^2+16^2	<473, 864, 985>	(5, 2, 3,	(5, 2, 1, 2,	[5, 2, 1, 1,	T
= 5 × **197**	29^2+12^2	<697, 696, 985>	(2, 2, 2, 1, 1,	(2, 2, 2, 4,	[2, 2, 2, 2,	W
997	31^2+6^2	<925, 372, **997**>	(6, 4, 1,	(6, 10,	[6, 5,	W

$p \equiv 1 \pmod 4$ and the composites of those p's, which appear as the hypotenuse c. Prime p's are printed in bold.

$m^2 + n^2 = c$ (See (2.1)). Non co-prime pairs, which do not give a pPT are parenthesized.

Examples of C(a), C(b), and C(c) are given for p=317=14^2+11^2 : C(a)=(2, 1, 7, = C_5(2, 1, 7, 1, 2), C(b)=(2, 1, 3, 2, = C_7(2, 1, 3, 2, 3, 1, 2), and C(c) = [2, 1, 3, 1, = C_8(2, 1, 3, 1, 1, 3, 1, 2).

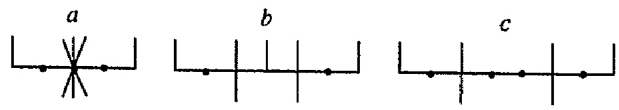


Table 2. Examples of primitive Pythagorean triples and the corresponding caterpillar graphs whose hypotenuses are highly composite

p	m^2+n^2	<a, b, c>	C(a)	C(b)	C(c)	Type
1105	33^2+4^2	<1073, 264, 1105>	(4, 7, 1,	(4, 16,	[4, 8,	W
$=5\times13$	32^2+9^2	<943, 576, 1105>	(4, 1, 1, 2, 1,	(4, 1, 1, 6,	[4, 1, 1, 3	W
$\times17$	31^2+12^2	<817, 744, 1105>	(2, 2, 1, 1, 1, 1,	(2, 2, 1, 1, 4,	[2, 2, 1, 1, 2	W
	24^2+23^2	<47, 1104, 1105>	(47,	(23, 2,	[23, 1,	T
1445	38^2+1^2	<1443, 76, 1445>	(37, 1,	(76,	[38,	W
$=5\times17^2$	31^2+22^2	<477, 1364, 1445>	(4, 2, 5,	(4, 2, 2, 2,	[4, 2, 2, 1	T
	(34^2+17^2)	17^2<3, 4, 5>				
2197	46^2+9^2	<2035, 828, 2197>	(9, 4, 1,	(9, 10,	[9, 5,	W
$=13^3$	(39^2+26^2)	13^2<5, 12, 13>				
2873	53^2+8^2	<2745, 848, 2873>	(2, 1, 1, 1, 5, 1,	(2, 1, 1, 1, 12,	[2, 1, 1, 1, 6,	W
$=13^2\times17$	43^2+32^2	<825, 2752, 2873>	(10, 1, 5,	(10, 1, 2, 2,	[10, 1, 2, 1,	T
	(52^2+13^2)	13^2<15, 8, 17>				
4225	63^2+16^2	<3713, 2016, 4225>	(15, 1, 2, 1,	(15, 1, 6,	[15, 1, 3,	W
$=5^2\times13^2$	56^2+33^2	<2047, 3696, 4225>	(3, 3, 2, 3,	(3, 3, 2, 1, 2,	[3, 3, 2, 1, 1,	T
	(60^2+25^2)	5^2<129, 120, 169>				
	(52^2+39^2)	13^2<7, 24, 25>				

$32045=5\times13\times17\times29$

m^2+n^2	<a, b, c>	C(a)	C(b)	C(c)	Type
179^2+2^2	<32037, 716, 32045>	(2, 88, 1,	(2, 178,	[2, 89,	W
178^2+19^2	<31323, 6764, 32045>	(2, 2, 1, 2, 8, 1,	(2, 2, 1, 2, 18,	[2, 2, 1, 2, 9,	W
173^2+46^2	<27813, 15916, 32045>	(2, 5, 3, 1, 2, 1,	(2, 5, 3, 1, 6,	[2, 5, 3, 1, 3,	W
166^2+67^2	<23067, 22244, 32045>	(2, 1, 10, 2, 1, 1,	(2, 1, 10, 2, 4,	[2, 1, 10, 2, 2,	W
163^2+74^2	<21093, 24124, 32045>	(2, 8, 1, 3, 1, 1,	(2, 8, 1, 3, 4,	[2, 8, 1, 3, 2,	W
157^2+86^2	<17253, 27004, 32045>	(3, 1, 2, 1, 4, 3,	(3, 1, 2, 1, 4, 1, 2,	[3, 1, 2, 1, 4, 1, 1,	T
142^2+109^2	<8283, 30956, 32045>	(3, 3, 3, 7,	(3, 3, 3, 3, 2,	[3, 3, 3, 3, 1,	T
131^2+122^2	<2277, 31964, 32045>	(4, 1, 1, 27,	(4, 1, 1, 13, 2,	[4, 1, 1, 13, 1,	T

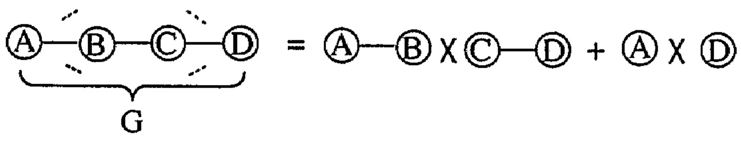
See the captions of Table 1.

Appendix

Two fundamental recursive relations for calculating TopIces of caterpillar graphs.[4,5]

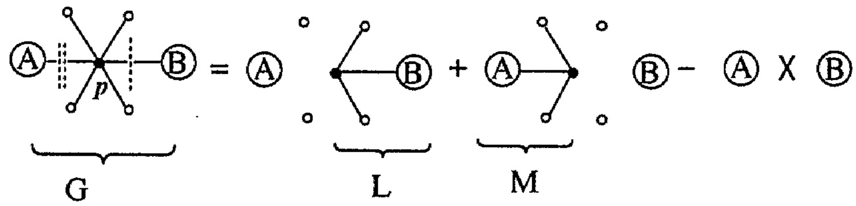
a) Edge deletion. A, B, C, and D are stars. *b*) Vertex deletion. A and B are stars.

a)



$$G = A{-}B{-}C{-}D = (A{-}B) \times (C{-}D) + A \times D$$

b)



$$G = A \times L + M \times B - A \times B$$