Pythagorean Triples. I. Classification and systematization

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Abstract A Pythagorean triple (PT) composed of a pair of legs, a and b, and hypotenuse, c, is trebly classified and registered by using three differences (Δ, δ, d) among (a b c). Recursive relations among the members of the same group thus classified are extensively studied. These recursive properties are found to be related to the Hall's matrices, **U**, **A**, and **D**, by the operator technique which has been proposed by the present author in 1983. The complicated mathematical structure of PT's under **U**, **A**, and **D** can be greatly simplified by using $\mathbf{U}^{1/n}$ and $\mathbf{D}^{1/n}$, which were found to have simple analytical expressions. Further, a number of newly found interesting mathematical properties of PT's are introduced.

1. Introduction

A Pythagorean triple (PT) is a rectangular triangle composed of integral edges and has continuously been a target of professional and amateur mathematicians for more than 4000 years.[1-3] A PT whose edges have no common factor is called a primitive PT and denoted here as pPT. It has long been known that a pPT (a, b, c) can be represented by a pair of integers, or an (m, n)-code, as

$$\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2, \end{aligned} \tag{1.1}$$

where a and b are called legs and c hypotenuse.

Hall has shown that by operating the following three matrices, **U**, **A**, and **D**, on the column vector, $(3\ \ 4\ \ 5)^{\mathrm{T}}$, of the smallest pPT (called the progenitor),

$$\mathbf{U} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix} \tag{1.2}$$

three pPT children are derived as

$$\mathbf{U}(3\ \ 4\ \ 5)^{\mathrm{T}} = (5\ \ 12\ \ 13)^{\mathrm{T}}, \quad \mathbf{A}(3\ \ 4\ \ 5)^{\mathrm{T}} = (21\ \ 20\ \ 29)^{\mathrm{T}}, \quad \text{and} \quad \mathbf{D}(3\ \ 4\ \ 5)^{\mathrm{T}} = (15\ \ 8\ \ 17)^{\mathrm{T}}.$$ [4]

Further, by doing the same process to each of these three children nine grand children of the pPT family are generated. By repeating this procedure he could obtain the genealogy, or the family tree, of pPT's as shown in Fig. 1.

(3 4 5) (2, 1)
- U — (5 12 13) (3, 2)
  - U — (7 24 25) (4, 3)
  - A — (55 48 73) (8, 3)
  - D — (45 28 53) (7, 2)
- A — (21 20 29) (5, 2)
  - U — (39 80 89) (8, 5)
  - A — (119 120 169) (12, 5)
  - D — (77 36 85) (9, 2)
- D — (15 8 17) (4, 1)
  - U — (33 56 65) (7, 4)
  - A — (65 72 97) (9, 4)
  - D — (35 12 37) (6, 1)

Fig. 1. The family tree of the Pythagorean triples (a b c) and their (m, n)-codes. The **U**, **A**, and **D** indicate that the PT which follow can be derived by operating them on the parent triple.

The necessary and sufficient conditions for (m, n) to represent a pPT are:

i) m and n belong to different parities (1.3)

ii) m and n are prime to each other (1.4)

All the (m, n)-codes appearing in Fig. 1 are shown to fulfill (1.3) and (1.4). By imposing another condition, $m > n \geq 1$, the whole family of pPT's can be represented by the set of black dots in the (m, n)-plane in Fig. 2, where two sets of (horizontal and slanted) parallel lines forming a slant grid are added for convenience sake of the later discussion.

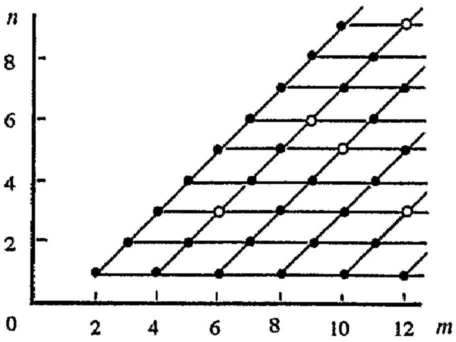


Fig. 2. The family of pPT dots on the (m, n)-grid.
● : pPT's, ●+○ : gpPT's.

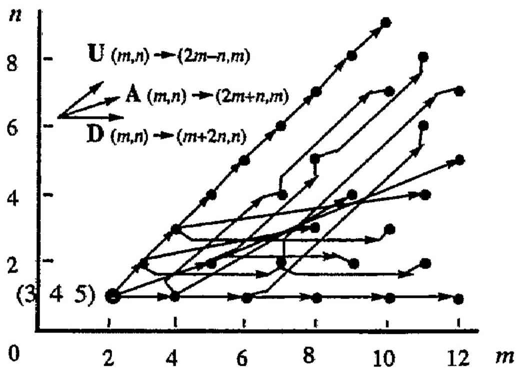


Fig. 3. The pPT genealogy of Fig. 2 mapped on the (m, n)-plane. Many arrows (see text) are omitted from the figure.

Note that the white points in the slant grid in Fig. 2 correspond to those non-pPT's which do not meet just condition ii), or (1.4). For example, they are (27 36 45) (6,3), (45 108 117) (9,6), *etc.*

The weakest point in the Hall's theory is demonstrated in Fig. 3, where lower members of the nodes and branches of the family tree of Fig. 2 are mapped on the grid of the (m, n)-plane. While the actions (represented by arrows) of the two matrices, **U** and **D**, are directed, respectively, 45 degree ascending and horizontally creeping toward right, that of **A** is unpredictably quaking in between the angle determined by the other two matrices. Further, since a single action of **U** shifts (m, n) to $(2m-n, m)$, the length of the arrow for **U** in Fig. 3 is rapidly increasing with the values of m and n. This is also the cases with **A** and **D**, whose corresponding shifts are, respectively, $(2m+n, m)$ and $(m+2n, n)$. Thus, although it has been proved that the (m, n)-point of any pPT can be reached from the progenitor (3 4 5) through a unique path, *i.e.*, a finite product of **U**, **A** and **D**,[4,5] the (m, n)-map of the family tree of Fig. 3 becomes rapidly entangled with the increase of the values of m and n.

One of the main purposes of the present paper is to obtain mathematically clearer relations among the whole family of pPT's. In this respect a possible reason for making the family tree of pPT thus complicated is suspected to be the segregation of the white points in Fig. 2. Then let us try to deem all the lattice points in the grid of the (m, n)-plane of Fig. 2 as equal members of the greater pPT family, or simply gpPT.

It is quite an easy task to obtain the following three 2×2 matrices, **u**, **a**, and **d**, to change the column vector $(m\ n)^T$ corresponding to the (m, n)-point to what **U**, **A** and **D** matrices cause as explained above:

$$\mathbf{u}=\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix},\quad \mathbf{a}=\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix},\quad \mathbf{d}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}. \tag{1.5}$$

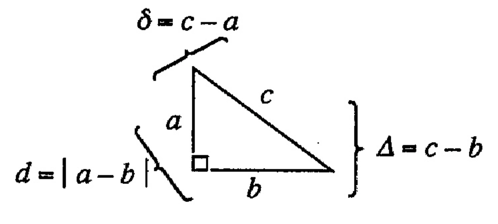
The (m, n)-codes attached to the pPT's in the family tree of Fig. 1 can, of course, be reproduced by operating (1.5) to the progenitor (2, 1). These matrices will play an important role in the later discussion.

2. Classification of PT

First consider a PT $(a\ \ b\ \ c)$, and take the three kinds of differences, (Δ, δ, d), between pairs of edges as shown in Fig. 4.

$$\Delta = c - b, \qquad \delta = c - a, \qquad d = |a - b| . \tag{2.1}$$

Note that as long as the condition i), or (1.3), holds, a and c are odd, while b is even. Further, it can easily be shown that Δ and δ, respectively, take only square of odd number and double of square number, each of which forms a group of pPT with this property. It is not so straightforward but has already been known that for a pPT d takes only the limited numbers as given in Fig. 4. However, this is not the



a, c : odd,
b : even

			group code
Δ = 1, 9, 25, 49, 81, 121, 169, ⋯	$(2k+1)^2$	square of an odd integer	Δk
δ = 2, 8, 18, 32, 50, 72, 98, ⋯	$2k^2$	double of a square number	δk
d = 1, 7, 17, 23, 31, 41, 47, ⋯	p (prime) $\equiv \pm 1 \pmod 8$		dp
	and the product of these numbers (49, 119, ⋯)		

Fig. 4. Definitions of three different edge differences for assigning a gpPT to its group codes.

case with a non-primitive gpPT. Thus for a given pPT three group codes are assigned, while for a non-primitive gpPT only Δ and δ are assigned. For a member of Δ*k* and δ*k* groups a serial number is added following the group code and hyphen. For example, (5 12 13) is assigned as Δ1-2 and δ2-1 meaning the second and first smallest members of Δ1 and δ2 groups, respectively.

Except for *d*1 group all the groups of *d* have more than one sub-groups. In this case the serial number is assigned in each sub-group. This serial number is often omitted in the discussion on the *d* group.

By these assignment rules and (1.2) there is one-to-one correspondence between (*m*, *n*)-code and Δ- and δ-codes as shown in Table 1 (Distinguish between minus sign and hyphen).

Table 1. Interrelationship between the (*m*, *n*)-code and family register codes

$$(m, n) \longrightarrow \Delta(m-n)\text{-}n, \quad \delta n\text{-}(m-n+1)/2$$
$$(k+j, j) \longleftarrow \Delta k\text{-}j$$
$$(2j+k-1, k) \longleftarrow \delta k\text{-}j$$

Thus by using these three edge differences a given pPT is trebly assigned and given three family register codes, whereas a non-primitive gpPT is assigned only Δ and δ codes. In Table 2 sixteen pPTs whose *c* is smaller than 100 are listed together with their (*m*, *n*)- and register codes. In this region there is only one non-primitive gpPT, (27 36 45) (6,3), whose register codes are Δ3-3, δ3-2, and no *d* code, although the actual value of *d* is 9 in this case.

In a later discussion there appear trios of (*X*, *Y*, *Z*) satisfying $X^2+Y^2=Z^2$ but containing zero or a negative integer. Although these trios are not the members of gpPT defined above, they will be included in a formal discussion of Pythagorean triples.

Table 2. Smaller members of pPT's and their (*m*, *n*)- and family register codes

a	*b*	*c*	*m*	*n*	Δ	δ	*d*	*a*	*b*	*c*	*m*	*n*	Δ	δ	*d*
3	4	5	2	1	1-1	1-1	1-1	63	16	65	8	1	7-1	1-4	47-1
5	12	13	3	2	1-2	2-1	7-1	21	20	29	5	2	3-2	2-2	1-2
7	24	25	4	3	1-3	3-1	17-1	45	28	53	7	2	5-2	2-3	17-2
15	8	17	4	1	3-1	1-2	7-2	33	56	65	7	4	3-4	4-2	23-2
9	40	41	5	4	1-4	4-1	31-1	77	36	85	9	2	7-2	2-4	41-1
11	60	61	6	5	1-5	5-1	49-1	39	80	89	8	5	3-5	5-2	41-2
35	12	37	6	1	5-1	1-3	23-1	55	48	73	8	3	5-3	3-3	7-3
13	84	85	7	6	1-6	6-1	71-1	65	72	97	9	4	5-4	4-3	7-4

3. Recursive relations of PT

That the classification criterion adopted above is mathematically meaningful can be supported by the universality of the recursive relations among the family members in each group. See Table 3, where smaller members of gpPT's are listed in the six groups, Δ1, Δ3, δ1, δ3, *d*1, and *d*7, from which several recursive properties can be observed. Non-primitive gpPT's are italicized in the table. Several important properties of the recursive nature of PT's can be deduced not only from Table 3 but also from larger data set of gpPT's.

i) Within each group a pair of recursive formulas of second and third orders hold for the three edges (*a*, *b*, *c*).

ii) For the edge not relevant to assign the group code a second-order recursive formula holds.

iii) For the remaining edges a third-order recursive formula commonly holds.

iv) The pair of recursive formulas above are common for the big groups of Δ and δ.

v) The pair of recursive formulas for the big group of *d* are different from the case of Δ and δ.

vi) While all the members of *d*1 group are related by the pair of recursive formulas as mentioned in v), all the members of other groups of *d* are divided into two or more sub-groups within which the pair of recursive formulas in v) respectively hold.

It is then remarked here that, as long as the recursive relations given in Table 3 are applied, the (*m*, *n*)-codes of the two Δ group members rise along the slanted lines in Fig. 2 with an equal step of the unit grid, and those of the two δ group members creep along the horizontal lines toward right similarly with an equal step. Further, in the cases of Δ3 and δ3 all the non-primitive gpPT members (marked with a white dot in Fig. 2) behave equally as a member in the recursive chain of their respective group.

Table 3. Several groups of gpPT's and their recursive relations (Italicized are non-pPT's)

Δ1			Δ3			δ1			δ3			*d*1			*d*7		
a	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*
3	4	5	15	8	17	3	4	5	7	24	25	3	4	5	15	8	17
5	12	13	21	20	29	15	8	17	*27*	*36*	*45*	21	20	29	65	72	97
7	24	25	*27*	*36*	*45*	35	12	37	55	48	73	119	120	169	403	396	565
9	40	41	33	56	65	63	16	65	91	60	109	697	696	985			
11	60	61	39	80	89	99	20	101	*135*	*72*	*153*				5	12	13
13	84	85	*45*	*108*	*117*	143	24	145	187	84	205				55	48	73
															297	304	425

Δ1, Δ3: $a: f_n = 2f_{n-1} - f_{n-2}$; $b, c: f_n = 3f_{n-1} - 3f_{n-2} + f_{n-3}$; $\Delta = c - b$

δ1, δ3: $a, c: f_n = 3f_{n-1} - 3f_{n-2} + f_{n-3}$; $b: f_n = 2f_{n-1} - f_{n-2}$; $\delta = c - a$

*d*1, *d*7: $a, b: f_n = 5f_{n-1} + 5f_{n-2} - f_{n-3}$; $c: f_n = 6f_{n-1} - f_{n-2}$; $d = |a - b|$

On the other hand, the behavior of each *d* group in the (*m*, *n*)-grid is different from that of Δ and δ groups. Although the set of the recursive formulas is common to each sub-group, it does not seem to be possible to combine the two sub-groups of *d*7 into a single recursive chain. Before discussing this point let us consider the relation between the recursive properties observed above and the three matrices proposed by Hall using the operator technique proposed by the present author in 1983[6,7].

4. Relation between the recursive properties and Hall's matrices

Try to define a step-up operator $\boldsymbol{O}$ with the following property:

$$\boldsymbol{O} f_n = f_{n+1}, \tag{4.1}$$

Where f_n can be a series of functions or variables.

When this $\boldsymbol{O}$ is applied to a PT (a, b, c), we have

$$\boldsymbol{O}\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & 0 & 0 \\ 0 & \boldsymbol{O} & 0 \\ 0 & 0 & \boldsymbol{O} \end{pmatrix}\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix}. \tag{4.2}$$

However, we already know the following relation for some special pair of pPT's connected by a matrix proposed by Hall, for example, $\mathbf{U}$, as

$$\mathbf{U}\,(a_n\ b_n\ c_n)^{\mathrm{T}} = (a_{n+1}\ b_{n+1}\ c_{n+1})^{\mathrm{T}}. \tag{4.3}$$

Then by combining (4.2) and (4.3) we have

$$(\mathbf{U} - \boldsymbol{O}\mathbf{E})\,(a_n\ b_n\ c_n)^{\mathrm{T}} = 0, \tag{4.4}$$

where $\mathbf{E}$ is the 3×3 unit matrix.

In order for (4.4) to have a non-trivial solution the following equation should hold,

$$\det(\mathbf{U} - \boldsymbol{O}\mathbf{E}) = 0. \tag{4.5}$$

Actually the following result can be obtained,

$$\begin{aligned} \det(\mathbf{U} - \boldsymbol{O}\mathbf{E}) &= -(\boldsymbol{O} - 1)^3 \\ &= -(\boldsymbol{O} - 1)(\boldsymbol{O}^2 - 2\,\boldsymbol{O} + 1) \\ &= -(\boldsymbol{O}^3 - 3\,\boldsymbol{O}^2 + 3\,\boldsymbol{O} - 1) = 0. \end{aligned} \tag{4.6}$$

By operating the operator polynomials of the second and third orders in (4.6) on f_n in (4.1) one can obtain the pair of recursive formulas for the Δ group in Table 3 as

$$\begin{aligned} f_n &= 2 f_{n-1} - f_{n-2}, \\ f_n &= 3 f_{n-1} - 3 f_{n-2} + f_{n-3}. \end{aligned} \tag{4.7}$$

Note that $\mathbf{D}$ has the same property as $\mathbf{U}$,

$$\det(\mathbf{D} - \boldsymbol{O}\mathbf{E}) = -(\boldsymbol{O} - 1)^3 = 0, \tag{4.8}$$

whereas $\mathbf{A}$ has a different property as

$$\begin{aligned} \det(\mathbf{A} - \boldsymbol{O}\mathbf{E}) &= -(\boldsymbol{O} + 1)(\boldsymbol{O}^2 - 6\,\boldsymbol{O} + 1) \\ &= -(\boldsymbol{O}^3 - 5\,\boldsymbol{O}^2 - 5\,\boldsymbol{O} + 1) = 0, \end{aligned} \tag{4.9}$$

giving the recursive formulas for the d group given in Table 3 as

$$\begin{aligned} f_n &= 6 f_{n-1} - f_{n-2} \\ f_n &= 5 f_{n-1} + 5 f_{n-2} - f_{n-3}. \end{aligned} \tag{4.10}$$

That (4.3) is applicable also to gpPT's can be demonstrated for many cases. Now mathematical relation between the recursive formulas and the set of matrices proposed by Hall for the gpPT family is clarified

5. Recursive structure of d groups of gpPT family

By operating matrix $\mathbf{A}$ repeatedly all the members of $d1$ group are generated from the progenitor as follows:

A $(3\ 4\ 5)^T = (21\ 20\ 29)^T$, A $(21\ 20\ 29)^T = (119\ 120\ 169)^T$, A $(119\ 120\ 169)^T = (697\ 696\ 985)^T$, *etc.*, just in accordance with the recursive chain given in Table 3. However, as have already been shown in Table 3, other higher d groups are divided into more than two sub-groups, in each of which successive operation of **A** gives the recursive chain correctly.

Instead of exploring the descendants of the PT family let us go back to its prehistoric roots. For this purpose one can use the recursion formulas in the opposite direction, and also use the reciprocal matrices,

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ -2 & -2 & 3 \end{pmatrix}, \quad \mathbf{U}^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ -2 & -1 & 2 \\ -2 & -2 & 3 \end{pmatrix}, \quad \mathbf{D}^{-1} = \begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & -2 \\ -2 & -2 & 3 \end{pmatrix} \tag{5.1}$$

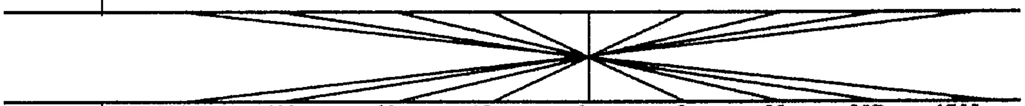
for $(a\ \ b\ \ c)$ and

$$\mathbf{a}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{u}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{d}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \tag{5.2}$$

for (m, n)-codes. The results of the two sub-groups of $d7$ are given in Table 4.

Table 4. Mirror relation between the two sub-groups of $d7$.

l	−4	−3	−2	−1	0	1	2	3	4
a	−1755	−297	−55	−5	−3	15	65	403	2325
b	−1748	−304	−48	−12	4	8	72	396	2332
c	2477	425	73	13	5	17	97	565	3293
m	−19	8	−3	2	1	4	9	22	53
n	46	−19	8	−3	2	1	4	9	22



a	−2325	−403	−65	−15	3	5	55	297	1755
b	−2332	−396	−72	−8	−4	12	48	304	1748
c	3293	565	97	17	5	13	73	425	2477
m	22	−9	4	−1	2	3	8	19	46
n	−53	22	−9	4	−1	2	3	8	19

c: $f_n = 6 f_{n-1} - f_{n-2}$, a,b: $f_n = 5 f_{n-1} + 5 f_{n-2} - f_{n-3}$
m, n: $f_n = 2 f_{n-1} + f_{n-2}$

As evident from Table 4, the two sub-groups are found to be the twins with pseudo-mirror symmetry. Two more examples are shown in Tables 5 and 6 for $d49$ and $d127$ groups, respectively. The latter number belongs to p as given in Fig. 4, while the former number is a square of p. On the other hand, $d119$ has a pair of twins, *i.e.*, four sub-groups in all. This number is not a prime, but the smallest composite number (7×17), a product of two different p's specified in Fig. 4. In those d groups of such composite numbers an interesting property has been observed as follows:

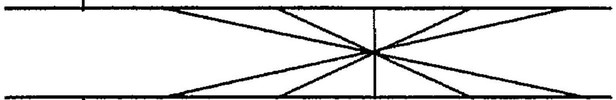
[Observation]

The number of sub-groups of *dk* group is given as

1: $k=1$, 2 : $k=p, p^n$, 4 : $k=pp'$, *etc.*, where $p, p' \equiv \pm 1$ (mod. 8).

Table 5. Twin sub-groups of *d*49

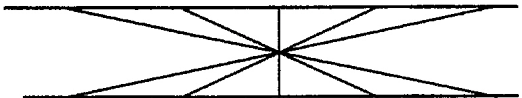
l	−2	−1	0	1	2
a	−253	−11	−9	153	731
b	−204	−60	40	104	780
c	325	61	41	185	1069
m	−6	5	4	13	30
n	17	−6	5	4	13



l	−2	−1	0	1	2
a	−731	−153	9	11	253
b	780	−104	−40	60	204
c	1069	185	41	61	325
m	13	−4	5	6	17
n	−30	13	−4	5	6

Table 6. Twin sub-groups of *d*127

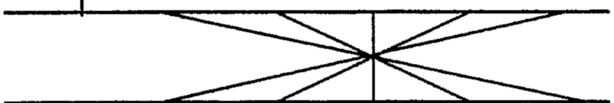
l	−2	−1	0	1	2
a	−2117	−435	15	17	595
b	−2244	−308	−112	144	468
c	3085	533	113	145	757
m	22	−7	8	9	26
n	−51	22	−7	8	9



l	−2	−1	0	1	2
a	−595	−17	−15	435	2117
b	−468	−144	112	308	2244
c	757	145	113	533	3085
m	−9	8	7	22	51
n	26	−9	8	7	22

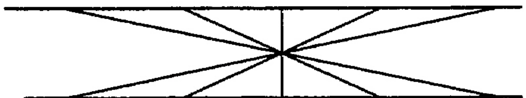
Table 7. A pair of twin sub-groups of *d*119

l	−2	−1	0	1	2
a	−1357	−299	39	57	779
b	−1476	−180	−80	176	660
c	2005	349	89	185	1021
m	18	−5	8	11	30
n	−41	18	−5	8	11



l	−2	−1	0	1	2
a	−779	−57	−39	299	1357
b	−660	−176	80	180	1476
c	1021	185	89	349	2005
m	−11	8	5	18	41
n	30	−11	8	5	18

l	−2	−1	0	1	2
a	−1943	−261	−99	143	481
b	−1824	−380	20	24	600
c	2665	461	101	145	769
m	−19	10	1	12	25
n	48	−19	10	1	12



l	−2	−1	0	1	2
a	−481	−143	99	261	1943
b	−600	−24	−20	380	1824
c	769	145	101	461	2665
m	12	−1	10	19	48
n	−25	12	−1	10	19

6. Complicated structure of Δ and δ groups of PT family

As already mentioned that all the members of Δ1 group can be generated from the progenitor by successive operation of **U** matrix as

$$\mathbf{U}(3\ 4\ 5)^T=(5\ 12\ 13)^T,\ \mathbf{U}(5\ 12\ 13)^T=(7\ 24\ 25)^T,\ \mathbf{U}(7\ 24\ 25)^T=(9\ 40\ 41)^T,\ etc.$$

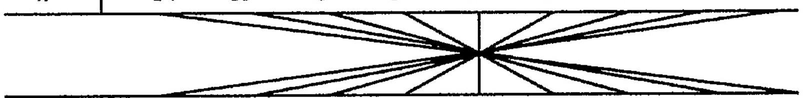
This is also the case with δ1 group by using **D** matrix as

$$\mathbf{D}(3\ 4\ 5)^T=(15\ 8\ 17)^T,\ \mathbf{D}(15\ 8\ 17)^T=(35\ 12\ 37)^T,\ \mathbf{D}(35\ 12\ 37)^T=(63\ 16\ 65)^T,\ etc.$$

However, this is not the case with all other Δ and δ groups. Let us ake Δ3 group as an example. Although all the members are nicely connected by the pair of recursive formulas given in Table 3, **U** matrix divides them into two primitive and one non-primitive sub-groups. Table 8 demonstrates the interesting property of the pair of primitive sub-groups each of which is individually governed by **U** matrix (See the right half of the table). Further, it is found by operating $\mathbf{U}^{-1}$ matrix successively the members of another sub-group are generated but with a minus sign on edge *a*. In this case also this pair of primitive sub-groups are twins.

Table 8. Relationship between the two sub-groups of Δ3 through matrices **U** and $\mathbf{U}^{-1}$

l	−4	−3	−2	−1	0	1	2	3	4
a	−75	−57	−39	−21	−3	15	33	51	69
b	308	176	80	20	−4	8	56	140	260
c	317	185	89	29	5	17	65	149	269
m	−11	−8	−5	−2	1	4	7	10	13
n	−14	−11	−8	-5	−2	1	4	7	10



l	−4	−3	−2	−1	0	1	2	3	4
a	−69	−51	−33	−15	3	21	39	57	75
b	260	140	56	8	−4	20	80	176	308
c	269	149	65	17	5	29	89	185	317
m	−10	−7	−4	−1	2	5	8	11	14
n	−13	−10	−7	−4	−1	2	5	8	11

$$\mathbf{U}=\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & 2\\ 2 & -2 & 3\end{pmatrix}$$

$$\mathbf{U}^{-1}=\begin{pmatrix}1 & 2 & -2\\ -2 & -1 & 2\\ -2 & -2 & 3\end{pmatrix}$$

On the other hand, the non-primitive sub-group of Δ3, namely,

(27 36 45), *(45 108 117)*, *(63 216 225)*, etc.

is isolated in Δ3. In fact, successive operation of $\mathbf{U}^{-1}$ matrix on this non-primitive sub-group generates the same sub-group members but with a minus sign on edge *a*.

Now go on to Δ5 group. Table 9 illustrates the structure of Δ5 group including its ancestors, which are composed of four primitive and one non-primitive sub-groups. The four sub-groups are found to form two twins. All the members are sequentially connected by the recursive formulas (4.7), while both **U** and **u** matrices can connect the members within each sub-group as indicated by horizontal arrows. Note also that the primitive (bold) and non-primitive (italic) PT's beginning from (35 12 37) occupy all the grid points of (n+5, n) on the slanted line starting from (6, 1) in Fig. 2 together with the members with negative n.

The structure of other Δk groups is similar to the cases with Δ3 and Δ5. The numbers of primitive twin and non-primitive sub-groups depend on the composite character of Δk.

The structure of each group of δk is almost similar to those of Δk, and becomes complicated with the value of k. Anyway except for δ1 group, **D** matrix cannot reproduce the single sequence of their members as aligned by the recursive formulas in Table 3.

Table 9. Structure of the PT (a b c) family of Δ5 group and their (m,n)-codes
Primitive and *non-primitive* PT's form gpPT family.
① and ④, and, ② and ③ are, respectively, twins. →: U matrix, ↓ : recursive formulas

①	(–115 252 277)	→	(–65 72 97)	→	(–15 –8 17)	→	**(35 12 37)**	→	**(85 132 157)**
	(–9, –14)		(–4, –9)		(1, –4)		(6, 1)		(11,6)
	↓		↓		↓		↓		↓
②	(–105 208 233)	→	(–55 48 73)	→	(–5 –12 13)	→	**(45 28 53)**	→	**(95 168 193)**
	(–8, –13)		(–3, –8)		(2, –3)		(7, 2)		(12, 7)
	↓		↓		↓		↓		↓
③	(–95 168 193)	→	(–45 28 53)	→	(5 –12 13)	→	**(55 48 73)**	→	**(105 208 233)**
	(–7, –12)		(–2, –7)		(3, –2)		(8, 3)		(13, –8)
	↓		↓		↓		↓		↓
④	(–85 132 157)	→	(–35 12 37)	→	(15 –8 17)	→	**(65 72 97)**	→	**(115 252 277)**
	(–6, –11)		(–1, –6)		(4, –1)		(9, 4)		(14, 9)
	↓		↓		↓		↓		↓
⑤	*(–75 100 125)*	→	*(–25 0 25)*	→	*(25 0 25)*	→	*(75 100 125)*	→	*(125 300 325)*
	(–5, –10)		(0, –5)		(5, 0)		(10, 5)		(15, 10)
	↗		↗		↗		↗		↗

7. Search for $\mathbf{U}^{1/k}$ and $\mathbf{D}^{1/k}$

The failure of **U** and **D** matrices in sequential alignment of the members of Δ and δ groups might be ascribed to their leap-frog property. Further, it was inferred from an extended analysis as above that the k-th next larger member is generated by the operation of **U** on a member of Δk group. Thus the k-th roots of **U** and **D** matrices were searched for.

The most exciting finding in this study is the discovery of general expressions for the j/k-th roots of **U** and **D** as

$$\mathbf{U}^{j/k} = \frac{1}{k^2}\begin{pmatrix} k^2 & -2jk & 2jk \\ 2jk & k^2-2j^2 & 2j^2 \\ 2jk & -2j^2 & k^2+2j^2 \end{pmatrix} \tag{7.1}$$

and

$$\mathbf{D}^{j/k} = \frac{1}{k^2}\begin{pmatrix} k^2-2j^2 & 2jk & 2j^2 \\ -2jk & k^2 & 2jk \\ -2j^2 & 2jk & k^2+2j^2 \end{pmatrix}, \tag{7.2}$$

whose proof will be given in Appendix. Note that j and k may even take negative integers. By putting j=1 in (7.1) and (7.2) one gets

$$\mathbf{U}^{1/k} = \frac{1}{k^2}\begin{pmatrix} k^2 & -2k & 2k \\ 2k & k^2-2 & 2 \\ 2k & -2 & k^2+2 \end{pmatrix} \tag{7.3}$$

and

$$\mathbf{D}^{1/k} = \frac{1}{k^2}\begin{pmatrix} k^2-2 & 2k & 2 \\ -2k & k^2 & 2k \\ -2 & 2k & k^2+2 \end{pmatrix}. \tag{7.4}$$

Further, one can prove

$$\det(\mathbf{U}^{j/k} - O\mathbf{E}) = \det(\mathbf{D}^{j/k} - O\mathbf{E}) = -(O-1)^3. \tag{7.5}$$

For the later discussion let us derive the following matrices:

$$\mathbf{U}^{1/3} = \frac{1}{9}\begin{pmatrix} 9 & -6 & 6 \\ 6 & 7 & 2 \\ 6 & -2 & 11 \end{pmatrix},\quad \mathbf{U}^{1/5} = \frac{1}{25}\begin{pmatrix} 25 & -10 & 10 \\ 10 & 23 & 2 \\ 10 & -2 & 27 \end{pmatrix},\quad \mathbf{U}^{1/7} = \frac{1}{49}\begin{pmatrix} 49 & -14 & 14 \\ 14 & 47 & 2 \\ 14 & -2 & 51 \end{pmatrix},$$

$$\mathbf{U}^{-1/3} = \frac{1}{9}\begin{pmatrix} 9 & 6 & -6 \\ -6 & 7 & 2 \\ -6 & -2 & 11 \end{pmatrix},\quad \mathbf{U}^{-1/5} = \frac{1}{25}\begin{pmatrix} 25 & 10 & -10 \\ -10 & 23 & 2 \\ -10 & -2 & 27 \end{pmatrix},\quad \mathbf{U}^{-1/7} = \frac{1}{49}\begin{pmatrix} 49 & 14 & -14 \\ -14 & 47 & 2 \\ -14 & -2 & 51 \end{pmatrix}$$

$$\mathbf{D}^{1/2} = \frac{1}{4}\begin{pmatrix} 2 & 4 & 2 \\ -4 & 4 & 4 \\ -2 & 4 & 6 \end{pmatrix},\quad \mathbf{D}^{1/3} = \frac{1}{9}\begin{pmatrix} 7 & 6 & 2 \\ -6 & 9 & 6 \\ -2 & 6 & 11 \end{pmatrix},\quad \mathbf{D}^{1/4} = \frac{1}{16}\begin{pmatrix} 14 & 8 & 2 \\ -8 & 16 & 8 \\ -2 & 8 & 18 \end{pmatrix},$$

$$\mathbf{D}^{-1/2} = \frac{1}{4}\begin{pmatrix} 2 & -4 & 2 \\ 4 & 4 & -4 \\ -2 & -4 & 6 \end{pmatrix},\quad \mathbf{D}^{-1/3} = \frac{1}{9}\begin{pmatrix} 7 & -6 & 2 \\ 6 & 9 & -6 \\ -2 & -6 & 11 \end{pmatrix},\quad \mathbf{D}^{-1/4} = \frac{1}{16}\begin{pmatrix} 14 & -8 & 2 \\ 8 & 16 & -8 \\ -2 & -8 & 18 \end{pmatrix} \tag{7.6}$$

8. Systematic alignment of Δ and δ small groups with $\mathbf{U}^{1/k}$ and $\mathbf{D}^{1/k}$

Finally a recipe for generating the gpPT members from its smallest member was obtained by the use of $\mathbf{U}^{1/k}$ and $\mathbf{D}^{1/k}$ matrices as the following Theorems.

[Theorem A]

By successive operation of $U^{1/k}$ matrix on (k+1, 1) pPT of (m, n)-code one can stepwise obtain all the gpPT members of Δk group as illustrated in Fig. 5.

[Theorem B]

By successive operation of $D^{1/k}$ matrix on (k+1, k) pPT of (m, n)-code one can stepwise obtain all the gpPT members of δk group as illustrated in Fig. 6.

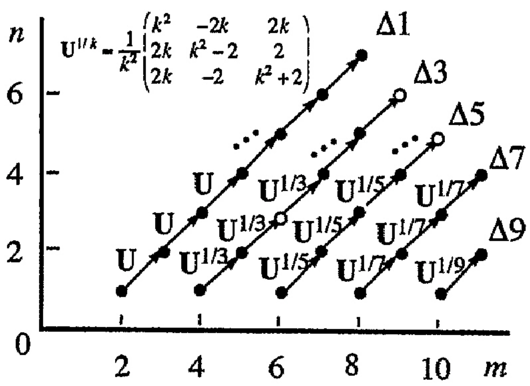


Fig. 5. Stepwise generation of Δ groups of gpPT.

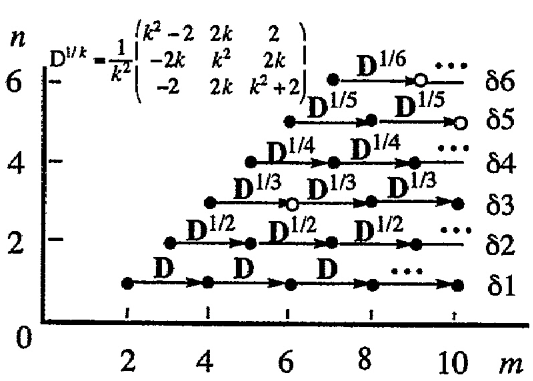


Fig. 6. Stepwise generation of δ groups of gpPT.

Theorem A will be demonstrated by taking Δ3 as an example. By using $U^{1/3}$ matrix, given in (7.6), we have

$$U^{1/3}\,(15\ 8\ 17)^T=(21\ 20\ 29)^T,\quad U^{1/3}\,(21\ 20\ 29)^T=(27\ 36\ 45)^T,\ etc.$$

The Δ3 group of gpPT family is generated from the vertex (4, 1) in (m, n)-grid in Fig. 2 along the slanted line stepwise toward top-right by $U^{1/3}$ matrix as seen in Fig. 5, just as the pair of recursive formulas (4.7) apply to the three edges of gpPT's under the condition of $c-b=3^2$. Further, with the help of $U^{-1/3}$ one can generate all the members of Δ3 group including non-pPT's sequentially as the recursion formulas (4.7) as given in Table 10.

Quite similarly all the members of other Δk groups including non-pPT's can sequentially be generated from their smallest members by $U^{1/k}$ matrix as seen in Fig. 5.

Now turn to δk groups. As has already been shown that all the members of δ1 group can be generated from (2,1) as in Fig. 3, while for other δk groups all the members including non-pPT's can sequentially be generated from their smallest members by $D^{1/k}$ matrix as seen in Fig. 6.

Therefore by the use of $U^{1/k}$ and $D^{1/k}$ matrices all the members of gpPT family can not only be generated from their smaller members but also can be related with all other members of the family. The mathematical structure of the gpPT family is thus clarified and understood systematically by their classification scheme proposed in this paper.

Table 10. Interrelation among the PT-family members of Δ3 group through matrices $U^{1/3}$ and $U^{-1/3}$
bold: pPT, bold+italic : gpPT

j	−5	−4	−3	−2	−1	0	**1**	**2**	***3***	**4**	**5**	***6***	**7**	**8**
a	−21	−15	−9	−3	3	9	**15**	**21**	***27***	**33**	**39**	***45***	**51**	**57**
b	20	8	0	−4	−4	0	**8**	**20**	***36***	**56**	**80**	***108***	**140**	**176**
c	29	17	9	5	5	9	**17**	**29**	***45***	**65**	**89**	***117***	**149**	**185**
m	−2	−1	0	1	2	3	**4**	**5**	***6***	**7**	**8**	***9***	**10**	**11**
n	−5	−4	−3	−2	−1	0	**1**	**2**	***3***	**4**	**5**	***6***	**7**	**8**

$a=(3+j)^2-j^2,\quad b=2j(3+j),\quad c=(3+j)^2+j^2$

$m=j+3,\quad n=j$ $\qquad\qquad U^{1/3}\,\mathrm{PT}(\Delta 3\text{-}j)=\mathrm{PT}(\Delta 3\text{-}j+1)$

$a: f_j=2f_{j-1}-f_{j-2}\quad b,\ c: f_j=3f_{j-1}-3f_{j-2}+f_{j-3}$

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Appendix

Derivation of $U^{j/k}$ and $D^{j/k}$ matrices

First consider U matrix proposed by Hall,

$$\mathbf{U}=\begin{pmatrix}1 & -2 & 2\\ 2 & -1 & 2\\ 2 & -2 & 3\end{pmatrix}. \tag{A.1}$$

By calculating its *n*-th powers as

$$\mathbf{U}^2=\begin{pmatrix}1 & -4 & 4\\ 4 & -7 & 8\\ 4 & -8 & 9\end{pmatrix},\quad \mathbf{U}^3=\begin{pmatrix}1 & -6 & 6\\ 6 & -17 & 18\\ 6 & -18 & 19\end{pmatrix},\quad \mathbf{U}^4=\begin{pmatrix}1 & -8 & 8\\ 8 & -31 & 32\\ 8 & -32 & 33\end{pmatrix},\quad \cdots, \tag{A.2}$$

one can conjecture a very simple general expression for U^n as

$$\mathbf{U}^n = \begin{pmatrix} 1 & -2n & 2n \\ 2n & 1-2n^2 & 2n^2 \\ 2n & -2n^2 & 1+2n^2 \end{pmatrix}, \tag{A.3}$$

whose validity can easily be proved by induction.

Then try to put $n=j/k$, and one gets

$$\mathbf{U}^{j/k} = \frac{1}{k^2}\begin{pmatrix} k^2 & 2jk & 2jk \\ -2jk & k^2-2j^2 & -2j^2 \\ 2jk & 2j^2 & k^2+2j^2 \end{pmatrix}. \tag{A.4}$$

Although rigorous mathematical proof of (A.4) seems to be time-consuming, one can at least assert that this is a sufficient condition.

Then for **D** one can similarly get

$$\mathbf{D}^n = \begin{pmatrix} 1-2n^2 & 2n & 2n^2 \\ -2n & 1 & 2n \\ -2n^2 & 2n & 1+2n^2 \end{pmatrix}, \tag{A.5}$$

and

$$\mathbf{D}^{j/k} = \frac{1}{k^2}\begin{pmatrix} k^2-2j^2 & 2jk & 2j^2 \\ -2jk & k^2 & 2jk \\ -2j^2 & 2jk & k^2+2j^2 \end{pmatrix}. \tag{A.6}$$