Pythagorean Triples. II. Growing caterpillar graphs generating Pythagorean triples

Haruo Hosoya

(Received September 30, 2008)

Abstract Various triplet series of caterpillar graphs are discovered whose topological indices (Z's) represent certain series of mostly primitive Pythagorean triples (PT's). Then quite a new type of method for generating PT's is proposed as follows: Namely one can draw triplet series of caterpillar graphs with mirror symmetry according to the specified recipe, whose Z indices correspond to the edges of a series of PT's. The proof and examples of this theorem are given.

1. Introduction

The present author proposed a new criterion for the classification of the large family of primitive Pythagorean triples and succeeded in their systematization in I of the present series of papers.[1] On the other hand, he has already shown that a special series of Pythagorean triples (PT's) have a triplet series of caterpillar graphs whose topological indices (Z's) just correspond to these PT's,[2,3] and has also shown the systematic relation between PT's and the Pell equations.[3] Along with these studies a number of results have been accumulated supporting the close relationship between PT's and caterpillar graphs through Z indices.

The purpose of the present paper is to propose quite a new type of algorithms for generating PT's. Draw triplet series of caterpillar graphs with mirror symmetry according to the specified recipe, and one can obtain the edges of a series of PT's as the Z indices of those caterpillar graphs. Before presenting these theorems some preliminary features of PT's will be given.

2. Preliminaries

A PT is a rectangular triangle composed of integral edges,[4-6] and if they have no common factor, it is called a primitive PT and denoted here as pPT. It has long been known that a pPT (a, b, c) can be represented by a pair of integers, or an (m, n)-code, as

$$\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2, \end{aligned} \tag{2.1}$$

where a and b are called legs and c hypotenuse, and $m>n$ is assumed for all positive edges. The necessary and sufficient conditions for (m, n) to represent a pPT are:

i) m and n belong to different parities, (2.2)

ii) m and n are prime to each other. (2.3)

In I[1)] it was shown that by omitting condition (2.3) one can enlarge the family of pPT to greater pPT family (gpPT) for systematic classification of PT's. In this paper even those PT's may also appear which violate condition (2.2) for global understanding of the PT family.

For classification purpose let us consider a PT (a b c), and take the three kinds of differences, (Δ, δ, d), between pairs of edges as

$$\Delta = c - b, \qquad \delta = c - a, \qquad d = |a - b| \, . \tag{2.4}$$

Note that as long as the condition i), or (2.2), holds, a and c are odd, while b is even. Further, it can easily be shown that Δ and δ, respectively, take only square of odd number, $(2k+1)^2$, and double of square number, $2k^2$. By using this property a given pPT is assigned to belong both to $\Delta(2k+1)$ and δk groups. On the other hand, it is known that for pPT's d takes only the limited numbers as 1, 7, 17, 23, *etc.*[1)] Thus by using these three edge differences a given pPT is trebly assigned and given three family register codes. For example, the register codes of PT (21 20 29) are [3, 2, 1], which are derived from Δ=9, δ=8, and d=1. It is also described by its (m,n) expression as (5, 2). On the other hand, although a non-primitive gpPT, such as (27 36 45), is formally given the register codes [3, 3, 9], $d9$ group does not have such an important role as those for the pPT family.

Hall has shown that by operating the following three matrices, **U**, **A**, and **D**, on the column vector, $(3\ \ 4\ \ 5)^{\mathrm{T}}$, of the smallest pPT (called the progenitor),[7,8)]

$$\mathbf{U} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix} \tag{2.5}$$

three pPT children are generated as

$$\mathbf{U}(3\ \ 4\ \ 5)^{\mathrm{T}} = (5\ \ 12\ \ 13)^{\mathrm{T}}, \quad \mathbf{A}(3\ \ 4\ \ 5)^{\mathrm{T}} = (21\ \ 20\ \ 29)^{\mathrm{T}}, \quad \text{and} \quad \mathbf{D}(3\ \ 4\ \ 5)^{\mathrm{T}} = (15\ \ 8\ \ 17)^{\mathrm{T}}.$$ [4)]

Further, by doing the same operation on each of these three children nine grand children of the pPT family are generated. By repeating this operation he could obtain the genealogy, or the family tree, of pPT's without omission and duplication.

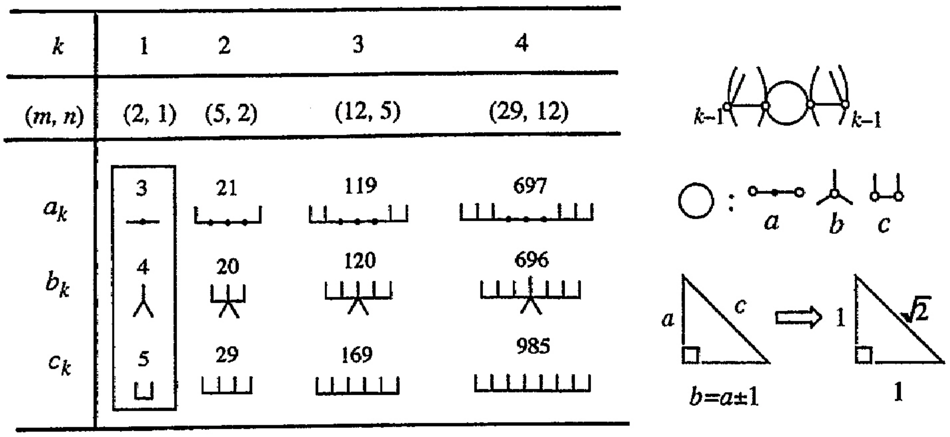
In the graph theory a caterpillar graph is defined as a tree graph that contains a path graph such that every edge has one or both endpoints in that path. Suppose a path graph S_n with n vertices and prepare a set of n star graphs, $\{K_m\}$, each of which is composed of the central vertex and $m-1$ edges of unit length, where the value of m may differ for each of n vertices. Then mount the central vertices of $\{K_m\}$ one by one on each vertex of S_n yielding a caterpillar graph.[9,10)]

The topological index Z,[11)] now called "Hosoya index" [12,13)] or Z-index, was originally proposed and coined by the present author in 1971 for characterizing a graph. The Z-indices of the series of path graphs and monocyclic graphs are, respectively, Fibonacci and Lucas numbers.[14,16)] For the later discussion let us paraphrase this fact as follows: The Z-graphs of Fibonacci and Lucas numbers are path and monocyclic graphs, respectively.

For larger graphs many useful recursive relations for calculating the Z-index have been known.[11,16,17)] In Appendix useful recursion formulas for calculating the Z-index of a caterpillar graph is given. The readers may refer to the brief account for the Z-index given in Refs. 14, 15).

3. Examples of smaller PT's

The present author has already shown that the Z-graphs of PT's with consecutive legs can be generated from the Z-graphs of the progenitor and sets of L-shaped parentheses graphs as illustrated in Fig. 1.[2,3]



$a, b : f_k = 5f_{k-1} + 5f_{k-2} - f_{k-3}$ $\qquad$ $c, a{+}b : f_k = 6f_{k-1} - f_{k-2}$

Fig. 1 Three series of caterpillar graphs whose Z values give the PT's with consecutive edges.

Namely, first consider the three graphs in the square box, whose Z-indices are those of the progenitor (3 4 5), or the smallest member with k=1. Note that several vertices are marked with white circles denoting the "buds" at which branches are going to grow for generating larger members. Next attach a pair of L-shaped parentheses to the buds of triplet graphs to generate the next larger caterpillar triplets with mirror symmetry. They are the Z-graphs of (21 20 29) with k=2. Then attach a pair of L-shaped parentheses to the white circles as shown in the top-right of Fig. 1 to form the Z-graphs with k=3. By consecutive attachment of a pair of L-shaped parentheses one can continue to generate the whole members of the Z-graphs of the PT's with consecutive edges as in Fig. 1.

Here it must be emphasized two points. First, the coincidence between the Z values and PT's seems to be systematic but not accidental. Secondly, all the Z-graphs are caterpillar graphs with mirror symmetry. It is important to note that all the PT's with consecutive legs, which is denoted by *d1* group by the present author, can systematically be generated either by one of the Hall matrices A in (2.5) or by the series of caterpillar graphs as given in Fig. 1. As this series of PT's approach an isosceles right triangle, the ratio $(a+b)/c$ is approaching the square root of two.[3]

Encouraged by the above finding and observation let us try to find the Z-graphs of all the 16 pPT's whose edges are smaller than 100. The results are demonstrated as a flow diagram in Fig. 2, where the numerals in ellipses indicate the value of a, and several non-pPT's (*italicized*) and larger members are also included to help understand the mathematical structure of the PT family.

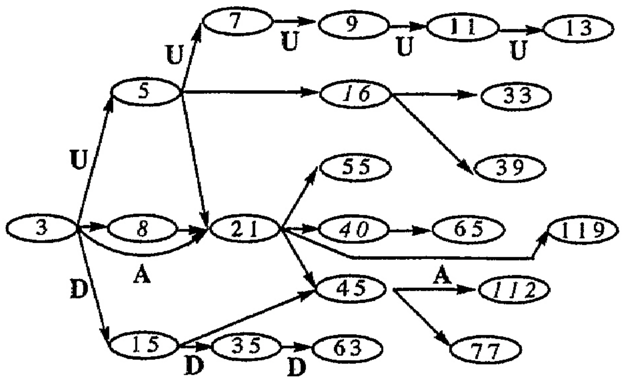


Fig. 2 Diagram showing the relationship between the *Z*-graphs and PT's.

Although three big flows of **U**, **A**, and **D** seem to dominate the whole tree structure in Fig. 2 as in the genealogy of Hall, it will be shown that here **U** and **D** do not play such an important role in larger members of the PT family. Let us take a close look at several flows in the diagram.

First see Figs. 3a and b, where the **U** and **D** flows are picked out from Fig. 2 and [Δ, δ, *d*] and (*m*,*n*) codes are supplemented for each triplet graph.

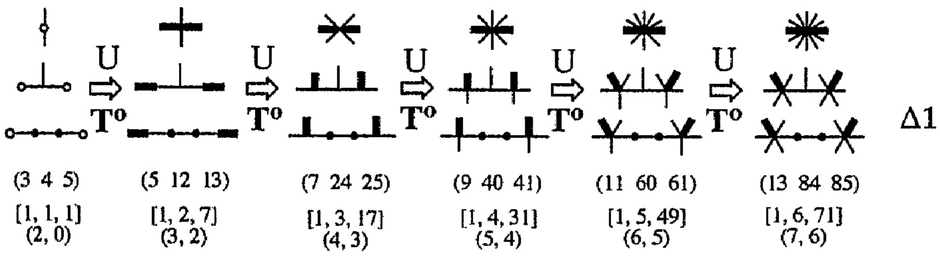


Fig. 3a Caterpillar *Z*-graphs with mirror symmetry for Δ1 group generated by **U** matrix. PT's are represented by (*a b c*), [Δ, δ, *d*], and (*m*,*n*) codes.

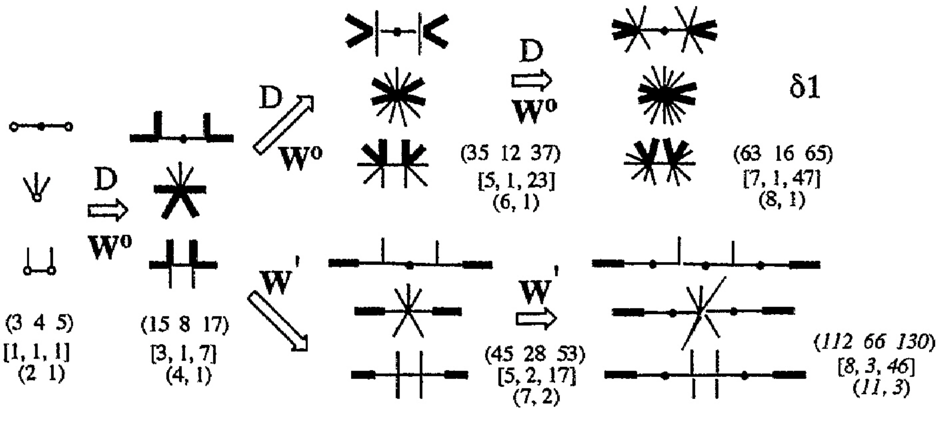


Fig. 3b Caterpillar *Z*-graphs with mirror symmetry for δ1 group generated by **D** matrix.

In the flow of Δ1 group (having 1 as the Δ code in the square brackets) each operation of **U** matrix is found to grow a pair of edges of unit length at all the buds of the triplet graphs. On the other hand, in the flow of δ1 group each operation of **D** matrix is found to grow a pair of two edges of unit length. Note that in this case the vertices of the buds in the starting triplet graphs of the progenitor are different from the case of **U** matrix. The meaning of **T** and **W** attached to the arrow will be explained later.

See Fig. 3b. From (15 8 17) another PT (45 28 53) is generated by growing a pair of edges of unit length at a pair of terminal vertices in order to yield longer triplet graphs with mirror symmetry. Then similar growth of a pair of edges generates non-primitive PT (112 66 30), which, however, is found to generate larger pPT members with longer *Z*-graphs.

Generation of larger pPT members with longer *Z*-graphs is also observed for the *d*1 group by the stepwise operation of **A** matrix, as already shown in Fig. 1 (See Fig. 3c). In this respect there are found two types of growth of *Z*-graphs by the addition of a pair of graphs to the predecessor, fattening (^{0}) and elongation ('). Examples of the former and the latter, respectively, are brought, for example, by **U** and **D** matrices and by **A** matrix. The *Z*-graph growth of elongation type as **A** matrix is also observed in Figs. 3d and e, where growth of fattening is also shown by the branching of the flow diagram. Note that PT (45 28 53) which was generated from (15 8 17) by elongation is also generated from (21 20 29) by fattening.

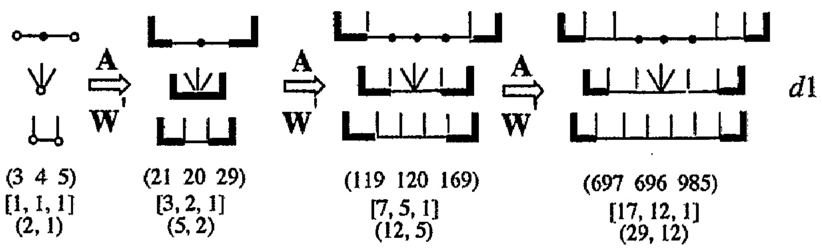


Fig. 3c Caterpillar *Z*-graphs with mirror symmetry for *d*1 groups generated by **A** matrix.

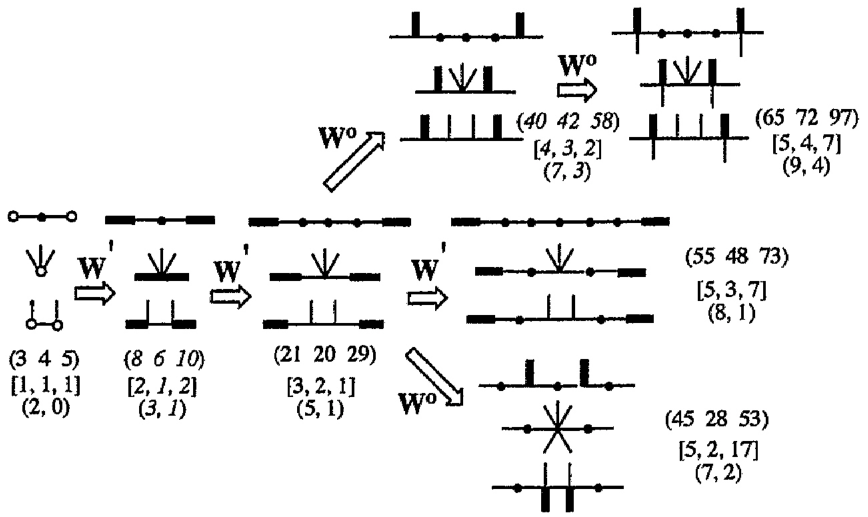


Fig. 3d Caterpillar *Z*-graphs with mirror symmetry for PT's in the central flow in Fig. 2.

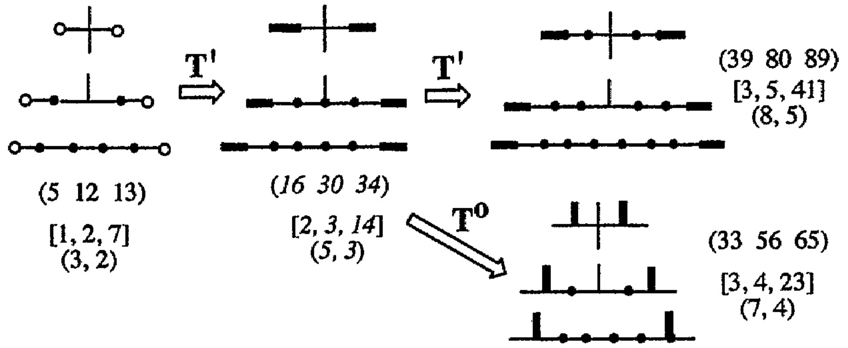


Fig. 3e Caterpillar Z-graphs with mirror symmetry for PT's in a flow branched from (5 12 13).

While the Z-graph growth of fattening type gives only slight or no increase in the [Δ, δ, d] codes, converging to slender PT's with smaller Δ and δ values, that of elongation type gives larger increase, generating "normal" fat PT's.

Enormous amount of observation as Figs. 3a-e has been accumulated supporting the following conjecture:

"For any PT one can draw a caterpillar Z-graph with mirror symmetry." (*)

Before giving an explicit statement extreme examples will be shown in Figs. 4a and b, where such two interesting series of PT's are introduced that give rational number approximation of square root of 13 as an outcome of the solutions of Pell equations.[3)]

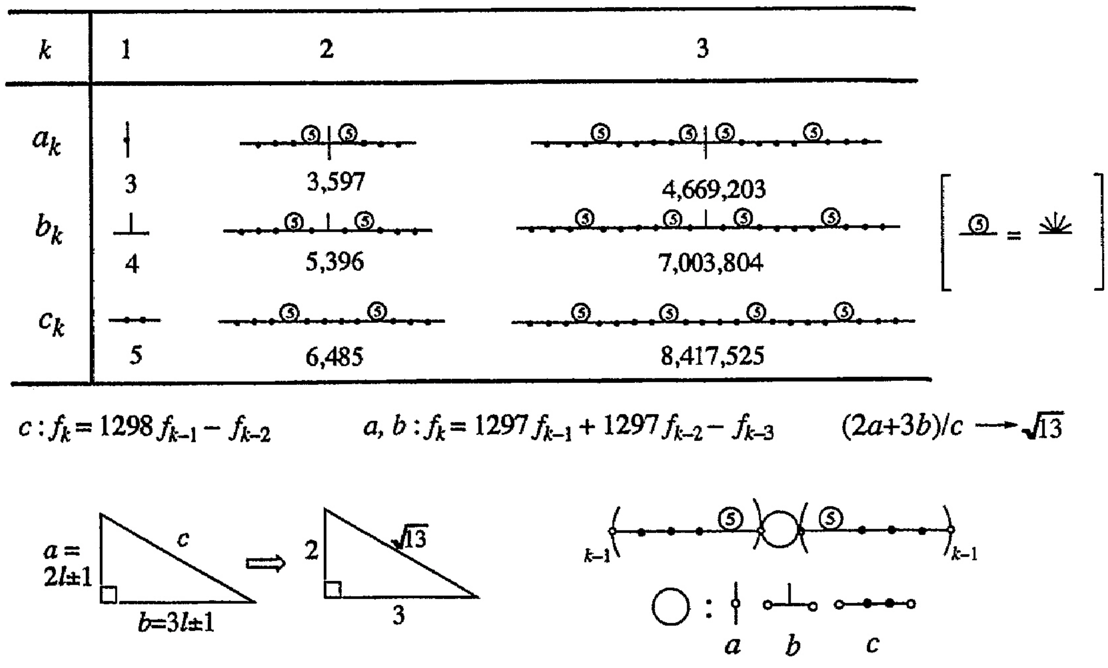


Fig. 4a The caterpillar Z-graphs with mirror symmetry corresponding to the PT's involving square root of 13.[3)]

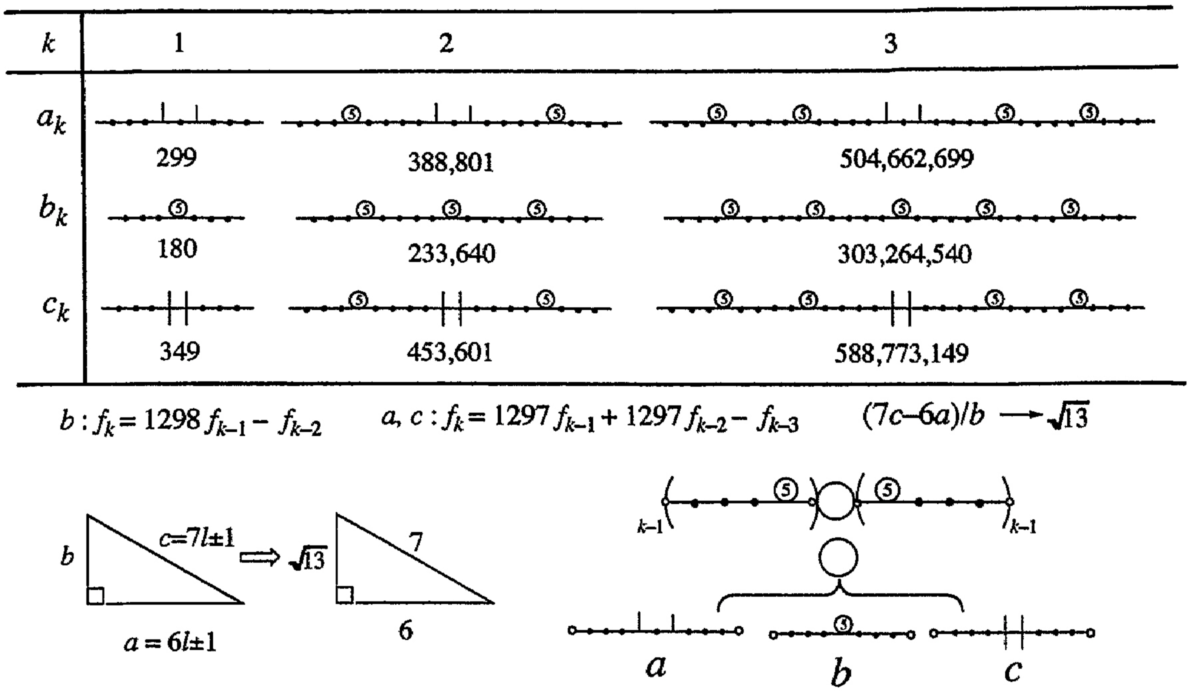


Fig. 4b Another series of the caterpillar Z-graphs with mirror symmetry corresponding to the PT's involving square root of 13.[3)]

Although the Z-values for only three k's are given in Figs. 4a and b, the validity of the recursive relations have been checked for larger k's. Finding such large Z-graphs as in Figs. 4a and b is not so easy but actually could be performed as shown above. Supported by the enormous amount of such results the above conjecture (*) can be formulated as follows.

4. Conjecture and Theorems

Our aim is to generate those triplet Z-graphs representing a PT which may or may not be primitive. The followings are the conjectured steps for designing a series of those graphs.

i) Given those special triplet caterpillar graphs with mirror symmetry whose Z indices form some special Pythagorean triplet.

ii) Assign such bud vertex or vertices to each of the triplet graphs.

iii) Prepare a pair of edges of unit length, or a pair of connected graphs with a root vertex.

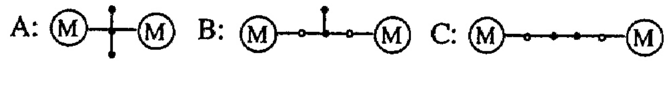
iv) Attach a pair of graphs of iii) to the bud vertex or vertices of ii) to get larger triplet caterpillar graphs with mirror symmetry.

v) Continue this process using the same pair of graphs to get a growing family of triplet series of caterpillar graphs.

At this stage it is not yet known what kinds of triplet graphs of i) and how to choose bud vertices of ii) from them. However, as a special case of i) the following theorems were obtained by choosing the progenitor of the big PT family.

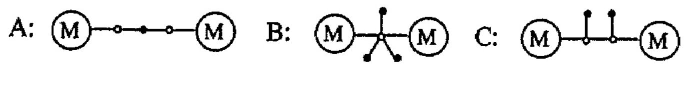
[Theorem 1] (T-type Z-caterpillars generating Pythagorean triples)

The Z-indices of the triplet caterpillar graphs, A, B, and C, with any graph M represent a PT as $Z_A{}^2 + Z_B{}^2 = Z_C{}^2$.



[Theorem 2] (W-type Z-caterpillars generating Pythagorean triples)

The Z-indices of the triplet caterpillar graphs, A, B, and C, with any graph M represent a PT as $Z_A{}^2 + Z_B{}^2 = Z_C{}^2$.



Their proofs are given in Appendix. Here supplemental explanation will be given. The kernels of the triplet caterpillars of T- and W-types are given in Fig. 5, where the bud vertices are marked with while circles.

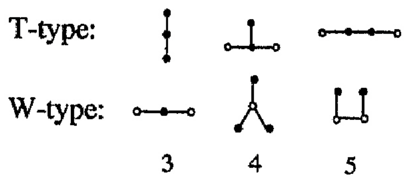


Fig. 5 The kernels of the T- and W-types of PT-generating caterpillars.
The numerals are the Z-values of the graphs and white circles are the buds.

After a pair of graphs, M's, are added to the buds of these kernels, the bud vertices for generating larger caterpillars may be moved toward both the ends of the graphs as long as the mirror symmetry is preserved. As already noticed in the explanation of Figs. 3a-e, there are two types of growing, *i.e.*, fattening and elongation both for T- and W-types. These operations are, respectively, designated as, T^0 and W^0, and T' and W' at each arrow in Figs. 3a-e.

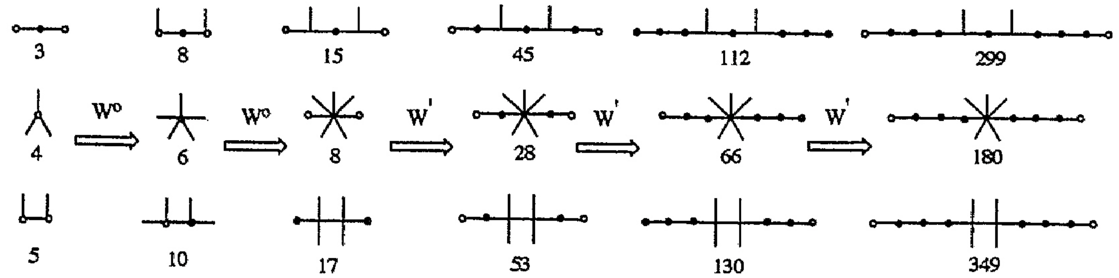


Fig. 6 The kernels in Fig. 4b can be generated from W-type progenitor.

See Figs. 4a and b, where rather big caterpillars are rapidly growing. In Fig. 4a the kernels are T-type progenitor, while those in Fig. 4b are very large and its growing scheme does not seem to obey either Theorem especially in the first step from k=1 to 2. However, see Fig. 6, where the kernels of the series of triplet caterpillars in Fig. 4b are shown to be generated stepwise from the progenitor of W-type within the scheme of Theorem 2.

Now it is not yet known if there are kernels which cannot be generated from the progenitor or are not caterpillars to give some series of PT's by successive addition of certain graphs. At least any positive evidence has not yet been obtained. If this issue is negatively clarified, the importance and effectiveness of Theorems 1 and 2 would be more increased.

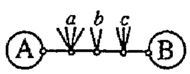
Refernces

1) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **59** (2009) 1.
2) H. Hosoya, *Croat. Chem. Acta*, **80** (2007) 239.
3) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **59** (2008) 19.
4) W. Sierpinski, Pythagorean Triangles, Graduate School of Science, Yeshiva Univ., New York (1962).
5) E. Maor, The Pythagorean Theorem, Princeton Univ. Press, Princeton (2007).
6) A. H. Beiler, Recreations in the Theory of Numbers, Dover, New York (1966).
7) A. Hall, *Mathematical Gazette*, **54** (1970) 377.
8) J. Roberts, Elementary Number Theory, MIT Press, Cambridge, MS (1977).
9) K. H. Rosen, J. G. Michaels, J. L. Gross, J. W. Grossman, and D. R. Shier (Eds.), Handbook of Discrete and Combinatorial Mathematics, CRC Press, Boca Raton, FL (2000), p. 498.
10) E. W. Weisstein, CRC Concise Encyclopedia of Mathematics, Chapman & Hall/CRC, Boca Raton, FL (2003), p. 351.
11) H. Hosoya, *Bull. Chem. Soc. Jpn.*, **44** (1971) 2332.
12) M. Trofimov, J. Math. Chem., **8** (1991) 327.
13) http://en.wikipedia.org/wiki/Hosoya index
14) T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley, New York (2001).
15) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **57** (2006) 35.
16) H. Hosoya, Fibonacci Quart., **11** (1973) 255.
17) H. Hosoya and A. Motoyama, J. Math. Phys., **26** (1985) 157.
18) H. Hosoya, *Natl. Sci. Rept. Ochanomizu Univ.*, **58** (2007) 15.

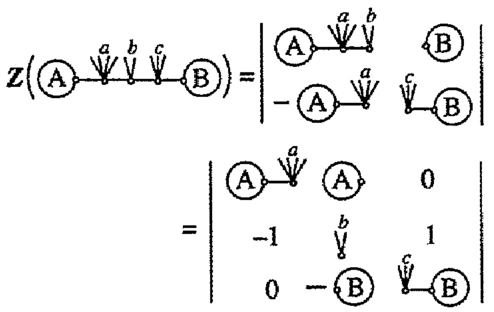
Appendix.

A. Calculation of the Z index of caterpillar graphs

Consider a caterpillar graph C as given below,

C = 

where the moieties A and B are also parts of a caterpillar similar to the structure of the central part, in which a, b, and c, respectively, represent the number of the emanating edges of unit length plus one. According to Theorem 8 in Ref. 18) the Z index of C can be obtained by either of (α) or (β) in Fig. A1.



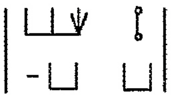
(α)

(β)

Fig. A1 Degradation formula for the Z index of a caterpillar graph.

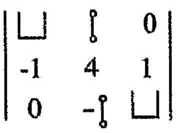
Let us take the following graph

as an example. Application of (α) and (β), respectively, gives the same results as



$$= \begin{vmatrix} 22 & 2 \\ -5 & 5 \end{vmatrix} = 120$$

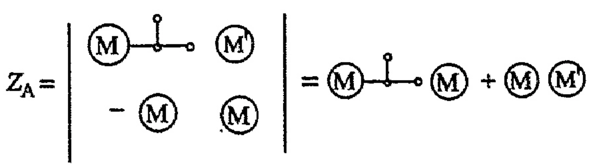
and



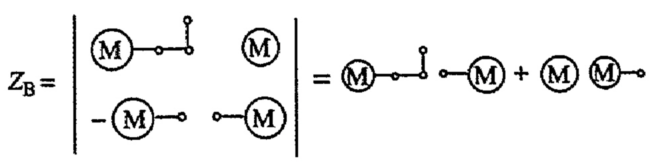
$$= \begin{vmatrix} 5 & 2 & 0 \\ -1 & 4 & 1 \\ 0 & -2 & 5 \end{vmatrix} = 120 .$$

B. Proof of Theorem 1

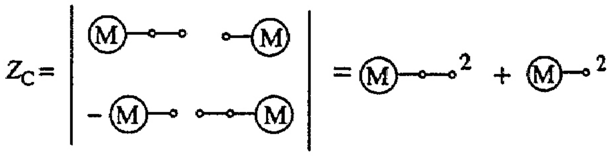
Consider the progenitor in T mode and the triplet, A, B, and C, which are obtained from T by attaching two M graphs at the buds in each of the triplet as shown in Theorem 1. By using (α) in Appendix A the Z-indices of A, B, and C can be obtained as follows:



$$= M(3M + M') + MM' = 3M^2 + 2\,MM'$$



$$= (3M + 2M')\,(M + M') + M(M + M') = 4M^2 + 6\,MM' + 2M'^2$$



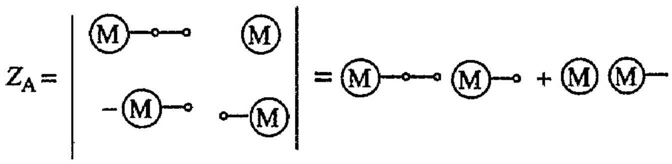
$$= (2M + M')^2 + (M + M')^2 = 5M^2 + 6\,MM' + 2M'^2$$

Then one can obtain

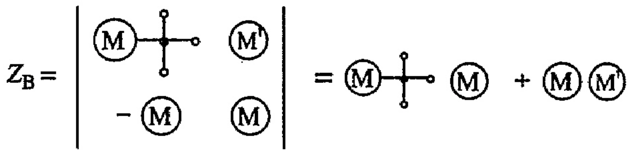
$$Z_A{}^2 + Z_B{}^2 = Z_C{}^2. \qquad \square$$

C. Proof of Theorem 2

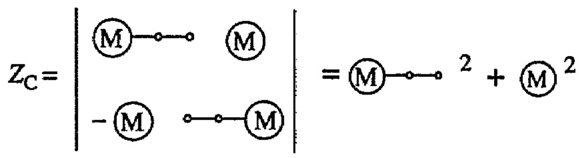
Consider the progenitor in W mode and the triplet, A, B, and C, which are obtained from W by attaching two M graphs at the buds in each of the triplet as shown in Theorem 2. By using (α) in Appendix A the Z-indices of A, B, and C can be obtained as follows:



$$= (2M + M')(M + M') + M\,(M + M') = 3M^2 + 4\,MM' + M'^2$$



$$= M\,(4M + M') + MM' = 4M^2 + 2\,MM'$$



$$= (2M + M')^2 + M^2 = 5M^2 + 4\,MM' + M'^2$$

Again one can obtain

$$Z_A{}^2 + Z_B{}^2 = Z_C{}^2. \qquad \square$$