A Numerical Method for Flow in a Two-Dimensional Channel of Large Aspect Ratio

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(Received October 3, 2007)
(Revised January 21, 2008)

Abstract

A new numerical method which is suitable for calculations of incompressible fluid flow with high aspect ratio such as a blood vessel or a river flow is presented. The flow variables are expressed as sum of those of the main flow and its variations. The former is obtained by solving one-dimensional equation analytically. Coupling of two-directional main flow equations enables a long duct of large curvature to be solved using Cartesian coordinate system. Two-dimensional flows in a long duct inclined at varying angles are performed to study the effectiveness of the present method. Furthermore, a flow in an orthogonal channel using two sub-domain grids is examined. These results indicate that the present method is effective in conservation of mass of fluid in a long region that is very difficult to satisfy the equation of continuity by other method such as the standard MAC method.

1 INTRODUCTION

When a very long region such as a river or a blood vessel is computed, it is quite difficult to satisfy the equation of continuity precisely. Though using the method based on the stream-function and vorticity resolves the above problem, this method works only on the two dimensional or axi-symmetric flows.

Owing to these shortcomings involved in stream-function vorticity formulation, the MAC method [1] and its variations are preferred to treat three-dimensional flow and the flow with curved geometry often appearing in a river or a blood vessel. On the other hand, the conservation of mass is only approximately satisfied by the method based on the Poisson equation of pressure appearing in the MAC method that is difficult to converge by using the iterative method. This disadvantage is emphasized in solving of a complex geometry where the generalized coordinate system is used and/or aspect ratio is remarkably high. In addition, an unsteady flow such as a blood vessel with pumping is difficult to compute precisely.

Kawamura and Miyashita [2] presented a new numerical method that is suitable for calculations of incompressible fluid flow in a very long region. However, their computational method is limited one-directional long region otherwise a polar coordinate system is required to treat a curved region. The method developed in the present work enables us to treat highly bended region using Cartesian coordinate system only.

In this paper, we present a developed numerical method based on the one presented in [2] and report several test calculations of flow in long bent and unbent channels.

2 NUMERICAL METHOD

Kawamura and Miyashita [2] have developed a new numerical method for a calculation of incompressible fluid flow with high aspect ratio such as a blood vessel or a river.

This development is summarized below. First, flow in a long region is divided into basic flow and its variation. The basic flow is defined as one-dimensional time-dependent flow $U_B(t)$ which is governed by the one-dimensional Navier-Stokes equation. Using this method, only variations $\tilde{u}(t,x,y)$ are required to be solved in the computation since the basic flow $U_B(t)$ can be obtained analytically.

While evolutionary, this method was difficult to handle a long region with greatly bended, since flow directions are strayed from those of the basic flow $U_B(t)$ in the greatly bended region. In order to handle such cases, i.e. multi-directional basic flows, we divided the whole computational domain into sub-domains where the basic flow is one-dimensional. Introduction of sub-domains requires to take into account some cases that the main flow is directed diagonally to coordinates(Fig.2). In order to overcome this problem, two-directional basic flows are introduced into the basic equations in this study. Using this method, we can handle a long channel of arbitrary shaped geometry by combining computational grids with different basic values. In this study, two-dimensional Cartesian coordinate system is considered.

When the geometry of a long duct is linearly extended, the basic flow is along the x-direction, y-direction(Fig.1) or directed diagonally to the axis(Fig.2).

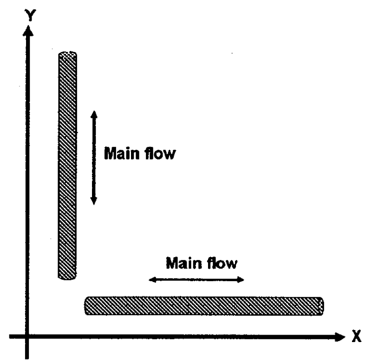

Figure 1: Illustration of a channel which is along the axis.

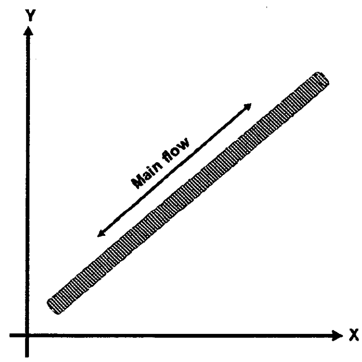

Figure 2: Illustration of a channel which is directed diagonally to the axis.

First, the basic flow is assumed to be along the x-direction. The flow can be treated as the basic flow U_B and its difference $\tilde{u}$. The basic flow is one-dimensional time-dependent which is governed by the one-dimensional Navier-Stokes equation as follows:

$$\frac{\partial U_B}{\partial x} = 0 \tag{1}$$

$$\frac{\partial U_B}{\partial t} + U_B \frac{\partial U_B}{\partial x} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 U_B}{\partial x^2}\right) \tag{2}$$

From eq.(1) the velocity of the basic flow is spatially homogeneous, i.e. $U_B(t)$. We can obtain the basic pressure P_B by substituting this into equation (2)

$$\frac{\partial U_B}{\partial t} = -\frac{\partial P_B}{\partial x}. \tag{3}$$

Then, the basic velocity of y-direction in the above manner is

$$\frac{\partial V_B}{\partial t} = -\frac{\partial P_B}{\partial y}. \tag{4}$$

From eq.(3) and eq.(4),

$$P_B = -\frac{\partial U_B}{\partial t}x - \frac{\partial V_B}{\partial t}y + c \qquad (c : constant). \tag{5}$$

Therefore, we can define the variation of the velocity and pressure by

$$u(t,x,y) = U_B(t) + \tilde{u}(t,x,y) \tag{6}$$

$$v(t,x,y) = V_B(t) + \tilde{v}(t,x,y) \tag{7}$$

$$p(t,x,y) = -\frac{\partial U_B(t)}{\partial t}x - \frac{\partial V_B(t)}{\partial t}y + c + \tilde{p}(t,x,y). \tag{8}$$

Substituting eq.(6), eq.(7) and eq.(8) into the Navier-Stokes equation yields following equations.

The conservation of mass is

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0. \tag{9}$$

The equations of motion are

$$\frac{\partial \tilde{u}}{\partial t} + (U_B + \tilde{u})\frac{\partial \tilde{u}}{\partial x} + (V_B + \tilde{v})\frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right) \tag{10}$$

$$\frac{\partial \tilde{v}}{\partial t} + (U_B + \tilde{u})\frac{\partial \tilde{v}}{\partial x} + (V_B + \tilde{v})\frac{\partial \tilde{v}}{\partial y} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right). \tag{11}$$

In these equations, u and v are the $x-$ and $y-$ components of the velocity. These equations are solved by the standard method such as the MAC method. The function $U_B(t)$ is determined by the boundary condition at the inlet. Note that the boundary conditions on fixed walls become

$$\tilde{u} = -U_B(t), \qquad \tilde{v} = -V_B(t), \tag{12}$$

since no-slip condition is imposed on the wall.

3 RESULTS

We have compared the present method with the standard MAC method by computing two types of channels: the geometry of the channel is unbent and bent.

3.1 Two-dimensional flow in a non-uniform unbent channel

At first, we show results of the flow in a non-uniform duct inclined at various angles to the x-axis. The shape of the wall is wavy which changes sinusoidally in the flow direction(fig.3).

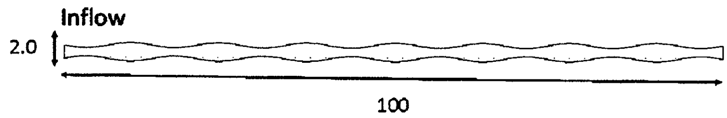

Figure 3: The computational domain of the non-uniform unbent channel.

Therefore, we use generalized coordinate system

$$\xi = \xi(x,y), \qquad \eta = \eta(x,y) \tag{13}$$

for eqs. (9), (10) and (11) so that the independent variables become ξ and η. Flow variables are obtained by solving basic equations in the transformed plane. We examined four cases which directions of the channel is tilted at 0 degrees, 30 degrees, 45 degrees and 60 degrees to the x-axis, respectively. Grids used in these four cases are those rotated of the one used in 0 degrees case (fig. 3). The iteration number of the Poisson equation is 20 in all computations. The flux at the inlet oscillates between 0.5 and 1.5 temporally as

$$flux(t) = 1.0 + 0.5\sin(\omega t) \tag{14}$$

where ω is an angular velocity of the oscillating inlet flow, t is an elapse time. The inlet velocity of the main flow $U_B(t)$ and $V_B(t)$ are canonicalized by its width.

Figure 4 show instantaneous velocity vectors obtained by the standard MAC method and the present method, respectively when the channel is inclined 30 degrees to the x-axis. Both vectors are captured at the same calculation time.

In this case, the basic flux is defined as follows:

$$flux_x(t) = \frac{\sqrt{(3)}}{2}\{1.0 + 0.5\sin(\omega t)\}, \qquad flux_y(t) = \frac{1}{2}\{1.0 + 0.5\sin(\omega t)\}. \tag{15}$$

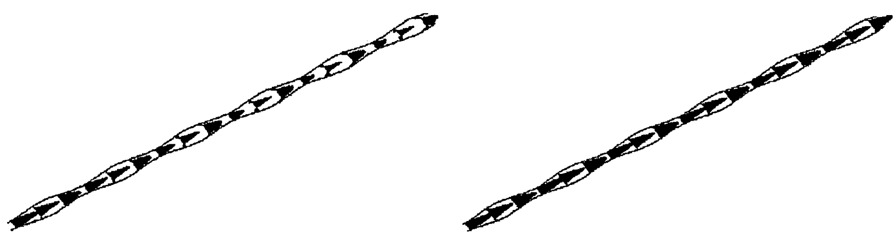

Figure 4: Instantaneous velocity vectors when the channel is tilted 30 degrees to the x-axis. The left vectors are obtained by the standard MAC method, the right are the present method.

Comparing these two figures, the flow rate obtained by the present method is almost the same in every cross section of the channel. This implies those results with the present method are much better consistent with the assumption of the incompressibility.

Figure 5 shows the time history of the flux at inflow and at half length of the channel with the standard MAC method and the present method. It is apparent that the flux obtained by the present method satisfies the equation of continuity more precisely than those obtained by the standard MAC method.

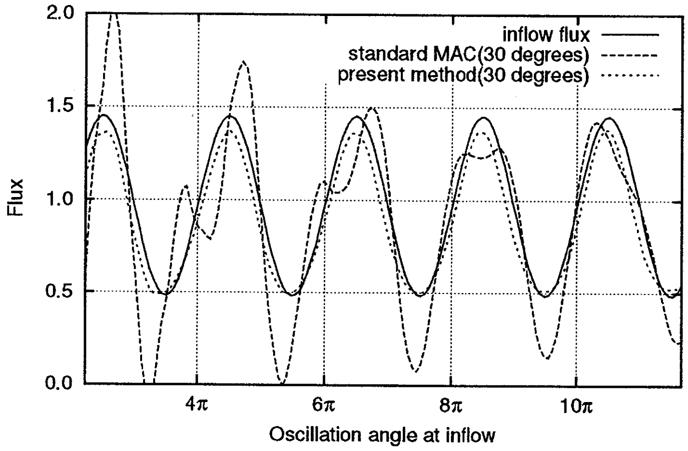

Figure 5: Time history of the flux at half length of the channel with comparison of the standard MAC method and the present method.

Figure 6 shows the flux errors relative to the inflow flux. In this figure, absolute differences between the inflow flux and every cross-sectional flux are summed from the inlet to the outlet. Then, this total flux differences are averaged by the number of cross-section grid at every time step.

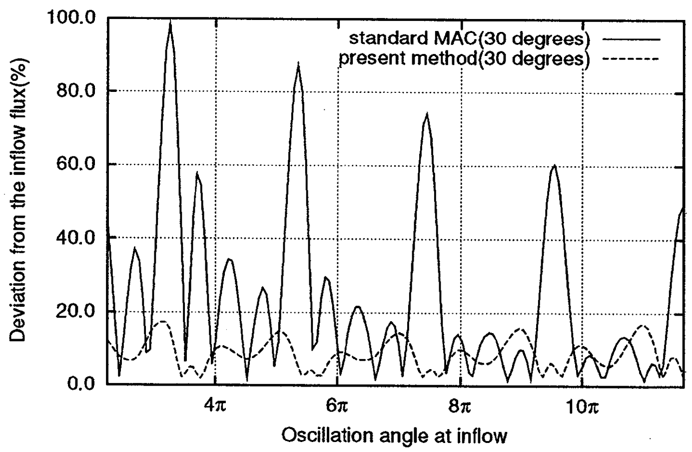

Figure 6: Time history of deviation from the inflow flux of the unbent channel.

From this figure, the flux errors fluctuate within a certain low range from the beginning by the present method, while those of the standard MAC method decreases with time. This result seems the effect of introduction basic flows U_B and V_B, since the flow field obtained by the present method are given approximate analytical solutions from the beginning Figure

7 shows the comparison of flux errors with various degrees of the channel. Those values coincide with each other completely.

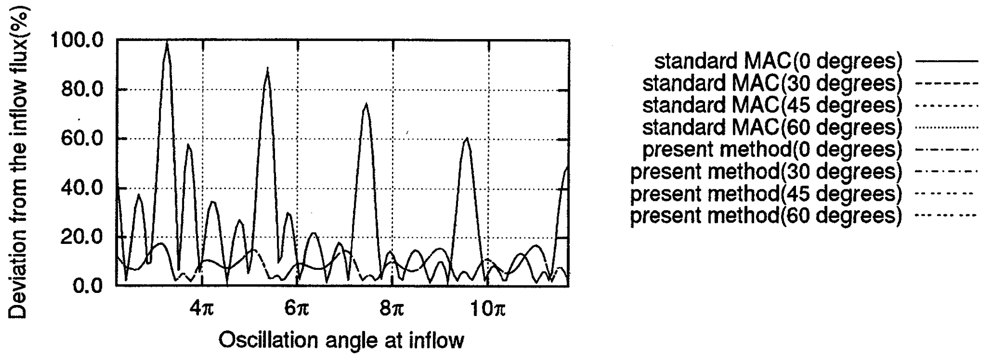

Figure 7: Time history of deviation from the inflow flux comparing between channels of various degrees.

All tilted channels are just rotated of the channel used in the 0 degrees case, so do the basic flux. Therefore, those results obtained by various tilted channels are expected to coincide with each other regardless of those angles to the axis. From this point of view, it is confirmed that the superposition of two basic flows U_B and V_B is valid.

3.2 Two-dimensional flow in an orthogonally bent channel

Second, we examined an application of an orthogonally bent channel using two sub-domain grids. In this case, the two sub-domains where the direction of the basic flow is different are connected as shown in fig.8.

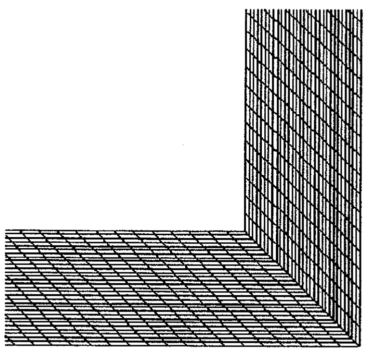

Figure 8: Connection of two sub-domains.

In every sub-domain, the basic flux is defined as follows, respectively:

$$flux_x(t) = 1.0 + 0.5\sin(\omega t), \qquad flux_y(t) = 0.0 \tag{16}$$

$$flux_x(t) = 0.0, \qquad flux_y(t) = 1.0 + 0.5\sin(\omega t). \tag{17}$$

Figure 9 are the instantaneous velocity vectors of the connected channel. The physical values on the connected grids are given averaged values of grid points on both sides as boundary conditions at every time step.

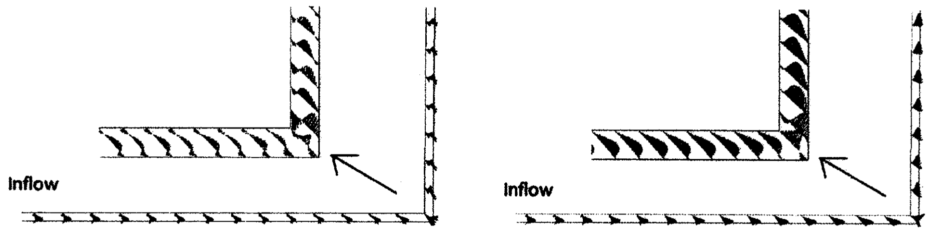

Figure 9: Instantaneous velocity vectors of connected channel. Zoomed-in figures of the connected region are also plotted simultaneously. The left vectors are obtained by the standard MAC method, the right are the present method.

Judging from the velocity distributions, the flow rate obtained by the present method is almost the same in every cross section of the channel.

Figure 10 is time history of the flux at 3/4 length of the duct with comparison of both methods and different iteration numbers. In this figure, the number noted in bracket represents the iteration number of the Poisson equation.

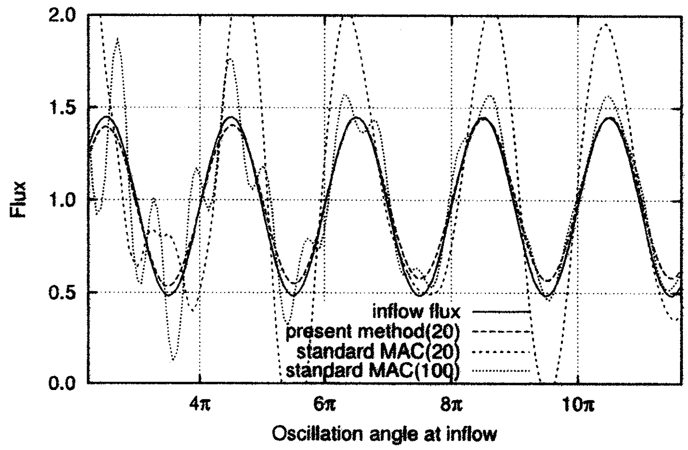

Figure 10: Time history of the flux at 3/4 length of the whole channel with comparison of the standard MAC method and the present method.

The flux differences between inflow and every cross section are plotted in fig.11. From this figure, results obtained by the present method with 20 times of iteration seem better than that of the standard MAC method with 100 times of iteration.

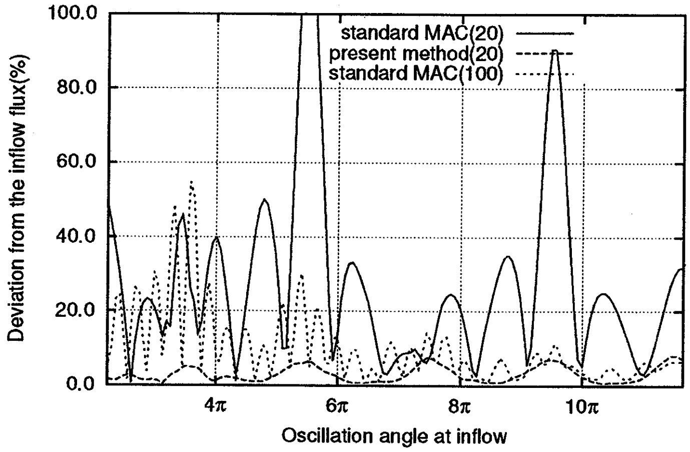

Figure 11: Time history of deviation from the inflow flux of the connected channel.

Seen from radius changes along the flow direction, shown in fig.7 and fig.11, the less the radius changes, the better the present method is profitable for a long flow because the main flow is defined as homogeneous spatially.

4 SUMMARY

In the present study a new numerical method that is suitable for calculations of incompressible fluid flow in a channel of large aspect ratio has been developed and evaluated. The numerical method for an incompressible fluid flow based on solving the Poisson equation of pressure such as the standard MAC method has a weak point that it is difficult to satisfy the equation of continuity precisely. This tendency becomes more serious when the region is very long and/or the flow is unsteady.

In order to improve these shortcomings, we proposed a new method. At first, the original problem is modified as simple as possible. Flow variables are expressed as the sum of those of simplified values and their variations. The simplified values which can be solved analytically are defined in x-direction and y-direction. In the calculation process, those two-directional simplified values are defined proportional to the inclination of the objective channel from an inflow flux. Using this method, a flow in a long channel with wavy wall of various titled angles is computed in order to confirm effectiveness of the present method. Comparing the flux between the present method and the standard MAC method, the conservation of mass of fluid is satisfied more precisely by using the present method. A series of results coincide with each other, which confirms the validation of superposition of two-directional basic flows.

Next, a connected channel is examined in order to validate effectiveness of the present method using multi-domain. The computed channel is orthogonally connected. Those results have indicated that multi-domain method using the present method satisfies the flux conservation more precisely than the standard MAC method even when an objective geometry is markedly curved.

Complex regions which is intended for practical cases, however, have not yet been thoroughly examined. This will necessitate future work on this concept that flow variables are expressed as the sum of a simplified problem and their variations. We can use numerical solutions as the simplified problem, which will make further progress towards more general method.

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