Fractal Tilings and developments of doubly-covered squares

Nobue Negishi, Nao Kamio and Fukiko Takeo
Department of Information Sciences, Ochanomizu University,
2-1-1, Otsuka, Bunkyo-ku, Tokyo 112-8610, Japan

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Abstract

By considering a 2×2 integer matrix and a digit set, we investigate a tiling problem and also the relationship between a tiling and a development of a doubly-covered square.

1 Introduction

A tiling is a locally finite covering of the plane by compact sets, such that the interiors of any two tiles are disjoint. For an expanding 2×2 matrix M and a digit set D, let the 'fraction' part be $\mathbf{A} = \left\{ \sum_{j=1}^{\infty} M^{-j}\mathbf{d}_{i_j} \in \mathbb{R}^2 \quad | \quad \mathbf{d}_{i_j} \in D \right\}$ and the 'integer' part be the lattice $\mathbf{W}$ spanned by the set $\left\{ \sum_{j=0}^{k} M^{j}\mathbf{d}_{i_j} \in \mathbb{Z}^2 \quad | \quad \mathbf{d}_{i_j} \in D, k \in \mathbb{N} \right\}$[6]. Then the boundary of $\mathbf{A}$ is often fractal [8], that is, its fractal dimension is more than 1. For some M and D, it is known [4][5] that $\mathbf{A} + \mathbf{w}_i$ and $\mathbf{A} + \mathbf{w}_j$ have no interior points in common for all $\mathbf{w}_i, \mathbf{w}_j (\mathbf{w}_i \neq \mathbf{w}_j)$ in $\mathbf{W}$ and $\bigcup_{\mathbf{w} \in \mathbf{W}} (\mathbf{A} + \mathbf{w})$ fills the plane, that is, $\{\mathbf{A} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}}$ is a tiling. By the property of matrix M, the form of the fraction part is sometimes characterized such as twin dragon, tame dragon or so on in [5] . It is also known [1], [2], [3] that the fraction part $\mathbf{A}$ sometimes becomes a development of a doubly-covered square.

As for the condition for M and D when $\{\mathbf{A} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}}$ is a tiling, the following theorem is known.

Theorem A [[7], Theorem 10] Given an expanding $\varphi \in N_n(\mathbb{Z})$ and a set $D \subset \mathbb{R}^n$ lying on a discrete φ-invariant lattice, with $|D| = |\det \varphi|$ and $0 \in F$, there is a corresponding self-similar tiling of $\mathbb{R}^n$ iff R_φ is injective on D^*, where $R_\varphi(x) = d_{i_0} + \varphi(d_{i_1}) + \cdots + \varphi^k(d_{i_k})$ and D^* is the set of finite sequences of elements of D.

In this paper, we shall consider an expanding 2×2 integer matrix M with $|\det M| = 2$ and a digit set $D = \{\mathbf{d}_0 = 0, \mathbf{d}_1 \in \mathbb{Z}^2$ with $\mathbf{d}_1 \neq \mathbf{d}_0\}$. The resulting fraction part $\mathbf{A}$ derived from M and D gives a tiling system with the integer part $\mathbf{W}$ by Theorem A. We investigate a condition when the set $\mathbf{A}$ is a development of a doubly-covered square if $\{\mathbf{A} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}}$ is a tiling. At first we show that the set $\mathbf{A}$ has point symmetry in Proposition 1 and show the property of the set with point symmetry in Lemma 1. In order to consider the development, we treat the folding, which corresponds to the reflection. In Lemmas 2 and 3, we treat the property of reflection. By using the lemmas, we show a condition that the set with point symmetry is a development of a doubly-covered square (Proposition 2). We apply Proposition 2 to the fraction part $\mathbf{A}$ and give a condition that $\mathbf{A}$ is a development of a doubly-covered square (Theorem 1). Furthermore, we consider whether the condition in Theorem 1 is also a necessarily condition for the set $\mathbf{A}$ to be a development and obtain an equivalent condition about a development of a doubly-covered square in Theorems 2 and 3.

2 Notations and terminology

In this paper, we will use the following notations.

Notation 1. Let M be an expanding 2×2 integer matrix with $|\det M| = 2$, where all eigenvalues of M have modulus > 1. Let

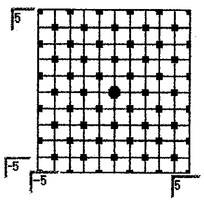
$$\begin{aligned} D &= \left\{ \mathbf{d}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{d}_1 \in \mathbb{Z}^2, \mathbf{d}_1 \neq \mathbf{d}_0 \right\} \\ \mathbf{A} &= \left\{ \sum_{j=1}^{\infty} M^{-j} \mathbf{d}_{i_j} \in \mathbb{R}^2 \quad | \quad \mathbf{d}_{i_j} \in D \right\}, \end{aligned}$$

and $\mathbf{W}_A$ be the lattice spanned by the set $\left\{ \sum_{j=0}^{k} M^j \mathbf{d}_{i_j} \in \mathbb{Z}^2 \quad | \quad \mathbf{d}_{i_j} \in D, k \in \mathbb{N} \right\}$.

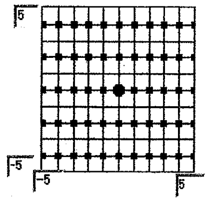
To characterize the lattice, we define the following.

Notation 2. Let the lattices $\mathbf{W}_1$ and $\mathbf{W}_2$ be defined by

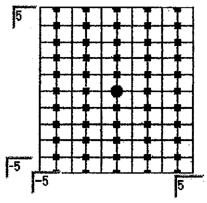
$$\begin{aligned} \mathbf{W}_1 &= \left\{ \begin{pmatrix} i+j \\ i-j \end{pmatrix} | \, i,j \in \mathbb{Z} \right\} \\ \mathbf{W}_2 &= \left\{ \begin{pmatrix} i \\ 2j \end{pmatrix} | \, i,j \in \mathbb{Z} \right\} \text{ and } \mathbf{W}_2' = \left\{ \begin{pmatrix} 2i \\ j \end{pmatrix} | \, i,j \in \mathbb{Z} \right\}. \end{aligned}$$



$\mathbf{W}_1$



$\mathbf{W}_2$



$\mathbf{W}_2'$

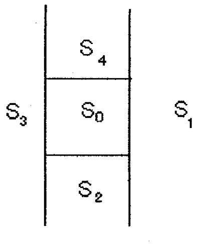
To classify the set $\mathbf{A}$ and to simplify the notation, we need the following map.

Notation 3. Let the map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\varphi \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i+j \\ i-j \end{pmatrix}.$$

To consider the development of a doubly-covered square V, we consider a face of V as S_0 or $\tilde{S}_0$. We also consider reflections and rotations as follows.

Notation 4. Let S_0 be a square with vertices $P_1 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $P_2 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$, $P_3 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$, $P_4 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. We divide the plane $\mathbb{R}^2$ as follows;



$$S_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} |x > \tfrac{1}{2} \right\}, \; S_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | \, |x| < \tfrac{1}{2}, \; y < -\tfrac{1}{2} \right\},$$

$$S_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} |x < -\tfrac{1}{2} \right\}, \; S_4 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | \, |x| < \tfrac{1}{2}, \; y > \tfrac{1}{2} \right\}.$$

Let $\tilde{S}_0 = S_0 + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$ be a square with vertices $\tilde{P}_j = P_j + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$, for $j \in \{1,2,3,4\}$ and let $\tilde{S}_j = S_j + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$, for $j \in \{1,2,3,4\}$

Notation 5. For $j \in \{1,2,3,4\}$, we shall define the **reflection** F_j, $\tilde{F}_j$ and the **rotation** R_j, $\tilde{R}_j$ as follows.

(1) let $F_j : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection with the line including the points P_j and P_{j+1}, where $P_5 = P_1$,

(2) let $\tilde{F}_j : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection with the line including the points $\tilde{P}_j$ and $\tilde{P}_{j+1}$, where $\tilde{P}_5 = \tilde{P}_1$,

(3) let $R_j : \mathbb{R}^2 \to \mathbb{R}^2$ be a 180° rotation about the point P_j,

(4) let $\tilde{R}_j : \mathbb{R}^2 \to \mathbb{R}^2$ be a 180° rotation about the point $\tilde{P}_j$.

Definition 1. For a compact set $\mathbf{A}$ and a lattice $\mathbf{W}$, if $\mathbf{A}+\mathbf{w}_i$ and $\mathbf{A}+\mathbf{w}_j$ have no interior points in common for all $\mathbf{w}_i, \mathbf{w}_j (\mathbf{w}_i \neq \mathbf{w}_j)$ in $\mathbf{W}$ and $\bigcup_{\mathbf{w}\in\mathbf{W}} (\mathbf{A}+\mathbf{w})$ fills the plane, then $\{\mathbf{A}+\mathbf{w}\}_{\mathbf{w}\in\mathbf{W}}$ is called a **tiling** with $\mathbf{W}$.

The following definition is due to [2].

Definition 2. (1) A flat polygon V is called a **doubly-covered square** if it consists of two congruent square faces joined together along each of the corresponding edges.

(2) A connected plane figure $\mathbf{A}$ is called a **development of doubly-covered square** V if it is obtained by cutting the surface of V and opening up its faces. The resulting figure is bounded by a simple closed curve.

3 Tilings and developments

Let M be an expanding 2×2 integer matrix with $|\det M| = 2$, where all eigenvalues of M have modulus > 1. Then M^{-1} can be considered as a contracting map of $\mathbb{R}^2$. For $\mathbf{d}_1 \in \mathbb{Z}^2$ with $\mathbf{d}_1 \neq \mathbf{d}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, let $D = \{\mathbf{d}_0,\ \mathbf{d}_1\}$ and $\mathbf{A} = \left\{\sum_{j=1}^{\infty} M^{-j}\mathbf{d}_{i_j} \in \mathbb{R}^2 \mid \mathbf{d}_{i_j} \in D\right\}$. Then $\mathbf{A}$ is a compact set. We shall consider conditions when $\mathbf{A}$ becomes a development of a doubly-covered square. At first, we show the property of $\mathbf{A}$.

Proposition 1. *Let* $\mathbf{A}$ *and* $\mathbf{W}_A$ *be defined as in Notation* 1. *Then the following holds.*

(1) *The set* $\mathbf{A}$ *has point symmetry.*

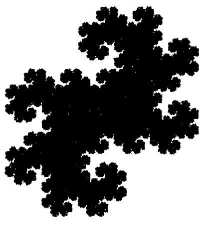
(2) $\{\mathbf{A}+\mathbf{w}\}_{\mathbf{w}\in\mathbf{W}_A}$ *is a tiling.*

Proof. (1) Put $\mathbf{a}_1 = \sum_{j=1}^{\infty} M^{-j}\mathbf{d}_1$ and $\mathbf{c} = \frac{\mathbf{a}_1}{2}$. For all $\mathbf{b}_0 = \sum_{j=1}^{\infty} M^{-j}\mathbf{d}_{i_j} \in \mathbf{A} = \left\{\sum_{j=1}^{\infty} M^{-j}\mathbf{d}_{i_j} \in \mathbb{R}^2 | \, \mathbf{d}_{i_j} \in D\right\}$, let $\mathbf{b}_1 = \sum_{j=1}^{\infty} M^{-j}(\mathbf{d}_1 - \mathbf{d}_{i_j})$. Since $\mathbf{d}_1 - \mathbf{d}_{i_j} = \mathbf{d}_0$ or $\mathbf{d}_1 - \mathbf{d}_{i_j} = \mathbf{d}_1$, we have $\mathbf{b}_1 \in \mathbf{A}$. Moreover, since $\mathbf{b}_0 + \mathbf{b}_1 = \sum_{j=1}^{\infty} M^{-j}(\mathbf{d}_{i_j} + \mathbf{d}_1 - \mathbf{d}_{i_j}) = \sum_{j=1}^{\infty} M^{-j}\mathbf{d}_1 = \mathbf{a}_1$, we have $\frac{\mathbf{b}_0+\mathbf{b}_1}{2} = \frac{\mathbf{a}_1}{2} = \mathbf{c}$.

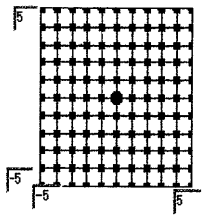
Therefore, for all $\mathbf{b}_0 \in \mathbf{A}$ there is a point $\mathbf{b}_1 \in \mathbf{A}$ such that $\dfrac{\mathbf{b}_0 + \mathbf{b}_1}{2} = \mathbf{c}$, which means $\mathbf{A}$ has point symmetry with respect to $\mathbf{c}$.

(2) In this case, D consists of two elements and so every element of $\mathbf{W}_A$ is uniquely expressed. So by Theorem A, $\{\mathbf{A} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}_A}$ is a tiling. □

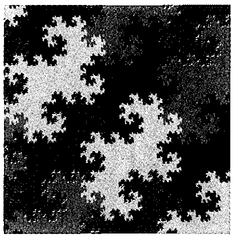
Example 1.
Let $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Then the sets $\mathbf{A}$, $\mathbf{W}_A$ and $\bigcup_{\mathbf{w} \in \mathbf{W}_A} (\mathbf{A} + \mathbf{w})$ are shown in the following.



the set $\mathbf{A}$



the lattice $\mathbf{W}_A$



$\bigcup_{\mathbf{w} \in \mathbf{W}_A} (\mathbf{A} + \mathbf{w})$

Since the set $\mathbf{A}$ has point symmetry, we shall consider the property of the set $\mathbf{B}$ which has point symmetry, where the set $\mathbf{B}$ has no relation with the matrix.

Lemma 1. *Let $\mathbf{W}_1$, $\mathbf{W}_2$, P_j and $\tilde{P}_j$ be defined as in Notations 2 and 4. Let the compact set $\mathbf{B}$ has point symmetry with respect to the origin and the area $|\mathbf{B}|$ of $\mathbf{B}$ is 2.*

(1) *If $\{\mathbf{B} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}_1}$ is a tiling, then the point P_j ($j \in \{1, 2, 3, 4\}$) is not an interior point of $\mathbf{B}$.*

(2) *If $\{\mathbf{B} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}_2}$ is a tiling, then the point $\tilde{P}_j$ ($j \in \{1, 2, 3, 4\}$) is not an interior point of $\mathbf{B}$.*

Proof. (1) Since $\{\mathbf{B} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}_1}$ is a tiling, $\mathbf{B}^\circ \cap \left(\mathbf{B}^\circ + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \emptyset$, where $\mathbf{B}^\circ$ is interior of $\mathbf{B}$. If P_1 is an interior point of B, there is $\varepsilon > 0$ such that $\mathbf{B} \supset U_\varepsilon(P_1)$. Since $\mathbf{B}$ has point symmetry with respect to the origin, we have $\mathbf{B} \supset U_\varepsilon(-P_1) = U_\varepsilon(P_3)$. So $\mathbf{B} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \supset U_\epsilon(P_3) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = U_\epsilon(P_1)$. Therefore, we have $\mathbf{B} \cap \left(\mathbf{B} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \supset U_\epsilon(P_1)$, which contradicts to $\mathbf{B}^\circ \cap \left(\mathbf{B}^\circ + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \emptyset$, so P_1 and P_3 are not interior points of $\mathbf{B}$. Likewise, since $\mathbf{B}^\circ \cap \left(\mathbf{B}^\circ + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = \emptyset$, P_2 and P_4 are not interior points of $\mathbf{B}$.

(2) Since $\{\mathbf{B} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}_2}$ is a tiling, $\mathbf{B}^\circ \cap \left(\mathbf{B}^\circ + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = \emptyset$. If $\tilde{P}_1$ is an interior point of $\mathbf{B}$, there is $\varepsilon > 0$ such that $\mathbf{B} \supset U_\varepsilon(\tilde{P}_1)$. Since $\mathbf{B}$ has point symmetry with respect to the origin, we have $\mathbf{B} \supset U_\varepsilon(-\tilde{P}_1) = U_\varepsilon(\tilde{P}_1 - \begin{pmatrix} 1 \\ 2 \end{pmatrix})$. So $\mathbf{B} \cap \left(\mathbf{B} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) \supset U_\epsilon(\tilde{P}_1)$, which contradicts to $\mathbf{B}^\circ \cap \left(\mathbf{B}^\circ + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = \emptyset$. So $\tilde{P}_1$ is not an interior points of $\mathbf{B}$. In the same way, $\tilde{P}_4$ is not

an interior points of $\mathbf{B}$. In a similar way, we can prove that $\tilde{P}_2$ and $\tilde{P}_3$ are not interior points of $\mathbf{B}$ by using $\mathbf{B}^\circ \cap \left(\mathbf{B}^\circ + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \emptyset$. □

To prove that $\mathbf{B}$ is a development of a doubly-covered square, we need the following Lemmas 2 and 3.

Lemma 2. *Let $\mathbf{W}_1$, $\mathbf{W}_2$ F_j and $\tilde{F}_j$ be defined as in Notations 2 and 5 . Then for any $i, j \in \{1,2,3,4\}$ with $i \neq j$,*

(1) *there exist $k \in \{0,1\}$ and $\mathbf{w} \in \mathbf{W}_1 \backslash \mathbf{d}_0$ satisfying*

(a) *$F_i(\mathbf{x}) = F_j((-1)^k \mathbf{x} + \mathbf{w})$ for any $\mathbf{x} \in \mathbb{R}^2$ and*

(b) *$F_i F_j(\mathbf{x}) = (-1)^k \mathbf{x} + \mathbf{w}$ for any $\mathbf{x} \in \mathbb{R}^2$.*

(2) *there exist $k \in \{0,1\}$ and $\mathbf{w} \in \mathbf{W}_2 \backslash \mathbf{d}_0$ satisfying*

(a) *$\tilde{F}_i(\mathbf{x}) = \tilde{F}_j((-1)^k \mathbf{x} + \mathbf{w})$ for any $\mathbf{x} \in \mathbb{R}^2$ and*

(b) *$\tilde{F}_i \tilde{F}_j(\mathbf{x}) = (-1)^k \mathbf{x} + \mathbf{w}$ for any $\mathbf{x} \in \mathbb{R}^2$.*

Proof. (1) By using the relations $F_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+1 \\ y \end{pmatrix}$, $F_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y-1 \end{pmatrix}$, $F_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x-1 \\ y \end{pmatrix}$ and $F_4\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y+1 \end{pmatrix}$, we get the result.

(2) By using the relations $\tilde{F}_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+1 \\ y \end{pmatrix}$, $\tilde{F}_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\tilde{F}_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x-1 \\ y \end{pmatrix}$ and $\tilde{F}_4\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y+2 \end{pmatrix}$, we get the result. □

By using Lemma 2, we get the following Lemma.

Lemma 3. *Let F_j and $\tilde{F}_j$ be defined as in Notations 1 and 5. Let the compact set $\mathbf{B}$ have point symmetry with respect to the origin and the area $|\mathbf{B}|$ of $\mathbf{B}$ is 2. Then for any $i, j \in \{1,2,3,4\}$ with $i \neq j$, we have*

$$F_i(\mathbf{B}^\circ) \cap F_j(\mathbf{B}^\circ) = \emptyset \quad \textit{and} \quad F_i F_j(\mathbf{B}^\circ) \cap \mathbf{B}^\circ = \emptyset.$$

$$\tilde{F}_i(\mathbf{B}^\circ) \cap \tilde{F}_j(\mathbf{B}^\circ) = \emptyset \quad \textit{and} \quad \tilde{F}_i \tilde{F}_j(\mathbf{B}^\circ) \cap \mathbf{B}^\circ = \emptyset.$$

Now we give a sufficient condition for the set $\mathbf{B}$ to be a development of a doubly-covered square.

Proposition 2. *Let the compact set $\mathbf{B}$ have point symmetry with respect to the origin and the area $|\mathbf{B}|$ of $\mathbf{B}$ is 2. If $\{\mathbf{B} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}}$ is tiling with $\mathbf{W} = \mathbf{W}_1$, $\mathbf{W}_2$ or $\mathbf{W}'_2$, then $\mathbf{B}$ is a development of a doubly-covered square.*

Proof. (1) Suppose $\mathbf{W} = \mathbf{W}_1$. We shall show that some family $\{F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_{2l}}(\mathbf{B}) \cap S_1\}$ are mutually disjoint and their union covers the square S_0 and some family $\{F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_{2l+1}}(\mathbf{B}) \cap S_0\}$ are mutually disjoint and their union covers the opposite side of the square S_0.

Put $A_1 = \mathbf{B} \cap S_0^\circ$, $C_{1,1} = \mathbf{B}^\circ \cap S_1$, $C_{1,2} = \mathbf{B}^\circ \cap S_2$, $C_{1,3} = \mathbf{B}^\circ \cap S_3$ and $C_{1,4} = \mathbf{B}^\circ \cap S_4$.

Then by Lemma 3, we have $F_i(C_{1,i}) \cap F_j(C_{1,j}) = \emptyset$ for $i \neq j$. Put

$$D_1 = \bigcup_{j=1}^{4} F_j(C_{1,j}) \tag{3.1}$$

and $B_1 = D_1 \cap S_0^\circ$. Since $F_j(C_{1,j})$ is mutually disjoint, we have $|D_1| = \sum_{j=1}^4 |C_{1,j}|$ and so

$$|D_1| = |\mathbf{B}| - |A_1| \geq 1 \tag{3.2}$$

by $|\mathbf{B}| = 2$ and $|A_1| \leq 1$.
(i) If $D_1 \backslash B_1 = \emptyset$, then $|D_1| \leq 1$ and with (3.2), we have $|D_1| = 1$ and $|A_1| = 1$. So A_1 covers S_0 and D_1 covers the opposite side of S_0.
(ii) If $D_1 \backslash B_1$ is nonempty, put

$$C_{2,1} = D_1 \cap S_1,\ C_{2,2} = D_1 \cap S_2, C_{2,3} = D_1 \cap S_3 \quad \text{and} \quad C_{2,4} = D_1 \cap S_4.$$

We claim that $A_1^\circ \cap F_i(C_{2,i}) = \emptyset$ for $i \in \{1,2,3,4\}$. For if there exists $\mathbf{x} \in A_1^\circ \cap F_i(C_{2,i})$, then there exists $\mathbf{y} \in C_{2,i} \subset D_1$ such that $\mathbf{x} = F_i(\mathbf{y})$. By (3.1) there exists j such that $i \neq j$ and $\mathbf{y} \in F_j(C_{1,j}) \subset F_j(\mathbf{B}^\circ)$ and $\mathbf{z} \in \mathbf{B}^\circ$ such that $y = F_j(\mathbf{z})$. So $\mathbf{x} = F_iF_j(\mathbf{z})$ and $\mathbf{B}^\circ \cap F_iF_j(\mathbf{B}^\circ) \neq \emptyset$, which contradicts Lemma 3.

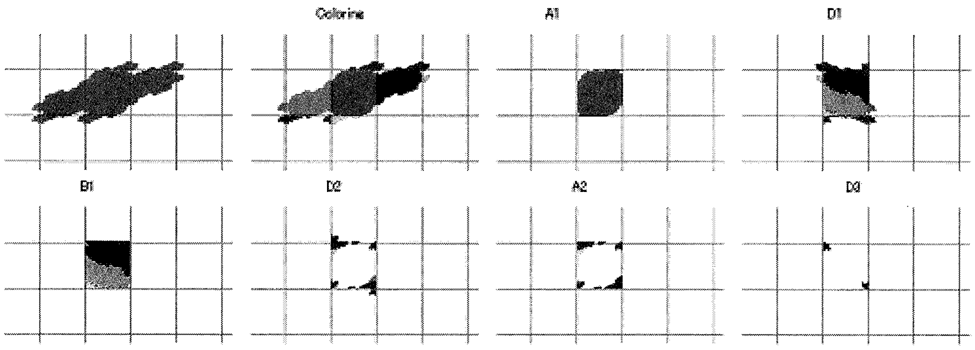
So A_1° and $F_j(C_{2,j})$ ($j \in \{1,2,3,4\}$) are mutually disjoint. Put $D_2 = \bigcup_{j=1}^{4} F_j(C_{2,j})$ and $A_2 = D_2 \cap S_0^\circ$. Then $|D_2| = \sum_{j=1}^4 |C_{2,j}| = |D_1| - |B_1| = 2 - |A_1| - |B_1|$. If $D_2 \backslash A_2 = \emptyset$, then $|A_1| + |A_2| = |A_1| + |D_2| = 2 - |B_1| \geq 1$. Since A_1 and A_2 are disjoint and are contained in S_0, $|A_1| + |A_2| \leq 1$ implies that $|A_1| + |A_2| = 1$. So the union of A_1 and A_2 covers S_0 and B_1 covers the opposite side of S_0.

If $D_2 \backslash A_2$ is nonempty, for $j \geq 2$ we continue this process inductively as follows.

$$A_j = D_{2j-2} \cap S_0^\circ,\ C_{2j-1,i} = D_{2j-2} \cap S_i,\ D_{2j-1} = \bigcup_{i=1}^{4} F_i(C_{2j-1,i})$$

$$B_j = D_{2j-1} \cap S_0^\circ,\ C_{2j,i} = D_{2j-1} \cap S_i,\ D_{2j} = \bigcup_{i=1}^{4} F_i(C_{2j,i})$$

Hence $\{A_j\}$ is mutually disjoint and $\bigcup A_j$ covers S_0. Also $\{B_j\}$ is mutually disjoint and $\bigcup B_j$ covers the opposite side of S_0.



(2) Suppose $\mathbf{W} = \mathbf{W}_2$. We shall prove in a similar way as in (1). Put $\tilde{D}_0 = \mathbf{B}$ and define $\tilde{A}_j$ and $\tilde{B}_j$ inductively for $j \geq 1$ as follows.

$$\tilde{A}_j = \tilde{D}_{2j-2} \cap \tilde{S}_0^\circ,\ \tilde{C}_{2j-1,i} = \tilde{D}_{2j-2} \cap \tilde{S}_i,\ \tilde{D}_{2j-1} = \bigcup_{i=1}^{4} \tilde{F}_i(\tilde{C}_{2j-1,i})$$

$$\tilde{B}_j = \tilde{D}_{2j-1} \cap \tilde{S}_0^\circ,\ \tilde{C}_{2j,i} = \tilde{D}_{2j-1} \cap \tilde{S}_i,\ \tilde{D}_{2j} = \bigcup_{i=1}^{4} \tilde{F}_i(\tilde{C}_{2j,i})$$

Then by using Lemma 3, $\{\tilde{A}_j\}$and $\{\tilde{F}_i(\tilde{C}_{2j-1,i})\}$ are mutually disjoint and $\bigcup \tilde{A}_j$ covers $\tilde{S}_0$. Also $\{\tilde{B}_j\}$ and $\tilde{F}_i(\tilde{C}_{2j,i})\}$ are mutually disjoint and $\bigcup \tilde{B}_j$ covers the opposite side of $\tilde{S}_0$. □

We apply Proposition 2 to the set $\mathbf{A}$ derived from 2×2 matrix M and a digit set D.

Theorem 1. *Let the map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$\varphi \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i+j \\ i-j \end{pmatrix}$$

and let the lattices $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_2'$ be defined by $\mathbf{W}_1 = \left\{ \begin{pmatrix} i+j \\ i-j \end{pmatrix} \mid i,j \in \mathbb{Z} \right\}$, $\mathbf{W}_2 = \left\{ \begin{pmatrix} i \\ 2j \end{pmatrix} \mid i,j \in \mathbb{Z} \right\}$ *and* $\mathbf{W}_2' = \left\{ \begin{pmatrix} 2i \\ j \end{pmatrix} \mid i,j \in \mathbb{Z} \right\}$. *Let M be an expanding 2×2 integer matrix with $|\det M| = 2$ and*

$$\begin{aligned} D &= \left\{ \mathbf{d}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{d}_1 \in \mathbb{Z}^2, \mathbf{d}_1 \neq \mathbf{d}_0 \right\} \\ \mathbf{A} &= \left\{ \sum_{j=1}^{\infty} M^{-j} \mathbf{d}_{i_j} \in \mathbb{R}^2 \quad | \quad \mathbf{d}_{i_j} \in D \right\}. \end{aligned}$$

If there exists $k \in \mathbb{N} \cup \{0\}$ satisfying

(1) *the area of $\mathbf{A}$ is 2^k and*

(2) *$\{\mathbf{A} + \mathbf{w}\}_{w \in \mathbf{W}}$ is a tiling with $\mathbf{W} = \varphi^{k-1}(\mathbf{W}_1)$, $\varphi^{k-1}(\mathbf{W}_2)$ or $\varphi^{k-1}(\mathbf{W}_2')$,*

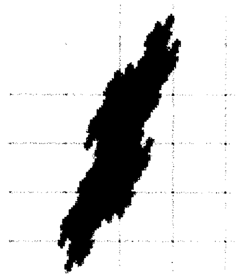
then $\mathbf{A}$ is a development of a doubly-covered square.

Proof. If the area of $\mathbf{A}$ is 2^k with some $k \in \mathbb{N} \cup \{0\}$, put $\mathbf{B} = \varphi^{-(k-1)}(\mathbf{A})$. Then the set $\mathbf{B}$ satisfies the assumption of Proposition 2 and so $\mathbf{B}$ is a development of a doubly-covered square. Hence $A = \varphi^{k-1}(\mathbf{B})$ is also a development of a doubly-covered square. □

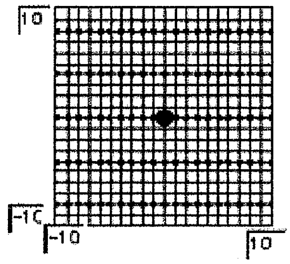
Remark 1. As shown in Proposition 1, $\{\mathbf{A} + \mathbf{w}\}_{w \in \mathbf{W}}$ is always a tiling with a lattice $\mathbf{W} = \mathbf{W}_A$. If the area of $\mathbf{A}$ is 2^k with $k \geq 2$, $\mathbf{W}_A$ is sometimes neither of $\varphi^{k-1}(\mathbf{W}_1)$, $\varphi^{k-1}(\mathbf{W}_2)$ nor $\varphi^{k-1}(\mathbf{W}_2')$. So even if $\{\mathbf{A} + \mathbf{w}\}_{w \in \mathbf{W}_A}$ is a tiling, $\mathbf{A}$ is not necessarily a development of a doubly-covered square as shown in the following example.

Example 2. Let $M = \begin{pmatrix} 2 & -1 \\ 4 & -1 \end{pmatrix}$ and $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

Then the area of $\mathbf{A}$ is 4 and the set $\mathbf{A}$ is not a development of a doubly-covered square, although $\{\mathbf{A} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}_A}$ is a tiling.



A



$$\mathbf{W}_A = \left\{ \left(\begin{array}{c} i \\ 4j \end{array} \right) \mid i, j \in \mathbb{Z} \right\}$$

Corollary 1. *Let M be an expanding 2×2 integer matrix with $|\det M| = 2$ and*

$$\begin{aligned} D &= \left\{ \mathbf{d}_0 = \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \mathbf{d}_1 \in \mathbb{Z}^2, \mathbf{d}_1 \neq \mathbf{d}_0 \right\} \\ \mathbf{A} &= \left\{ \sum_{j=1}^{\infty} M^{-j} \mathbf{d}_{i_j} \in \mathbb{R}^2 \quad | \quad \mathbf{d}_{i_j} \in D \right\}. \end{aligned}$$

If the area of $\mathbf{A}$ is 1 or 2, then $\mathbf{A}$ is a development of a doubly-covered square.

Proof. If the area of $\mathbf{A}$ is 1, then $\mathbf{W}_A$ is $\varphi^{-1}(\mathbf{W}_1)$. If the area of $\mathbf{A}$ is 2, then $\mathbf{W}_A$ is $\mathbf{W}_1$, $\mathbf{W}_2$ or $\mathbf{W}'_2$. So $\mathbf{A}$ is a development of a doubly-covered square. $\square$

Next we shall investigate a necessary condition for a compact set $\mathbf{B}$ to be a development of a doubly-covered square.

Lemma 4. *Suppose that a compact set $\mathbf{B}$ has point symmetry with $|\mathbf{B}| = 2$ and is a development of a doubly-covered square with a square face S .*

1. *If the center of $\mathbf{B}$ is the center of $S = S_0$ with vertices $\{P_j\}$, then*

$$\mathbf{B}^\circ \cap R_j(\mathbf{B}^\circ) = \emptyset \quad \text{for } j \in \{1, 2, 3, 4\},$$

where R_j is defined in Notation 5.

2. *If the center of $\mathbf{B}$ is the midpoint of an edge $\tilde{P}_1\tilde{P}_4$ of $S = \tilde{S}_0$, then*

$$\mathbf{B}^\circ \cap \tilde{R}_j(\mathbf{B}^\circ) = \emptyset \quad \text{for } j \in \{1, 2, 3, 4\},$$

where $\tilde{R}_j$ is defined in Notation 5

Proof. By the definition, $R_j = F_j F_{j+1}$ and $\tilde{R}_j = \tilde{F}_j \tilde{F}_{j+1}$ holds. So by Lemma3, the result is obtained. $\square$

Now we give an equivalent condition for the set $\mathbf{B}$ to be a development of a doubly-covered square.

Theorem 2. *Let the compact set $\mathbf{B}$ have point symmetry and the area $|\mathbf{B}|$ of $\mathbf{B}$ is 2. Then the following* (1) *and* (2) *are equivalent.*

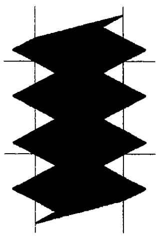
(1) $\{\mathbf{B} + \mathbf{w}\}_{\mathbf{w} \in \mathbf{W}}$ *is a tiling with* $\mathbf{W} = \mathbf{W}_1$, $\mathbf{W}_2$ *or* $\mathbf{W}'_2$.

(2) (i) *The set* $\mathbf{B}$ *is a development of a doubly-covered square* V *with a face* S *and*
(ii) *the center of* $\mathbf{B}$ *is either the center of* S *or the midpoint of some edge of* S.

Proof. (1) $\to$ (2): follows from Lemma 1 and Proposition 2.
(2) $\to$ (1): Let S be S_0 defined in Notation 4 with vertices $\{P_j\}_{j\in\{1,2,3,4\}}$.
(i) If the center of $\mathbf{B}$ is the center of S_0, that is, the origin, then $\mathbf{B}^\circ \cap R_1(\mathbf{B}) = \emptyset$ by Lemma 4. Since $\mathbf{B}$ has point symmetry with respect to the origin, $R_1(\mathbf{B}) = \mathbf{B} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ holds. So we have $\mathbf{B}^\circ \cap \left(\mathbf{B}^\circ + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \emptyset$.
In the same way, we have $\mathbf{B}^\circ \cap \left(\mathbf{B}^\circ + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \emptyset$ by $\mathbf{B}^\circ \cap R_2(\mathbf{B}) = \emptyset$. Hence $\{\mathbf{B}+\mathbf{w}\}_{\mathbf{w}\in\mathbf{W}_1}$ is a tiling.
(ii) If the center of $\mathbf{B}$ is the midpoint of some edge of S, we can prove in a similar way to (i) that $\{\mathbf{B}+\mathbf{w}\}_{\mathbf{w}\in\mathbf{W}}$ is a tiling with $\mathbf{W} = \mathbf{W}_2$ or $\mathbf{W}_2'$ by using $\tilde{R}_j$ instead of R_j. $\square$

Remark 2. In Theorem 2, the condition (ii) of (2) is necessary. Even if the set $\mathbf{B}$ is a development of a doubly-covered square, (1) does not hold without the condition (ii), as shown in the following example 3.

Example 3.



A development of doubly-covered square, which cannot be tiled by $\mathbf{W}_1$ nor $\mathbf{W}_2$.

By Theorems 1 and 2, we have the following theorem.

Theorem 3. *Let the map* $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$\varphi\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i+j \\ i-j \end{pmatrix}$$

and let the lattices $\mathbf{W}_1$, $\mathbf{W}_2$ *and* $\mathbf{W}_2'$ *be defined by* $\mathbf{W}_1 = \left\{\begin{pmatrix} i+j \\ i-j \end{pmatrix} \mid i,j \in \mathbb{Z}\right\}$, $\mathbf{W}_2 = \left\{\begin{pmatrix} i \\ 2j \end{pmatrix} \mid i,j \in \mathbb{Z}\right\}$ *and* $\mathbf{W}_2' = \left\{\begin{pmatrix} 2i \\ j \end{pmatrix} \mid i,j \in \mathbb{Z}\right\}$. *Let* M *be an expanding* 2×2 *integer matrix with* $|\det M| = 2$ *and*

$$\begin{aligned} D &= \left\{\mathbf{d}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{d}_1 \in \mathbb{Z}^2, \mathbf{d}_1 \neq \mathbf{d}_0\right\} \\ \mathbf{A} &= \left\{\sum_{j=1}^{\infty} M^{-j}\mathbf{d}_{i_j} \in \mathbb{R}^2 \mid \mathbf{d}_{i_j} \in D\right\}. \end{aligned}$$

Then the following (1) *and* (2) *are equivalent.*

(1) *There exists* $k \in \mathbb{N}\cup\{0\}$ *satisfying*

(a) *the area of* $\mathbf{A}$ *is* 2^k *and*

(b) $\{\mathbf{A}+\mathbf{w}\}_{w\in\mathbf{W}}$ *is a tiling with* $\mathbf{W}=\varphi^{k-1}(\mathbf{W}_1)$, $\varphi^{k-1}(\mathbf{W}_2)$ *or* $\varphi^{k-1}(\mathbf{W}_2')$.

(2) *The set* $\mathbf{A}$ *is a development of a doubly-covered square* V *with a face* S *and the center of* $\mathbf{A}$ *is either the center of* S *or the midpoint of some edge of* S.

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