Enumeration of Switching Graphs of Small Orders

by Akira Kaneko[1] and Rina Nagahama

Department of Information Sciences, Ochanomizu University
and
Department of Mathematics and Information Science
Graduate School of Humanities and Sciences, Ochanomizu University

(Received April 10, 2009, Revised July 27, 2009)

Abstract

Switching graphs describe the structure of the solution set for the two projection binary tomography. It is made of vertices corresponding to the reconstructions and edges corresponding to the switching operations. In our former works we examined basic properties of this graph. Here we prove an estimation theorem for the minimal size of figures corresponding to the given switching graph. We thereby execute enumeration of the switching graphs of small orders by means of computer search. These would serve as a basic data for the complete characterization of the switching graph.

1 Introduction

Two projection discrete tomography for discrete plane figures studies the relation of a plane figure with its x- and y-projections, that is the counting data of the constructing cells of the figure in these directions. As in our pevious works, we assume the cells to be fitted to the integral lattice, and refer to the positions by the coordinates of their lower-left corners.

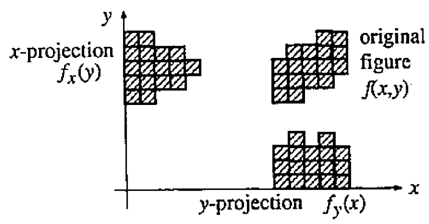


Fig. 1. x- and y-projection data.

Because of the fewness of the projection direction, the figure is generally not determined uniquely, and there are many reconstruction solutions for the same projection data. A switching graph was introduced to study the structure of the whole solution sets. It consists of vertices corresponding to the reconstruction solutions and of edges corresponding to the switching operations. Here a switching operation means the modification of the type of a switching component: A pair of cells (x_1, y_1), (x_2, y_2) in a figure F is called a switching component if the two positions (x_1, y_2), (x_2, y_1) are vacant in F. It is said to be of type 1 or type 2, respectively, if $x_2 - x_1$ and $y_2 - y_1$ have the same or the opposite sign.

Fig. 2. Switching operation, from type 2 to type 1.

The switching operation obviously preserves the two projection data, and conversely, the existence of such a pair is equivalent to the non-uniqueness of the reconstruction. Also, we can make the graph directed, by assigning the arrow of the edge to the above direction of modification.

This graph was first introduced by Brualdi [1] under the name of interchange graph. Later, [6] re-introduced it and called Ryser graph. Similar graphs are also used in statistics to study marginal distributions. Unaware of these, we called it the switching graph and studied its properties. The notion of switching digraph seems to have been introduced by us [3] for the first time. Also we considered a version with constraint [4].

[1] Partially supported by Grant-in-Aid for Scientific Research No. 20540157

In this article we prove an estimation theorem for the minimal size of figures which generate a given switching graph. This enables to confine the search region to a finite size. We thereby executed enumeration of the switching graphs up to order 20 with the aid of computers. The obtained list would serve as a basic data for the complete characterization of the switching graph. Though similar study is possible for the case with constraint or for the directed version. Here we only treat the original switching graphs.

For general exposition on discrete tomography see [8], although we do not employ any further knowledge about it.

2 Basic examples of general order

In this section we gather examples of switching graphs which constitute series of graphs with parameterized orders. in enumeration of switching graphs, we first have to take into account ones coming from these. In the description below, we denote the place of a unit cell by the coordinates of, say, its lower-left corner as in [3]. (1) to (5) are given there, and (6) to (8) are newly given here.

(1) The switching graph Δ_n of a simple diagonal linear figure $F = \{(i, i); i = 1, 2, \ldots, n\}$ (Fig. 3 left) is a bipartite regular graph of degree $\dfrac{n(n-1)}{2}$ with $n!$ vertices and $\dfrac{n^2(n-1)^2}{4}(n-2)!$ edges. ([3], Lemma3.1.) The order sequence for this is $2, 6, 24, \ldots$.

(2) The switching graph $Z_{m,n}$ of two slided shells each of size m, n, $F = \{(i, 1) \mid i = 1, \ldots, m\} \cup \{(m+j, 2) \mid j = 1, \ldots, n\}$ (Fig. 3 center), is a regular of degree mn with $\dfrac{(m+n)!}{m!n!}$ vertices, and $\dfrac{(m+n)!}{2(m-1)!(n-1)!}$ edges. ([3], Lemma3.4.) The order sequence for this (other than the special case $n = 1$) is $6(m = n = 2), 10(m = 3, n = 2), 15(m = 4, n = 2), 20(m = n = 3), \ldots$.

(3) As a special case of the above, $m \mapsto n-1$, $n \mapsto 1$ of the above, the switching graph $Z_{n-1,1}$ of $F = \{(i, 1) \mid i = 1, \ldots, n-1\} \cup \{(n, 2)\}$ (Fig. 3 right) is the complete graph K_n of order n. The order sequence is $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$.

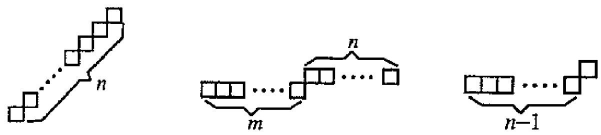


Fig. 3. Δ_n, $Z_{m,n}$ and K_n.

(4) The switching graph $S_{m,n}$ of $F = \{(i, j) \mid i = 1, \ldots, m, j = 1, \ldots, n\} \cup \{(m+1, n+1)\}$ (Fig. 4 left) is a suspension of a regular graph of order mn and degree $m+n-2$ by another vertex, hence has $mn+1$ vertices and $\dfrac{mn(m+n)}{2}$ edges ([3], Lemma 3.5). Especially, $S_{1,n} = K_{n+1}$. The order sequence (except for the special case $m = 1$) is 5 $(m = n = 2)$, 7 $(m = 3, n = 2)$, 10 $(m = n = 3)$, 13 $(m = 4, n = 3;\ m = 6, n = 2)$, 15 $(m = 7, n = 2)$, 16 $(m = 5, n = 3)$,.... By the invariance property of Lemma 3.1, the same graph is obtained by a figure of Fig. 4 center. Since this latter is simpler, we shall rather denote by $S_{m,n}$ this one in the sequel.

(5) The switching graph $L_{m,n}$ of $F = \{(i, 1) \mid 1 \le i \le m\} \cup \{(j+m, j+1) \mid 1 \le j \le n\}$ (Fig. 4 right) is a regular graph of degree $\dfrac{n(n+2m-1)}{2}$ with $\dfrac{(n+m)!}{m!}$ vertices and $\dfrac{n(n+2m-1)(n+m)!}{4m!}$ edges, obtained by contraction from Δ_{m+n} by reducing each subgraph Δ_m in the latter to one vertex. (See [3], Corollary 4.6. We changed the parameterization of this figure F there a little. Anyway the original definition was incorrectly starting from the cell $(0, 1)$, which should be $(1, 1)$.)The order sequence (except for the special cases $L_{0,n} = \Delta_n$, $L_{1,n} = \Delta_{n+1}$, $L_{m,1} = K_{m+1}$) is $12(m = 2, n = 2), 20(m = 3, n = 2), \ldots$.

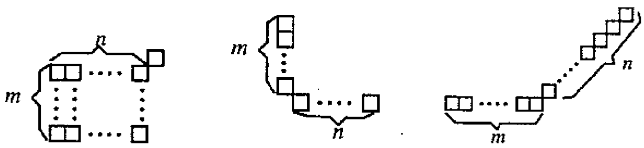


Fig. 4. $S_{m,n}$ and $L_{m,n}$.

(6) The switching graph $S_{m,n,k}$ of the figure to the left of Fig. 5 below consists of $(mn-n+1)k+mn+1$ vertices and has $\frac{k(k+1)}{2}(mn-n+1)+k\frac{mn(m+n)-n(n-1)}{2}+\frac{mn(m+n)}{2}$ edges. In fact, this is a composite graph made by setting $S_{m,n}$ to each vertex of K_{k+1}, pasting each other by the common subgraph K_n corresponding to the figures where the last $k+1$ cells are all in the ground state. See the right of Fig. 5 below, where the case $m=3$, $n=2$, $k=2$ is shown, and the shadowed cells do not move inside the respective groups. Thus denoting by $v(G)$ resp. $e(G)$ the vertex resp. edge sets of G and by $|S|$ the cardinal of the set S, the number of vertices is calculated as

$$(k+1)|v(S_{m,n})\setminus v(K_n)|+|v(K_n)|=(k+1)(mn+1-n)+n=k(mn-n+1)+mn+1,$$

and the number of edges is equal to

$$\begin{aligned}&|e(K_{k+1})||v(S_{m,n})\setminus v(K_n)|+|v(K_{k+1})||e(S_{m,n})\setminus e(K_n)|+|e(K_n)|\\&=\frac{k(k+1)}{2}(mn+1-n)+(k+1)\frac{mn(m+n)-n(n-1)}{2}+\frac{n(n-1)}{2}\\&=\frac{k(k+1)}{2}(mn-n+1)+k\frac{mn(m+n)-n(n-1)}{2}+\frac{mn(m+n)}{2}\end{aligned}$$

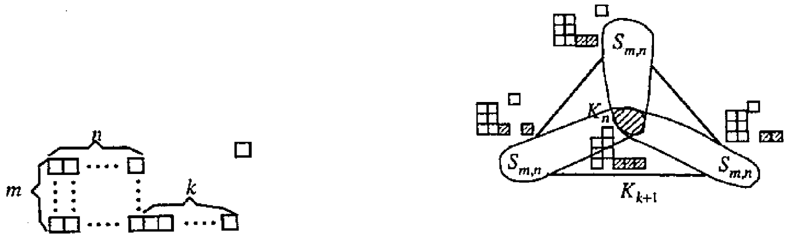


Fig. 5. $S_{m,n,k}$.

Obviously $S_{m,n,0}$ reduces to $S_{m,n}$ of (4). Also, $S_{m,1,k}$ is equivalent to $S_{m,k+1}$. Furthermore, $S_{m,n,1}$ is reduced to $S_{m,2,n-1}$ by taking the complement and switching once. Since the latter always has smaller number of cells for $m\ge 2$, $n\ge 3$, we shall abandon the former. Omitting these, the order sequence for this begins with 8 ($S_{2,2,1}$), 11 ($S_{2,2,2}$), 12 ($S_{3,2,1}$), 14 ($S_{2,2,3}$), 15 ($S_{2,3,2}$), 16 ($S_{4,2,1}$), 17 ($S_{2,2,4}$, $S_{3,2,2}$), 19 ($S_{2,3,3}$), 20 ($S_{5,2,1}$, $S_{2,2,5}$)....

(7) More generally, the switching graph $S_{m,n,l,k}$ of Fig. 6 left has $kl(mn-ln+1)+mn+1$ vertices and $\frac{lk(k+1)}{2}(mn-ln+1)+\frac{lk}{2}[-\{(m+n)n+n-1\}l+mn(m+n+1)+2n-1]+\frac{mn(m+n)}{2}$ edges. In fact, this is a composite graph of l copies of $S_{m-l+1,n,k}$ pasted together with the common subgraph $S_{m-l,n}$ corresponding to those in which the upper-right free cell is above the l-th row. Thus a calculation similar to the above yields

$$\begin{aligned}|v(G)|&=l(|v(S_{m-l+1,n,k})\setminus v(S_{m-l,n})|)+|v(S_{m-l,n})|\\&=l[\{(m-l+1)n-n+1\}k+(m-l+1)n+1-\{(m-l)n+1\}]+(m-l)n+1\\&=kl(mn-ln+1)+mn+1.\end{aligned}$$

The number of edges is more complicated. The first group of edges is l copies of those of $S_{m-l+1,n,k}$ except for those inside $S_{m-l,n}$ which appear only once. The second group is those between these independent l subgraphs which appear as switchings like Fig. 6 right.

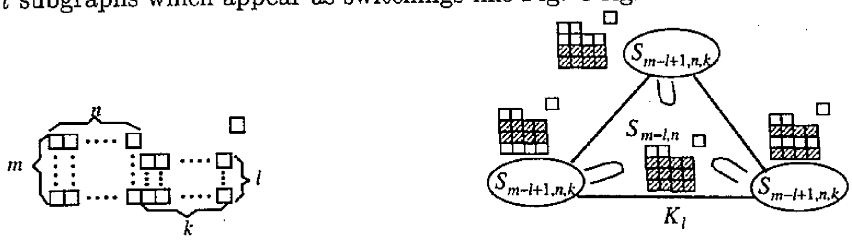


Fig. 6. $S_{m,n,l,k}$.

$$
\begin{aligned}
|e(G)| &= l|e(S_{m-l+1,n,k}) \setminus e(S_{m-l,n})| + |e(S_{m-l,n})| + \frac{l(l-1)}{2}|v(S_{m-l+1,n,k}) \setminus v(S_{m-l,n})| \\
&= l\Big[\frac{k(k+1)}{2}\{(m-l+1)n - n + 1\} + k\frac{(m-l+1)n(m-l+1+n) - n(n-1)}{2} \\
&\quad + \frac{(m-l+1)n(m-l+1+n)}{2} - \frac{(m-l)n(m-l+n)}{2}\Big] \\
&\quad + \frac{(m-l)n(m-l+n)}{2} \\
&\quad + \frac{l(l-1)}{2}[\{(m-l+1)n - n + 1\}k + (m-l+1)n + 1 - \{(m-l)n+1\}] \\
&= \frac{lk(k+1)}{2}(mn - ln + 1) \\
&\quad + \frac{lk}{2}[-\{(m+n)n + n - 1\}l + mn(m+n+1) + 2n - 1] \\
&\quad + \frac{mn(m+n)}{2}
\end{aligned}
$$

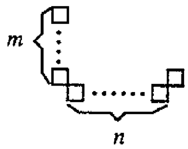
Obviously $S_{m,n,1,k} = S_{m,n,k}$. Also, by passing to the symmetry, $S_{m,n,l,k}$ changes to $S_{n+k,l,n,m-l}$. Especially, $S_{m,1,l,k}$ reduces to $S_{k+1,l,m-l}$.

(8) The switching graph of $T_{m,n} = \{(1, j+1) \mid 1 \le j \le m\} \cup \{(i+1, 1) \mid 1 \le i \le n\} \cup (n+2, 2)$ as in Fig. 7 has $\frac{(n+1)(2mn-n+2)}{2}$ vertices and $\frac{n(n+1)}{2}(m^2 + 2mn - n + 1)$ edges.
In fact, this is a concatenation of $n+1$ copies of S_{mn} which doubly have each of the figures with two cells at $(i, 2)$, $2 \le i \le n+2$. Thus

$$
|v(T_{m,n})| = |v(K_{n+1})||v(S_{m,n})| - |v(Z_{n-1,2})| = (n+1)(mn+1) - \frac{n(n+1)}{2} = \frac{(n+1)(2mn-n+2)}{2}.
$$

The edges come from $S_{m,n}$ except for the doubly counted ones plus those between the corresponding pairs of vertices in the different copies of $S_{m,n}$. Thus

$$
\begin{aligned}
|e(T_{m,n})| &= |v(K_{n+1})||e(S_{m,n})| - |e(Z_{n-1,2})| + |e(K_{n+1})|\{|v(S_{m,n})| - n\} + |e(Z_{n-1,2})| \\
&= (n+1)\frac{mn(m+n)}{2} + \frac{n(n+1)}{2}(mn - n + 1) = \frac{n(n+1)}{2}(m^2 + 2mn - n + 1)
\end{aligned}
$$



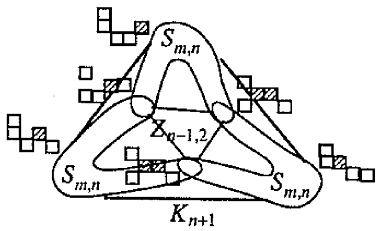


Fig. 7. $T_{m,n}$.

The order sequence of this series contains 12 $(m = n = 2)$, 18 $(m = 3, n = 2)$, 22 $(m = 2, n = 3)$, 24 $(m = 4, n = 2)$, ...

3 Basic invariance properties of switching graphs

In this section we gather basic properties of switching graphs which are useful to enumerate them. We also review some standard way of constructing switching graphs from known ones.

Lemma 3.1 *The switching graph of F does not change if we apply the following transformation to F:*

(i) *Translation to x or y directions.*

(ii) *Reflection with respect to a horizontal, vertical or diagonal line (equivalently, exchange of x and y coordinates).*

(iii) *Rotation by 90 degrees.*
(iv) *Permutation of columns or rows.*
(v) *The complement $\complement F$ of F, obtained by changing cells to blank and vice versa in its enclosing rectangle, has the same switching graph as F.*

Proposition 3.2 (Direct product; [3], Proposition 4.8) *The switching graph of the block figure as in the figure below, where G resp. H (by abuse of notation) stands for a figure corresponding to the switching graph G resp. H, is the direct product $G \times H$ of the two graphs: Its vertices are the product set $v(G) \times v(H)$, and its edges are those between (a,c) and (b,c) inherited from the one between a and b in G, and those between (a,c) and (a,d) inherited from the one between c and d in H.*

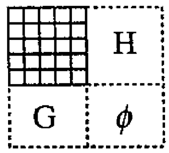


Fig. 8. Direct product

Remark 1. The transformations of Lemma 3.1 applied separately to the factors G, H obviously make the switching graph of the product $G \times H$ invariant. These transformations are, however, not realizable as those of Lemma 3.1 applied to the whole $G \times H$ except for the permutation of columns or rows. We shall extend the meaning of the transformations in Lemma 3.1 as allowed to apply separately to the components of the direct product.

Proposition 3.3 (Contraction; [3], Proposition 4.12) *Assume that a row of a pattern contains m cells, each of which is unique in their columns. Then the switching graph G for this pattern can be obtained from the graph H of the new pattern which has these m cells in new independent m rows as follows: Search the vertices of H which correspond to the permutations of m cells in these new m rows. Contract these to one vertex and make a reduced graph. If these m cells are not in separate columns, delete the corresponding vertices, together with the edges emanating from these to other vertices. The degree at each vertex is diminished by $\frac{m(m-1)}{2}$ by this contraction, and if there are deleted vertices, more edges are deleted. Thus a regular graph produces a regular graph again by this process if and only if the remaining cells are all in different columns. The same assertion holds for a column.*

4 Estimate of the minimal size of figures for a given switching graph

In this section, we give a fundamental estimate on the minimal size of figures which have a given switching graph.

First we prepare several terms. A column in a figure, of which all the cells are immobile throughout the switchings, will be called *redundant*. Obviously, we can delete redundant columns without changing the associated switching graph. (Note, however, that we cannot necessarily delete immobile cells individually, as those appearing in the product type figure.) A column which is not redundant will be called *effective*. We use the same terminology for rows.

It is clear that a column which contain a cell switching with another one in a figure is effective. The following converse, however, is not obvious, because a cell initially immobile might become movable after switchings of the cells in the other columns.

Lemma 4.1 *Assume that a column C of a figure F has no cells which can switch with another cell. Then C is redundant, that is, its cells do not switch in any figure obtained from F by a series of switching operations.*

Proof Assume that after a series of switching operations in the other columns, a cell P of C becomes switchable with, say Q. We consider the shortest one among such series of switchings. By permutations of columns and rows, we can assume without loss of generality that C is the first column, its cells filling the continuous positions from the bottom line of which P is the top, whereas Q is in the second column and above P. (We can also assume without loss of generality, that Q is one cell above P.) See Fig. 9 center.

Assume that just before the switching of P, Q, the cell R where Q should go switches with another cell S in a column other than the first two, thereby offering the place for the switching of P, Q. If S is below P as in Fig. 9 left (which we can also assume without loss of generality to be one cell below P), then the place where R should go is vacant, hence Q can switch with the cell in the first column at the height S at one step earlier. This contradicts to the choice of shortest series. If S is above P as in Fig. 9 right, then S can switch with P also at one step earlier, again contradicting to the assumption of shortest series. Thus P should be able to switch with another cell from the beginning.

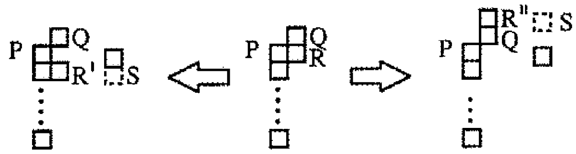


Fig. 9. Proof of Lemma 4.2.

It remains the case where the position R is vacant but Q comes to this place in one step earlier switching with say, T. Then T should be at the height of present Q. If the former place Q' of Q is below P, then the cell of C at this height switches with T at one step earlier. If Q' is above P, then it can switch with P at one step earlier, because there are no cells above P in C. Thus in any case they contradict to the shortestness assumption. QED

After [6] two figures F and F' are *equivalent* if they can be transformed from one to the other by a series of switching operations. We shall extend this notion and say that two figures F and F' are *T-equivalent* if after transformations of Lemma 3.1 (including Remark 1) and omission or adding of redundant columns or rows, they become equivalent. We shall further say that F and F' are *G-equivalent* if their switching graphs $G(F)$, $G(F')$ are isomorphic. We shall denote the equivalence, T-equivalence, and G-equivalence by $F \equiv F'$, $F \approx F'$, and $F \sim F'$, respectively. Obviously,

$$F \equiv F' \quad \Longrightarrow \quad F \approx F' \quad \Longrightarrow \quad F \sim F'.$$

One of the fundamental open questions about switching graphs is if conversely $F \sim F'$ implies $F \approx F'$.

Theorem 4.2 *A switching graph of order n can be realized by a figure of cells contained in $n \times n$ square.*

Proof We prove the following contraposition to the theorem, employing an induction on the number of effective columns n:

Claim *The switching graph $G(F)$ of a figure F with n effective columns has order $\geq n$.*

In the sequel, for a graph G, let $\mathrm{ord}(G)$ denote its order $|v(G)|$. For any figure F with $n = 2$ effective columns, $\mathrm{ord}(G(F)) \geq 2$ is obvious, because the existence of an effective column implies existence of at least two different figures. Assume that the Claim holds for any figure with up to $n-1$ effective columns, and take a figure F with n columns, all effective. Consider the first column C, and the figure F' obtained by deleting C from F. $G(F')$ can obviously be regarded as a full subgraph of $G(F)$.

If all the columns of F' are still effective, then by the induction hypothesis $G(F')$ has order $\geq n-1$. The fact that C is effective implies the existence of another figure with different pattern at the first column, which is clearly a vertex of $G(F)$ not contained in $G(F')$. Thus in this case $\mathrm{ord}(G(F)) \geq \mathrm{ord}(G(F')) + 1 \geq n$.

Assume next that F' contains a redundant column, say C'. This means that C' contains no cell switching with the other columns of F'. Since it was effective in F, this means, in view of Lemma 4.1, that a cell of C' must switch with some cell of C. If the figure, obtained from F by deleting C' instead of C, has no redundant columns, then the induction works just as above. If not, it means that the cells of C switches only with C' and vice versa. In this case, the figure F'' obtained from F by deleting the two columns C, C' must consist of $n-2$ effective columns. In fact, they were effective in F, hence in view of

Lemma 4.1 they must contain a cell switching with a cell in another column, which is neither C nor C' by what is mentioned above.

Thus two columns C, C' contain a switching component which switches independently of the cells in F''. Thus by the induction hypothesis $\operatorname{ord} G(F'') \geq n-2$, and $G(F)$ has at least $2 \times \operatorname{ord} G(F)$ vertices corresponding to the switching in the columns C, C'. Thus $\operatorname{ord} G(F) \geq 2(n-2)$. This is $\geq n$ if $n \geq 4$. On the other hand, in case $n=3$, the present situation does not occur, because the only remaining column would then be redundant in F.

Thus the claim was proved. It follows that we can realize a switching graph of order n by a figure with at most n effective columns. The argument applies also to the rows with obvious paraphrase. Thus the theorem is proved. QED

By this theorem, the list of examples of the switching graphs of small orders which we reported earlier was justified with the aid of the computer search. The above proof, when precisely examined, implies the following, which is less elegant but it is practically meaningful and makes further search more effective. In the description below, for the sake of simplicity we shall denote (2) and (3) for K_2 and K_3, respectively. (This is the label given in the list of switching graphs in §5.)

Corollary 4.3 *A switching graph of order n is realized by a figure in $\lfloor \frac{n}{2}+2 \rfloor \times \lfloor \frac{n}{2}+2 \rfloor$ square of cells except for the case of the complete graph of order n realized by the figure K_n. Furthermore, It is realized by a figure in $\lfloor \frac{n+5}{3} \rfloor \times \lfloor \frac{n+5}{3} \rfloor$ square of cells except for K_n, $K_{n/2+2} \times (2)$, $S_{(n-1)/2,2}$, $K_{n/3} \times (3)$, $K_{n/3} \times S_{1,2}$, $K_{n/4} \times (2) \times (2)$ for $n = 8, 12, 16, 20$, and $(2) \times (2) \times (2) \times (2)$. (The case where the fraction in the index is non-integer is omitted.)*

Proof We also prove the contraposition:

Claim *The order n of the switching graph $G(F)$ of a figure F with k effective columns is $\geq 3k-5$ except for the following (neglecting redundant rows):*

(i) *k of K_k, for $k \geq 3$,*

(ii) *$2k-4$ of $K_{k-2} \times (2)$, for $k \geq 4$,*

(iii) *$2k-1$ of $S_{k-1,2}$, for $k \geq 3$,*

(iv) *$3k-9$ of $K_{k-3} \times (3)$, for $k \geq 5$,*

(v) *$3k-6$ of $K_{k-2} \times S_{1,2}$, for $k \geq 4$,*

(vi) *$4k-16$ of $K_{k-4} \times (2) \times (2)$, for $6 \leq k \leq 10$,*

(vii) *$2^{k/2}$ of $(2) \times (2) \times (2) \times (2)$, for $k = 8$.*

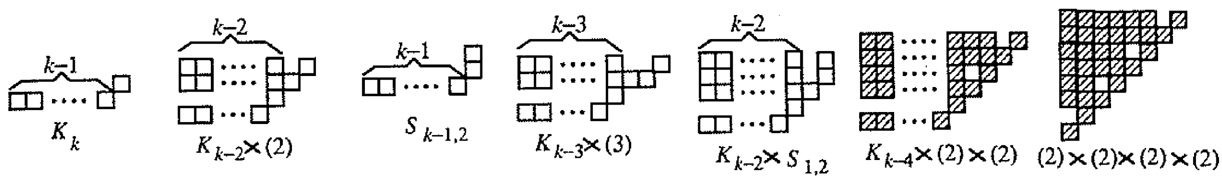


Fig. 10. Exceptional figures.

Practically we can omit (v) from our enumeration because it is essentially equivalent to (iv). (As remarked in Remark 1 after Proposition 3.2, the notion of T-equivalence is extended to these two figures.)

Now we prove Claim. For $k \leq 4$ the assertion is obvious from the list given in the next section (which is easily verified by the estimate of Theorem 4.2 only). Assume $k \geq 5$, and the assertion holds up to $k-1$.

Consider first the case where there is a column C such that the deletion of C produces a figure F' with no redundant columns. We divide the cases.

(i) Case where F' is equal to K_{k-1}.

(a) If the column C contains only one cell and it is at the height of the major group of cells of K_{k-1}, then F is equal to K_k, whence reduces to the exceptional case (i).

(b) If C contains only one cell but it is at the height of the minor one, then F is equivalent to $Z_{k-2,2}$, hence the switching graph $G(F)$ has order $\frac{k(k-1)}{2}$, which is $\geq 3k-5$ for $k \geq 5$. Thus the induction proceeds in this case. This estimate is valid even if there exist cells at the heights other than these two of K_{k-1}.

(c) If C is vacant at the two heights of K_{k-1}, then it contains at least one cell at the third height, and $G(F)$ contains a subgraph corresponding to $L_{k-2,2}$ of order $k(k-1)$, which is $\geq 3k-5$ for all k. Thus the induction proceeds also in this case.

(d) If C consists of a cell at the height of the major group and another cell at a third height, then after one switching operation the figure becomes equivalent to $S_{k-1,2}$, hence $G(F)$ has order $2k-1$, being the exceptional case (iii).

(e) If C has still other cells besides (d), then after one switching operation the figure contains a subfigure equivalent to $S_{3,k-1}$ whose switching graph has order $3k-2$. Hence the induction proceeds. Note that C does not have cells at both heights of F', because then it would be redundant.

(i)-(a) (i)-(b) (i)-(c) (i)-(d) (i)-(e)

Fig. 11. Proof of Claim; Case (i). (Cells of C are shadowed.)

(ii) Case where F' is equal to $K_{k-3} \times (2)$.

(a) If the deleted column C is such that it extends the part K_{k-3} to K_{k-2}, then $F = K_{k-2} \times (2)$ and the switching graph has order $2(k-2) = 2k-4$, reducing to the exceptional case (ii).

(b) If C is such that it extends the part (2) of F' to (3), then $F = K_{k-3} \times (3)$ and the switching graph has order $3(k-3) = 3k-9$, which is the exceptional case (iv).

(c) For all the other configurations of C, the switching graph of F has more vertices, as is verified in an elementary way. To verify this the most delicate case will be one giving rise to the product $Z_{k-4,2} \times (2)$. But then $G(F)$ has order $2(k-2)(k-3)$, which is $\geq 3k-5$ for $k \geq 5$.

(iii) Case where F' is equal to $S_{k-2,2}$. Then the switching graph of F' has at least $2(k-1)-1 = 2k-3$ vertices.

(a) If C has only one cell at the height of the major part of $S_{k-3,2}$, then F is equal to $S_{k-2,2}$, being the exceptional case (ii).

(b) If C has one cell at the height of one of the minor parts of $S_{k-3,2}$, then F contains a subfigure equivalent to $Z_{k-2,2}$, of which the switching graph has order $\frac{k(k-1)}{2} \geq 3k-5$ for $k \geq 5$ as before.

In all the other cases $G(F)$ has order $\geq 3k-5$ provided that C is effective.

(iv) Case where $F' = K_{k-4} \times (3)$. If C consists of a cell at the height of the major part of K_{k-4}, then F is equivalent to $K_{k-3} \times (3)$, the exceptional case (iv). For all the other configurations of C the resulting F has the switching graph of order $\geq 3k-5$, as is easily verified.

(v) Case where $F' = K_{k-3} \times S_{1,2}$. In this case, the adding of a cell in C to the height of the major part of the factor K_{k-3} gives an exception of the type $K_{k-2} \times S_{1,2}$. All the other cases, as is easily verified, produces a figure F such that $G(F)$ has order $\geq 3k-5$.

(vi) Case where $K_{k-5} \times (2) \times (2)$. In this case, the adding of a cell in C to the height of the major part of the factor K_{k-5} gives an exception of the type $K_{k-4} \times (2) \times (2)$. All the other cases, as is easily verified, produces a figure F such that $G(F)$ has order $\geq 3k-5$.

(vii) The other cases. By the induction hypothesis, the switching graph $G(F')$ of F' has at least $3(k-1)-5 = 3k-8$ vertices. Thus it suffices to show that $G(F)$ has at least 3 more vertices than $G(F')$. Assume that $G(F')$ has order $< 3k-5$, since otherwise we have nothing to do at this step.

(a) If there exist three cells in C which move by switching operations, then the obtained patterns are distinct from those of the subgraph $G(F')$ already at the first column, as in Fig. 12 (a). Thus we assume hereafter that at most two cells of C can switch.

(b) Since C is effective in F, there is at least one switching component P, Q with $P \in C$, $Q \in F'$, which, without loss of generality, we can assume to be at the upper-left corner as in Fig. 12 (b). Let $\widetilde{F'}$ be the figure obtained by deleting the first column after switching this. It differs from F' only by the position of Q. If $\widetilde{F'}$ admits two switching operations, we obtain three new patterns, and the induction proceeds. Thus assume that $\widetilde{F'}$ admits at most one switching.

(c) Assume first that $\widetilde{F'}$ has no switching components. The structure of such a figure is well known and re-described in Lemma 4.4 below. $\widetilde{F'}$ differs, however, with F' only by the position of Q, and the latter has no redundant columns. Thus F' should be T-equivalent to a figure as in Fig. 12 (c). Since the case $F' \approx K_{k-1}$ is now omitted, F thus contains at least a subfigure T-equivalent to $S_{2,1,k-2}$ whose switching graph has $(k-2)(k-1) \geq 3k-5$ vertices for $k \geq 5$.

(d) Assume next that $\widetilde{F'}$ has only one switching component. Such a figure is described in Lemma 4.4 below. Since again F', differing from it only by the position of Q, has no redundant columns, Q should belong to the unique switching component or the longest row, and it should occupy a place as in Fig. 12 (d), omitting redundant rows. Then it is easily verified that the switching graph of F has more vertices than the case (c), irrespective of the pattern of C.

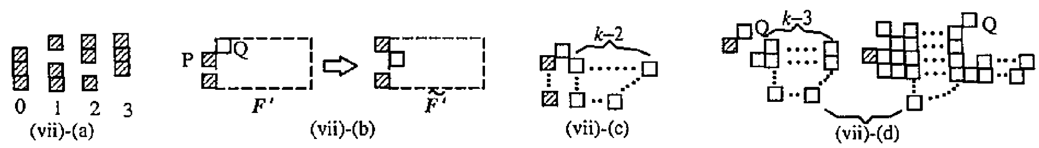


Fig. 12. Proof of Claim; Case (vii). (Cells of C are shadowed.)

Next consider the case where deletion of any column produces a redundant column. In this case, as was discussed in the proof of Theorem 4.2, the columns of F are grouped to pairs such that each pair contains an independent switching component. Thus $k = 2l$ and $G(F)$ has order at least $2^l = 2^{k/2}$. This is $\geq 3k-5$ for $k \geq 10$. The cases $k = 4$, $k = 6$ are contained in the exceptional cases (ii), (vi), respectively. Thus $k = 8$ is the final exception.

Thus we have proved the contraposition. The obtained inequality $n \geq 3k-5$, when solved for k becomes $k \leq \dfrac{n+5}{3}$. Since k is integer, this can be replaced by $k \leq \lfloor\dfrac{n+5}{3}\rfloor$. Thus the main part of Corollary is proved. For the first assertion of simpler estimate, it suffices to verify that the exceptional cases (ii) to (vii) all satisfies the weaker estimate $\geq 2k-4$. QED

Lemma 4.4 *A figure is unique, that is, admits no switching operations and hence has switching graph with only one vertex, if and only if via permutation of columns and of rows, it reduces to the union of rectangular rows with decreasing widths. A figure has switching graph with only one edge, that is, admits only one switching operation, if and only if via permutation of columns and of rows it reduces to the form as in Fig. 13 right. Namely, it contains a switching component, the columns and rows containing it consist of pairs of cells, and the remaining part at the lower-right corner is a unique figure which is filled with cells at the position of which the column and row both have cells at those of the switching component.*

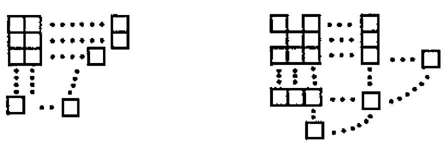


Fig. 13. Figures with zero or one switching.

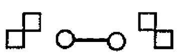
Proof The first assertion is already noticed in Lemma 2.1 of [4], with columns instead of rows. The second assertion follows by an elementary argument: Let the switching component be placed at the upper-left corner. Then the part of rows in the columns of these two should be either both filled with cells or both empty, otherwise a second switching occurs using the cells of the first switching component. Also the part with these two columns or rows omitted should be a unique figure. The final point to be noted is that if there is a hole at a position of which the column and row both have cells at those of the switching component, a new switching occurs using the cells of the first switching component, too. QED

We may continue to refine the classification of Corollary 4.3. But the above estimate suffices to do the search up to $n = 21$, since the range by the computer search can thereby be limited to 8×8, which is favorable for programming. Thus we applied our program developped for [3] to construct a switching graph to every figure realized in this square and picked up those with the indicated order n, besides those exceptional patterns listed in Corollary 4.3. In practice, to annihilate the duplication in search, we assumed that the height of the initial figure is $\leq$ the width, the Hamming weight of the rows are in non-increasing order upward, the lowest row has cells concentrated to the left, the second row has cells concentrated to the left in both regions above the cells of the first row and outside. We further avoided the transformation to the complement by assuming that the total number of cells does not exceed the half of that of the enclosing square. Also we can omit the case where the lowest row is full of cells, because then it would be redundant.

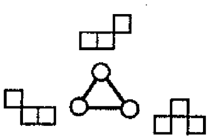
5 enumeration of switching graphs up to order 20

Now we list up switching graphs up to order 20 searched by means of both theory and computation as described at the end of the preceding section. We joined the total number of edges and arranged the graphs in the decreasing order thereof for each assigned order.

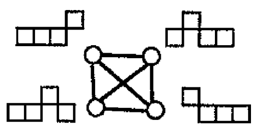
Order 2 to 4



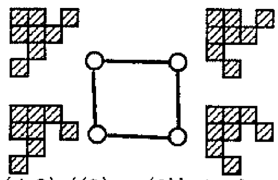
(2) ($K_2 = S_{1,1}$) 1 edge



(3) (K_3) 3 edges

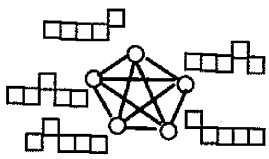


(4-1) (K_4) 6 edges

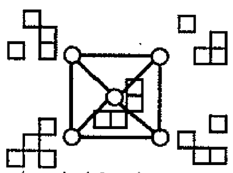


(4-2) ((2) $\times$ (2)) 4 edges

Order 5

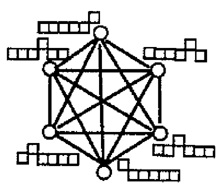


(5-1) (K_5) 10 edges

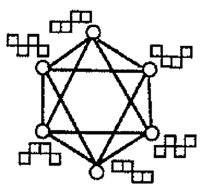


(5-2) ($S_{2,2}$) 8 edges

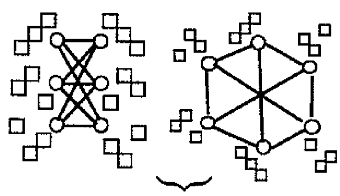
Order 6



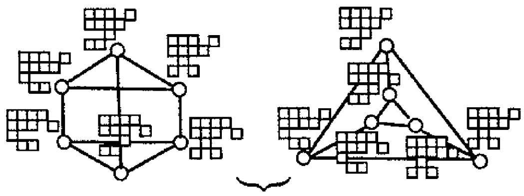
(6-1) (K_6) 15 edges



(6-2) ($Z_{2,2}$) 12 edges

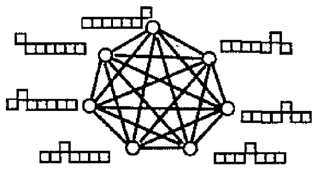


(6-3) (Δ_3) 9 edges

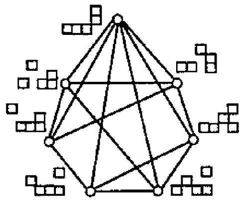


(6-4) ((3) × (2)) 9 edges

Order 7

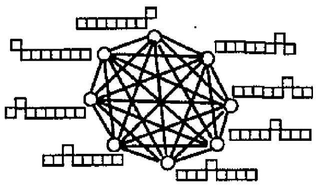


(7-1) (K_7) 21 edges

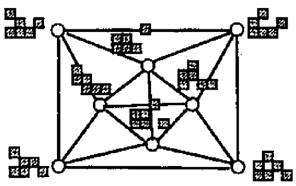


(7-2) ($S_{3,2}$) 15 edges

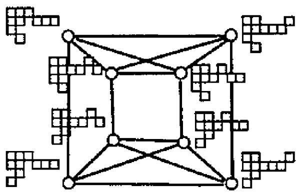
Order 8



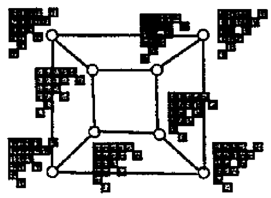
(8-1) (K_8) 28 edges



(8-2) ($S_{2,2,1}$) 18 edges

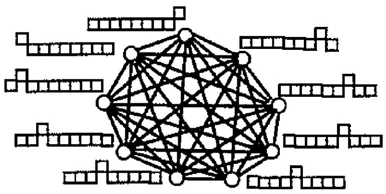


(8-3) ($K_4\times$ (2)) 16 edges

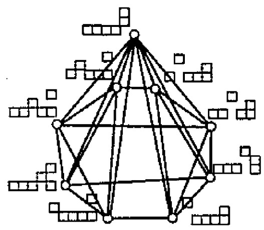


(8-4) ((2)×(2)×(2)) 12 edges

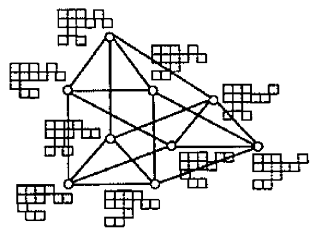
Order 9



(9-1) (K_9) 36 edges



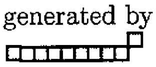
(9-2) ($S_{4,2}$) 24 edges



(9-3) ((3)×(3)) 18 edges

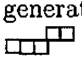
Order 10

K_{10} generated by



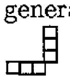
(10-1) 45 edges

$Z_{3,2}$ generated by



(10-2) 30 edges

$S_{3,3}$ generated by



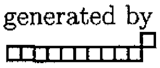
(10-3) 27 edges

$K_5\times(2)$ (10-4) 25 edges

$S_{2,2}\times(2)$ (10-5) 21 edges

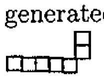
Order 11

K_{11} generated by

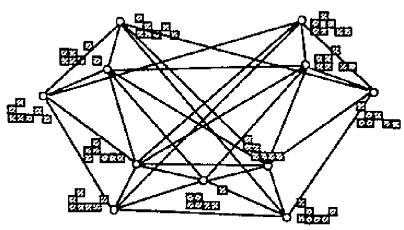


(11-1) 55 edges

$S_{5,2}$ generated by



(11-2) 35 edges

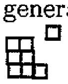


(11-3) ($S_{2,2,2} \approx S_{2,3,1}$) 31 edges

Order 12

K_{12} generated by  (12-1) 66 edges

$K_6 \times (2)$ (12-2) 36 edges

$S_{3,2,1}$ generated by  (12-3) 34 edges

$T_{2,2}$ generated by  (12-4) 33 edges

L_{22} generated by  (12-5) 30 edges

$Z_{2,2} \times (2)$ (12-6) 30 edges

$K_4 \times (3)$ (12-7) 30 edges

$\Delta_3 \times (2)$ (12-8) 24 edges

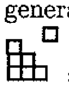
$(3) \times (2) \times (2)$ (12-9) 24 edges

Order 13

K_{13} (13-1) 78 edges

$S_{6,2}$ (13-2) 48 edges

$S_{4,3}$ (13-3) 42 edges

generated by  $\approx S_{3,2,2,1}$ (13-4) 38 edges

Order 14

K_{14} (14-1) 91 edges

$K_7 \times (2)$ (14-2) 49 edges

$S_{2,2,3}$ (14-3) 47 edges

$S_{3,2} \times (2)$ (14-4) 37 edges

Order 15

K_{15} (15-1) 105 edges

$S_{7,2}$ (15-2) 63 edges

$Z_{4,2}$ (15-3) 60 edges

$S_{2,3,2}$ (15-4) 51 edges

$K_5 \times (3)$ (15-5) 45 edges

$S_{2,2} \times (3)$ (15-6) 39 edges

Order 16

K_{16} (16-1) 120 edges

$K_8 \times (2)$ (16-2) 64 edges

$S_{5,3}$ (16-3) 60 edges

$S_{4,2,1}$ (16-4) 54 edges

$K_4 \times K_4$ (16-5) 48 edges

$S_{2,2,1} \times (2)$ (16-6) 44 edges

$K_4 \times (2) \times (2)$ (16-7) 40 edges

$(2) \times (2) \times (2) \times (2)$ (16-8) 32 edges

Order 17

K_{17} (17-1) 136 edges

$S_{8,2}$ (17-2) 80 edges

$S_{2,2,4}$ (17-3) 66 edges

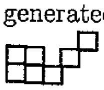
$S_{4,4}$ (17-4) 64 edges

$S_{3,2,2}$ (17-5) 58 edges

Order 18

K_{18} (18-1) 153 edges

$K_9 \times (2)$ (18-2) 81 edges

generated by  (18-3) 63 edges

$K_6 \times (3)$ (18-4) 63 edges

$T_{3,2}$ (18-5) 60 edges

$S_{4,2} \times (2)$ (18-6) 57 edges

$Z_{2,2} \times (3)$ (18-7) 54 edges

$\Delta_3 \times (3)$ (18-8) 45 edges

$(3) \times (3) \times (2)$ (18-9) 45 edges

Order 19

K_{19} (19-1) 171 edges

$S_{9,2}$ (19-2) 99 edges

$S_{6,3}$ (19-3) 81 edges

$S_{2,3,3}$ (19-4) 75 edges

$S_{3,2,2,2}$ (19-5) 67 edges

Order 20

K_{20} (20-1) 190 edges	$K_{10} \times (2)$ (20-2) 100 edges	$Z_{3,3}$ (20-3) 90 edges	$S_{2,2,5}$ (20-4) 88 edges	$S_{5,2,1} \approx S_{3,1,2,4}$ (20-5) 78 edges
$L_{3,2}$ (20-6) 70 edges	$Z_{3,2} \times (2)$ (20-7) 70 edges	$K_5 \times K_4$ (20-8) 70 edges	$S_{3,3} \times (2)$ (20-9) 64 edges	$S_{2,2} \times K_4$ (20-10) 62 edges
$K_5 \times (2) \times (2)$ (20-11) 60 edges	$S_{2,2} \times (2) \times (2)$ (20-12) 52 edges			

We shall make a little consideration on the obtained list.

Remark 2. (i) We cannot completely classify the switching graphs only by means of the order and the edge number, and even by the degree sequence at the vertices. This is seen from the two graphs (6-3), (6-4). They reappear at order 12 as factors, producing another example. Such an example might be seen exceptional when looking only at the small orders. But it might become abondant for larger orders, and we might need a deeper criterion for the isomorphism.

(ii) The figures (12-5) to (12-7) constitute another example of non-isomorphism. Although they are all regular of degree 5, their matroid structures are different, that is, in (12-5) the maximal size of the set of common vertices adjacent to two different vertices is 3, whereas it is 4 for the other. Besides, for the latter two the number of vertices with this maximal size of common adjacent vertices from a fixed vertex is different, being 6, 5, respectively. Similarly, we can see that although (18-3) and (18-4) are both regular of degree 7, they are non-isomorphic from the difference of the maximal size of common adjacency 5, 6, respectively. Also, although (20-6) $\sim$ (20-8) are all regular of degree 7, the first is distinguished from the other by the maximal size 4 vs 5. The latter two have the same maximal size, but the number of vertices with this maximal size of common adjacent vertices from a fixed vertex is different, being 7 and 5, respectively.

(iii) On the contrary, the figure of (13-4) and $S_{3,2,2,1}$ are T-equivalent, as shown in the figure below. This is why we omitted adding a new series of figures containing the former.

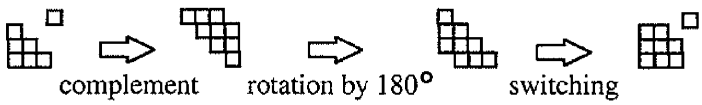


Fig. 14. Example of T-equivalent deformations.

(iv) For each order n, the graph with the maximal number of edges is obviously given by the complete graph K_n. There is an interesting problem of determining the graph with the minimal number of edges. As is expected from the above list, for composite n it seems to be given by the product of those factors with minimal number of edges. For a prime n, however, it will be much mysterious.

There are similar problems of enumerating switching graphs with constraint ([4]), and switching digraphs ([3]). We shall undertake these in a forthcoming paper.

Acknowledgements The authors are much obliged to the referee for the careful reading and valuable suggestions for improvement.

References

[1] Brualdi, R. A. : Matrices of zeros and ones with fixed row and column sum vectors, *Linear Alg. and Appl.* **33** (1980), 159 – 231.

[2] Chartrand, G. and L. Lesniak: *Graphs and Digraphs*, 4-th Ed., Chapman & Hall/CRC, 2004.

[3] Kaneko, A. and R. Nagahama: Switching graphs and digraphs associated with total reconstructed sets from two projection data, *Nat. Sci. Report Ochanomizu Univ.* **56**-2 (2005), 33–45.

[4] Kaneko, A. and R. Nagahama: Reconstruction algorithm and switching graph for two-projection tomography with prohibited subregion, in *Discrete Geometry for Computer Imagery*, Proceedings of DGCI 2006, Szeged, (Eds. A. Kuba, L. Nyúl and K. Palágyi), LNCS **4245**, Springer, 2006, pp.110–121.

[5] Kaneko, A. and R. Nagahama: Reconstruction from Two Projections with Prohibited Subregion. – Algorithm, Switching Graph and Consistency, *Nat. Sci. Report Ochanomizu Univ.* **57**-2 (2006), 7–18.

[6] Kong, T. Yung and T. Gabor Herman: Tomographic equivalence and switching operations, in *Discrete Tomopgraphy*, Proceedings, (Eds. G. Herman and A. Kuba), Birkhauser, 1999, pp.59 - 84.

[7] Kuba, A. and A. Volčič: Characterization of measurable plane sets which are reconstructable from their two projections, *Inverse Problems* **4** (1988), 513–527.

[8] Kuba, A. and T. G. Herman: Discrete tomography: a historical overview, in *Discrete Tomography*, Proceedings, (Eds. G. Herman and A. Kuba), Birkhauser, 1999, pp.3 - 34.

[9] Lorentz, G. G.: A problem of plane measure, *Amer. J. Math.* **71** (1949), 417–426.

[10] Ryser, H. J.: *Combinatorial Mathematics*, The Math. Association of America, Washington, 1963.