Weak type inequalities of general potentials in homogeneous spaces

Hisako Watanabe

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Abstract

In this paper we give weak type inequalities of potentials $I_\Phi * f$ for a generalized Riesz kernel Φ in homogeneous spaces. Especially, we also estimate $I_\Phi * f$ for the Riesz kernel Φ by the Hausdorff content.

1. Introduction

Sharper inequalities than the classical Sobolev inequality have been obtained by the mean of distribution functions in $\mathbf{R}^n$. For example the following estimate deduces from [7] and [8]:

There exists a constant C such that

$$\int_0^\infty t^{p-1}|\{x \in \Omega : |u(x)| > t\}|^{1-p/n} dt \le C$$

for all $u \in C_0^\infty(\Omega)$, with $\|\nabla u\|_{L^p(\Omega)} \le 1$ when $1 < p < n$.

The inequality of this type has been also treated by P. Hajłasz and P. Koskela in a metric space (cf. [3]).

Furthermore J. Malý and L. Pick considered this inequality in a homogeneous space (X, μ) satisfying a certain lower estimate for the measure μ of a ball. The lower estimate for the measure μ of a ball means that μ has the following property:

There exist constants $n > 1$ and $\gamma > 0$ such that for every $x \in X$ and $r \in (0, R]$

$$\mu(B(x,r)) \ge \gamma r^n.$$

they obtained the following result in [6]:

Theorem A. *Let $1 < p < n$ and put $R = 2diam\, X$. Then there exists a constant C such that for every nonnegative function $g \in L^p(X, \mu)$ with $\|g\|_p \le 1$ we have*

$$\int_0^\infty t^{p-1}\mu(\{x \in X : (I_1 g)(x) > t\})^{1-p/n} dt \le C,$$

where

$$(I_1 g)(x) = \int_0^R \Big(\frac{1}{\mu(B(x,t))}\int_{B(x,t)} g(y)d\mu(y)\Big)dt.$$

A. Iwamura obtained the following result corresponding to Theorem A for I_α instead of I_1.

Theorem B. *Let $1 < p < \infty$ and $\alpha > 0$. If $\alpha p < n$, then there exists a constant $C > 0$ such that for every nonnegative function $g \in L^p(X, \mu)$ with $\|g\|_p \le 1$ we have*

$$\int_0^\infty t^{p-1} \mu(\{x \in X : (I_\alpha g)(x) > t\})^{1-\alpha p/n} dt \le C,$$

where

$$(I_\alpha g)(x) = \int_0^R t^{\alpha-1} (\frac{1}{\mu(B(x,t))} \int_{B(x,t)} g(y) d\mu(y) dt.$$

(cf. [4])

We note that

$$c_1 (I_\alpha g)(x) \le \int \rho(x,y)^{\alpha-n} g(y) d\mu(y) \le c_2 (I_\alpha g)(x)$$

if the measure μ satisfies

$$\gamma_1 r^n \le \mu(B(x,r)) \le \gamma_2 r^n$$

for all balls $B(x,r)$, where c_1, c_2, γ_1 and γ_2 are all positive constants.

In this paper we investigate trace inequalities with weak type for the integral operator given the convolution with a general kernel Φ in a homogeneous space (X, ρ) instead of the kernel $|x|^{\alpha-n}$ $(0 < \alpha < n)$ in $\mathbf{R}^n$. This kernel includes the one that in a neighborhood of 0 behaves like $|x|^{\alpha-n}(\log \frac{e}{|x|})^\beta$, with $0 < \alpha \le n$ and $\beta \ge 0$.

More precisely, let X be a quasi-metric space with a mapping ρ from $X \times X$ to $[0, \infty)$ having the following properties:

(i) $\rho(x,y) = 0$ if and only if $x = y$,

(ii) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$,

(iii) There is a constant $K \ge 1$ such that

$$\rho(x,y) \le K(\rho(x,z) + \rho(z,y)) \quad \text{for all } x, y, z \in X. \tag{1.1}$$

The function ρ is called a quasi-metric.

Furthermore assume that there exists a doubling mearsure μ on X. A doubling measure μ on X means a Borel measure on X, satisfying that there exists a constant $C > 0$ (the doubling constant of μ) so that

$$\mu(B(x, 2r)) \le C\mu(B(x,r)) \tag{1.2}$$

for any $x \in X$ and $r > 0$. Here $B(x,r) = \{y \in X : \rho(x,y) < r\}$.

It is shown in [5] that for any quasi-metric ρ in a homogeneous space there exists an equivalent quasi-metric ρ' such that any ball $B(x,r)$ is open. So we may assume that all balls are open.

We also assume that $\mu(X) < \infty$ and $\operatorname{diam} X = R/2K$, where R is a positive constant.

We will make two assumptions on the homogeneous space X:

(iv) There exists a strictly increasing function h from $[0, +\infty)$ to $[0, +\infty)$ such that $h(r)$ is equivalent to $\mu(B(x,r))$ for any $x \in X$, $r > 0$, that is, there exist b_1, $b_2 > 0$ such that

$$b_1 h(r) \le \mu(B(x,r)) \le b_2 h(r) \tag{1.3}$$

for $x \in X$ and $R \ge r > 0$.

(v) There exists a nonincreasing function Φ from $(0, \infty)$ to $(0, \infty)$ such that Φh is a Dini function.

Recall that a nonnegative function g is a Dini function if g has the following two properties:

(d_1) g is nondecreasing.

(d_2) For any $r > 0$

$$\int_0^r g(t) \frac{dt}{t} \le C g(r),$$

where C is a constant independent of r.

Under these conditions Φ has the following doubling property:

There is a constant $c > 0$ such that

$$\Phi(2r) \geq c\Phi(r) \tag{1.4}$$

for every $r > 0$.

(cf. Lemma 2.4, (a) in [1])

Since Φh is a Dini function, there is a constant $C > 0$ such that

$$\int_0^r \Phi(t)h(t)\frac{1}{t}dt \leq C\Phi(r)h(r) \tag{1.5}$$

for every $r > 0$. The potential $(I_\Phi * f)(x)$ of a μ-measurable function f with respect to Φ is defined by

$$(I_\Phi * f)(x) = \int_X \Phi(\rho(x,y))f(y)d\mu(y)$$

for $x \in X$.

For the function Φ we can define a maximal function M_Φ with respect to Φ: Given a μ-measurable function f on X and $x \in X$,

$$(M_\Phi f)(x) = \sup\{\Phi(r)\int_{B(y,r)} |f|d\mu : x \in B(y,r)\}.$$

We note that for a μ-measurable nonnegative function f

$$M_\Phi f \leq C I_\Phi * f \tag{1.6}$$

for some $C > 0$ independent of f.

We also assume that for a nonnegative μ-measurable function f all sets $\{x \in X : (I_\Phi * f)(x) > t\}$ are open.

Under these assumptions we will prove the following estimate of $I_\Phi * f$ for a μ-measurable function f.

Theorem 1. *Let Φ be a positive nonincreasing function defined on $(0,\infty)$, such that Φh is a Dini function. Furthermore, let $1 < p < +\infty$, $p' = p/(p-1)$, $f \in L^p(X,\mu)$ and $f \geq 0$. Assume that the function $\Phi^{-p}h^{1-p}$ is nondecreasing. Put, for $t > 0$,*

$$\Omega_t = \{x \in X : (I_\Phi * f)(x) > t\}. \tag{1.7}$$

Then

$$\int_0^\infty \Big(\int_{h^{-1}(\mu(\Omega_t)/b_1)}^\infty \Phi(s)^{p'}h(s)\frac{1}{s}ds\Big)^{1-p}t^{p-1}dt \leq C\|f\|_p^p,$$

where C is a constant independent of f and t, and b_1 is the constant in (1.3).

Note that the L^p-norm of f is the one with respect to the measure μ.

Finally I am sorry to say that there are some mistakes in the proof of Lemma 3.3 in my paper [9]. Therefore the proof of Theorem, (i) in the paper is not correct. More precisely we consider the case where $h(r) = r^d$ and $\Phi(r) = r^{\alpha-d}$. Theorem 2 is a revision of the theorem and we will prove it in the last section.

Theorem 2. *Assume that $0 < \eta \le d$ and $0 \le \alpha < d$. Let f be a μ-measurable nonnegative function. Put, for $t > 0$,*

$$\Omega_{\alpha,t} = \{x : (I_\alpha * f)(x) > t\}. \tag{1.8}$$

If $\eta/d < p < \eta/\alpha$, then

$$\int_0^\infty t^{p-1}\mu(\Omega_{\alpha,t})^{(\eta-\alpha p)/d}dt \le c\int f^p dH_\infty^\eta.$$

Here

$$(I_\alpha * f)(x) = \int \rho(x,y)^{\alpha-d}f(y)d\mu(y). \tag{1.9}$$

Recall that the Hausdorff content H_∞^η is defined by

$$H_\infty^\eta(E) = \inf\{\sum_i (r_i)^\eta : E \subset \cup_{i=1}^\infty B(x_i, r_i)\}$$

for a subset E of X.

The integral with respect to the Hausdorff content H_∞^η is defined by

$$\int g dH_\infty^\eta = \int_0^\infty H_\infty^\eta(\{x : g(x) > t\})dt \tag{1.10}$$

for a nonnegative function g.

In the following, all positive constants appearing in each proof will be denoted by c_1, c_2, c_3 $\cdots$.

2. Estimates of $I_\Phi * f$

In this section we prepare fundamental lemmas which hold within our framework.

Let $B = B(x,r)$ be a ball and b be a positive real number. The notation bB stands for the ball of radius br centered at x and $r(B)$ stands for the radius of B.

We note that the following inequality also holds with respect to Φ.

$$\int_{\rho(x,y)<r} \Phi(\rho(x,y))d\mu(y) \le C\int_0^r \Phi(t)h(t)\frac{1}{t}dt, \tag{2.1}$$

where C is a constant independent of x and $r > 0$.

(cf. Lemma 2.4 in [1])

By the same method as in the proof of Lemma 2.4 in [1] we can prove the following inequality: There is a constant C, independent of x and r, such that

$$\int_{\rho(x,y)\ge r} \Phi(\rho(x,y))^{p'}d\mu(y) \le C\int_r^\infty \Phi(t)^{p'}h(t)\frac{1}{t}dt. \tag{2.2}$$

We also have the following lemma.

Lemma 2.1. *Let $0 < r \le R$, $x \in X$ and f be a μ-measurable nonnegative function. Then there exists a constant C, independent of f, x, r, such that*

$$\int_{\rho(x,y)<r} \Phi(\rho(x,y))f(y)d\mu(y) \le C(Mf)(x)\Phi(r)h(r),$$

where Mf is the Hardy-Littlewood maximal function,i.e.,

$$(Mf)(x) = \sup\{\frac{1}{\mu(B)}\int_B f d\mu : B \text{ is a ball, } x \in B\}.$$

Proof. Let $0 < \eta < 1$. Then

$$I_1 \equiv \int_{\rho(x,y)<r} \Phi(\rho(x,y))f(y)d\mu(y) \le c_1 \sum_{k\ge 0} \Phi(\eta^k r)h(\eta^k r)(Mf)(x).$$

Hence, by (1.5),

$$I_1 \le c_2(Mf)(x)\int_0^r \Phi(t)h(t)\frac{1}{t}dt \le c_3(Mf)(x)\Phi(r)h(r).$$

Q.E.D

Lemma 2.2. *Let f be a μ-measurable nonnegative function. Then there exists a constant v such that*

$$\mu(\Omega_t) \le \frac{1}{2}\mu(X)$$

for all $t \ge v$.

Proof. Note that, by the definition of Ω_t,

$$\mu(\Omega_t) \le \frac{1}{t}\int_{\Omega_t}(I_\Phi f)(\xi)d\mu(\xi) \le \frac{c_1}{t}\int f(y)d\mu(y)\int_{B(y,R)}\Phi(\rho(\xi,y))d\mu(\xi).$$

From (2.1) and (1.5) we deduce

$$\begin{aligned}\mu(\Omega_t) &\le \frac{c_2}{t}\|f\|_p\mu(X)^{(p-1)/p}\int_0^R \Phi(t)h(t)\frac{dt}{t}\\ &\le \frac{c_3}{t}\Phi(R)h(R)\mu(X)^{-1/p}\|f\|_p\mu(X).\end{aligned}$$

Set $v = 2c_3\Phi(R)h(R)\mu(X)^{-1/p}\|f\|_p$. If $t \ge v$, then

$$\mu(\Omega_t) \le \frac{1}{2}\mu(X).$$

Q.E.D

By Lemma 2.2 we see that the complement Ω_t^c of Ω_t is not empty for $t \ge v$. So, for $t \ge v$, there exists a sequence $\{B_j\}$ ($B_j = B(x_j, r_j)$) of balls of Whitney type having the following properties:

(w_1) $\Omega_t = \cup_j B_j$,

(w_2) There is a constant $S \ge 1$ such that $SB_j \cap \Omega_t^c \ne \emptyset$ for every j and Ω_t,

(w_3) $\sum_j \chi_{B_j} \le N$ for some constant N.

(cf. Theorem (1.3) in Chapter 3 in [2])

Lemma 2.3. *Let $t \ge v$ for v in Lemma 2.2 and $\{B_j\}$ be a sequenece of balls of Whitney type for the open set Ω_t. Then, there exists a constant $A_1 \ge 1$, independent of j and Ω_t, such that*

(i) $\mu(B_j \cap \Omega_{A_1 t}) \le (1/2)\mu(B_j)$ (ii) $\mu(B_j \setminus \Omega_{A_1 t}) \ge (1/2)\mu(B_j)$.

Proof. Let $a > 0$ and x_j be the center of B_j. Then

$$\begin{aligned} at\mu(B_j \cap \Omega_{at}) &\leq \int_{\Omega_{at}\cap B_j} (I_\Phi * f)(x) d\mu(x) \\ &\leq \int f(y) d\mu(y) \int_{B_j} \Phi(\rho(x,y)) d\mu(x) = I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_{\rho(x_j,y)<3KSr(B_j)} f(y) d\mu(y) \int_{B_j} \Phi(\rho(x,y)) d\mu(x)$$

and

$$I_2 = \int_{\rho(x_j,y)\geq 3KSr(B_j)} f(y) d\mu(y) \int_{B_j} \Phi(\rho(x,y)) d\mu(x).$$

Here K and S are constants in (1.1) and (w_2), respectively.

Using (2.1) and (1.5), we have

$$\begin{aligned} I_1 &\leq c_1 \int_0^{4KSr(B_j)} \Phi(t) h(t) \frac{dt}{t} \int_{\rho(x_j,y)<3KSr(B_j)} f(y) d\mu(y) \\ &\leq c_2 \Phi(4KSr(B_j)) h(4KSr(B_j)) \int_{\rho(x_j,y)<3KSr(B_j)} f(y) d\mu(y). \end{aligned}$$

Take $z_j \in SB_j \cap \Omega_t^c$. Then, by (1.3) and (1.6),

$$I_1 \leq c_3 \mu(B_j)(M_\Phi f)(z_j) \leq c_4 \mu(B_j)(I_\Phi * f)(z_j) \leq c_4 \mu(B_j) t. \tag{2.3}$$

We next consider I_2. Assume that $\rho(x_j,y) \geq 3KSr(B_j)$. For any $x \in B_j$ we have

$$\begin{aligned} \rho(z_j,y) &\leq K\rho(z_j,x_j) + K^2\rho(x_j,x) + K^2\rho(x,y) \\ &< (KS + K^2) r(B_j) + K^2 \rho(x,y). \end{aligned}$$

From

$$K\rho(x,y) \geq \rho(x_j,y) - K\rho(x_j,x) \geq K(3S-1) r(B_j)$$

we deduce

$$r(B_j) \leq \frac{\rho(x,y)}{3S-1}.$$

Hence

$$\rho(z_j,y) < \left(\frac{K(S+K)}{3S-1} + K^2\right) \rho(x,y).$$

Therefore, by (1.4),

$$\Phi(\rho(x,y)) \leq c_5 \Phi(\rho(z_j,y)).$$

From this inequality we deduce

$$I_2 \leq c_5 \int f(y) d\mu(y) \int_{B_j} \Phi(\rho(z_j,y)) d\mu(x) \leq c_5 \mu(B_j)(I_\Phi * f)(z_j) \leq c_5 \mu(B_j) t. \tag{2.4}$$

By (2.3) and (2.4) we obtain

$$at\mu(B_j \cap \Omega_{at}) \leq I_1 + I_2 \leq c_6 \mu(B_j) t,$$

whence

$$\mu(B_j \cap \Omega_{at}) \leq \frac{c_6}{a} \mu(B_j).$$

If we set

$$A_1 = 2c_6,$$

we have

$$\mu(B_j \cap \Omega_{A_1 t}) \leq \frac{\mu(B_j)}{2},$$

which is the assertion (i). From this the assertion (ii) is easily deduced.

Q.E.D

We often use the notation

$$\lambda = K + 2K^2.$$

Lemma 2.4. *Let $B = B(z,r) \subset \Omega_t$ and $x \in \Omega_t$. Put*

$$q_t = h^{-1}\left(\frac{\mu(\Omega_t)}{b_1}\right).$$

If $y,\ \xi \in B$, then

$$\Phi(\rho(x,y))\chi_{\{w\in B:\rho(x,w)\geq\lambda q_t\}}(y) \leq C\Phi(\rho(x,\xi))\chi_{\{\tau\in B:\rho(x,\tau)\geq q_t\}}(\xi), \tag{2.5}$$

where C is a constant independent of $x,\ y,\ \xi$ and b_1 is in (1.3).

Proof. From $h(q_t) = \mu(\Omega_t)/b_1$ and for $y,\ \xi \in B$

$$b_1 h(r) \leq \mu(B) \leq \mu(\Omega_t) \leq b_1 h(q_t)$$

we deduce $h(r) \leq h(q_t)$. Since h is strictly increasing, we have $r \leq q_t$.
It suffices to show (2.5) for $x,\ y$ satisfying $\rho(x,y) \geq \lambda q_t$. Then

$$\begin{aligned}\rho(x,\xi) &\leq K(\rho(x,y)+\rho(y,\xi)) \\ &< K(\rho(x,y)+2Kq_t) < (K+2K^2/\lambda)\rho(x,y).\end{aligned}$$

Using the doubling of Φ, we have

$$\Phi(\rho(x,y)) \leq \Phi\left(\frac{\rho(x,\xi)}{K+2K^2/\lambda}\right) \leq c_1\Phi(\rho(x,\xi)).$$

On the other hand, assume that $\rho(x,\xi) < q_t$. Then

$$\rho(x,y) \leq K\rho(x,\xi) + K^2\rho(\xi,z) + K^2\rho(z,y) < (K+2K^2)q_t = \lambda q_t.$$

So, we see that if $\rho(x,y) \geq \lambda q_t$, then $\rho(x,\xi) \geq q_t$. Thus we have the conclusion.

Q.E.D

Lemma 2.5. *There exists a constant $A_2 > 0$ such that*

$$\int_{\Omega^c_{t/A_2}} \Phi(\rho(x,y))f(y)d\mu(y) \leq \frac{t}{2}$$

for all $t \geq A_2 v$ and $x \in \Omega_t$.

Proof. Let $a \geq 1$ and $t \geq av$. By the aid of Lemma 2.2 we see that $\Omega_{t/a}^c \neq \emptyset$. So there is a $z \in \Omega_{t/a}^c$ such that $\rho(x,z) < 2\text{dist}\,(x, \Omega_{t/a}^c)$. For any $y \in \Omega_{t/a}^c$ we have

$$\rho(z,y) \leq K(\rho(z,x) + \rho(x,y)) < 3K\rho(x,y).$$

Hence, by the doubling of Φ,

$$\int_{\Omega_{t/a}^c} \Phi(\rho(x,y)) f(y) d\mu(y) \leq c_1 (I_\Phi * f)(z) \leq \frac{c_1 t}{a}.$$

We put $A_2 = \max\{2c_1, 1\}$ and have the conclusion.

Q.E.D

3. Main lemma and proof of Theorem 1

In this section we prove the main lemma (Lemma 3.1) and Theorem 1.

Lemma 3.1. *Put $A = \max\{A_1, A_2\}$ and let $t \geq Av$, $x \in \Omega_t$. Then*

$$\begin{aligned} & \int_{\rho(x,y) \geq \lambda q_t} \Phi(\rho(x,y)) f(y) d\mu(y) \\ \leq \; & \frac{t}{2} + C \int_{(\Omega_{t/A} \setminus \Omega_{At}) \cap \{y : \rho(x,y) \geq q_t\}} \Phi(\rho(x,y)) (Mf)(y) d\mu(y), \end{aligned}$$

where C is a constant independent of t, x and f, and A_1, A_2 are the constants in Lemma 2.3 and Lemma 2.5, respectively.

Proof. Let $t \geq Av$ and $\{B_j\}$ be a sequence of balls of Whitney type for Ω_t. Let $\xi \in B_j$. By Lemma 2.4 and the doubling of Φ we have

$$\begin{aligned} & \int_{\{\rho(x,y) \geq \lambda q_t\} \cap B_j} \Phi(\rho(x,y)) f(y) d\mu(y) \\ \leq \; & c_1 \int_{\{\rho(x,y) \geq \lambda q_t\} \cap B_j} \Phi(\rho(x,\xi)) \chi_{\{\tau \in B_j : \rho(x,\tau) \geq q_t\}}(\xi) f(y) d\mu(y) \\ \leq \; & c_1 \Phi(\rho(x,\xi)) \chi_{\{\tau \in B_j : \rho(x,\tau) \geq q_t\}}(\xi) \int_{B_j} f(y) d\mu(y) \\ \leq \; & c_1 \Phi(\rho(x,\xi)) \chi_{\{\tau \in B : \rho(x,\tau) \geq q_t\}}(\xi) \mu(B_j) (Mf)(\xi). \end{aligned}$$

Integrating over $B_j \setminus \Omega_{At}$ with respect to ξ and using Lemma 2.3 we have

$$\begin{aligned} & \int_{\{\rho(x,y) \geq \lambda q_t\} \cap B_j} \Phi(\rho(x,y)) f(y) d\mu(y) \\ \leq \; & c_1 \frac{\mu(B_j)}{\mu(B_j \setminus \Omega_{At})} \int_{(B_j \setminus \Omega_{At}) \cap \{\rho(x,\xi) \geq q_t\}} \Phi(\rho(x,\xi)) (Mf)(\xi) d\mu(\xi) \\ \leq \; & 2c_1 \int_{(B_j \setminus \Omega_{At}) \cap \{\rho(x,\xi) \geq q_t\}} \Phi(\rho(x,\xi)) (Mf)(\xi) d\mu(\xi). \end{aligned}$$

Hence

$$\int_{\{\rho(x,y) \geq \lambda q_t\} \cap \Omega_t} \Phi(\rho(x,y)) f(y) d\mu(y) \leq c_2 \int_{(\Omega_t \setminus \Omega_{At}) \cap \{\rho(x,\xi) \geq q_t\}} \Phi(\rho(x,\xi)) (Mf)(\xi) d\mu(\xi). \tag{3.1}$$

On the other hand, since $f \le Mf$ $\mu-$a.e., we have

$$\int_{(\Omega_{t/A}\setminus\Omega_{At})\cap\{\rho(x,y)\ge\lambda q_t\}} \Phi(\rho(x,y))f(y)d\mu(y) \tag{3.2}$$
$$\le \int_{(\Omega_{t/A}\setminus\Omega_{At})\cap\{\rho(x,y)\ge q_t\}} \Phi(\rho(x,y))(Mf)(y)d\mu(y).$$

From (3.1), (3.2) and Lemma 2.5 we deduce

$$\int_{\{\rho(x,y)\ge\lambda q_t\}} \Phi(\rho(x,y))f(y)d\mu(y) \le \frac{t}{2} + c_3 \int_{(\Omega_{t/A}\setminus\Omega_{At})\cap\{\rho(x,y)\ge q_t\}} \Phi(\rho(x,y))(Mf)(y)d\mu(y).$$

This is the conclusion.

Q.E.D

We note that the following inequality holds.

Lemma 3.2. *Let $p > 1$ and $p' = p/(p-1)$. Then*

$$\left(\int_r^\infty \Phi(t)^{p'} h(t)\frac{1}{t}dt\right)^{p/p'} \ge C\Phi(r)^p h(r)^{p-1}.$$

Proof. Since Φh is increasing, we have

$$\begin{aligned}\int_r^\infty \Phi(t)^{p'} h(t)\frac{1}{t}dt &\ge \Phi(r)^{p'}h(r)^{p'}\int_r^\infty \frac{1}{h(t)^{p'-1}t}dt \ge \Phi(r)^{p'}h(r)^{p'}\int_r^{2r} \frac{1}{h(t)^{p'-1}t}dt\\ &\ge c_1 \frac{\Phi(r)^{p'}h(r)^{p'}}{h(2r)^{p'-1}} \ge c_2\Phi(r)^{p'}h(r),\end{aligned}$$

whence

$$\left(\int_r^\infty \Phi(t)^{p'} h(t)\frac{1}{t}dt\right)^{p/p'} \ge c_3\Phi(r)^p h(r)^{p/p'} = c_3\Phi(r)^p h(r)^{p-1}.$$

Q.E.D

Proof of Theorem 1. Let $f \in L^p(X,\mu)$ and $f \ge 0$. Take the constants v and A in Lemma 2.2 and Lemma 3.1, respectively. For $t \ge Av$ and $x \in \Omega_t$ we write

$$\begin{aligned} t &< (I_\Phi * f)(x) = \int_{\rho(x,y)<\lambda q_t} \Phi(\rho(x,y))f(y)d\mu(y)\\ &+ \int_{\rho(x,y)\ge\lambda q_t} \Phi(\rho(x,y))f(y)d\mu(y) \equiv I_1 + I_2.\end{aligned}$$

Using Lemma 2.1 and the doubling of Φ and h, we have

$$I_1 \le c_1(Mf)(x)\Phi(q_t)h(q_t).$$

Lemma 3.1 implies

$$\begin{aligned} t &< I_1 + I_2 \le c_1(Mf)(x)\Phi(q_t)h(q_t)\\ &+ \frac{t}{2} + c_2\int_{(\Omega_{t/A}\setminus\Omega_{At})\cap\{\rho(x,y)\ge q_t\}} \Phi(\rho(x,y))(Mf)(y)d\mu(y),\end{aligned}$$

whence

$$t \le c_3(Mf)(x)\Phi(q_t)h(q_t) + c_3 \int_{(\Omega_{t/A}\setminus\Omega_{At})\cap\{\rho(x,y)\ge q_t\}} \Phi(\rho(x,y))(Mf)(y)d\mu(y). \tag{3.3}$$

Using (2.2), we have

$$\begin{aligned} &\int_{(\Omega_{t/A}\setminus\Omega_{At})\cap\{\rho(x,y)\ge q_t\}} \Phi(\rho(x,y))(Mf)(y)d\mu(y) \\ \le\ & \Big(\int_{\Omega_{t/A}\setminus\Omega_{At}} (Mf)^p d\mu\Big)^{1/p}\Big(\int_{\rho(x,y)\ge q_t} \Phi(\rho(x,y))^{p'} d\mu(y)\Big)^{1/p'} \\ \le\ & c_4\Big(\int_{\Omega_{t/A}\setminus\Omega_{At}} (Mf)^p d\mu\Big)^{1/p}\Big(\int_{q_t}^{\infty} \Phi(t)^{p'} h(t)\frac{1}{t}dt\Big)^{1/p'}. \end{aligned}$$

Hence

$$t^p \le c_5(Mf)(x)^p\Phi(q_t)^p h(q_t)^p + c_5\Big(\int_{\Omega_{t/A}\setminus\Omega_{At}} (Mf)^p d\mu\Big)\Big(\int_{q_t}^{\infty} \Phi(t)^{p'} h(t)\frac{1}{t}dt\Big)^{p/p'}.$$

Integrating over $\Omega_t \setminus \Omega_{At}$ with respect to x, we have

$$\begin{aligned} t^p \ &\le\ c_6\Phi(q_t)^p h(q_t)^{p-1}\frac{\mu(\Omega_t)}{\mu(\Omega_t\setminus\Omega_{At})}\int_{\Omega_t\setminus\Omega_{At}} (Mf)^p d\mu \\ &+\ c_6\Big(\int_{\Omega_{t/A}\setminus\Omega_{At}} (Mf)^p d\mu\Big)\Big(\int_{q_t}^{\infty} \Phi(t)^{p'} h(t)\frac{1}{t}dt\Big)^{p/p'}. \end{aligned}$$

With the aid of Lemma 2.3 and Lemma 3.2, we have

$$t^p \le c_7\Big(\int_{\Omega_{t/A}\setminus\Omega_{At}} (Mf)^p d\mu\Big)\Big(\int_{q_t}^{\infty} \Phi(t)^{p'} h(t)\frac{1}{t}dt\Big)^{p/p'}. \tag{3.4}$$

Let v be the number in Lemma 2.2. Since the function $\Phi^{-p}h^{-p+1}$ is increasing, we have, by Lemma 3.2,

$$\begin{aligned} &\int_0^{Av} t^{p-1}\Big(\int_{q_t}^{\infty} \Phi(s)^{p'} h(s)\frac{1}{s}ds\Big)^{-p/p'} dt \\ \le\ & c_8\int_0^{Av} \frac{t^{p-1}}{\Phi(q_t)^p h(q_t)^{p-1}}dt \le c_9\frac{A^p v^p}{\Phi(h^{-1}(\mu(X)/b_1))^p(\mu(X)/b_1)^{p-1}} \\ \le\ & c_{10}v^p = c_{11}\|f\|_p^p. \end{aligned}$$

On the other hand, by (3.4),

$$\begin{aligned} \int_{Av}^{\infty} t^{p-1}\Big(\int_{q_t}^{\infty} \Phi(s)^{p'} h(s)\frac{1}{s}ds\Big)^{-p/p'} dt \ &\le\ c_{12}\int_{Av}^{\infty} \frac{1}{t}\Big(\int_{\Omega_{t/A}\setminus\Omega_{At}} (Mf)^p d\mu\Big)dt \\ &\le\ c_{12}\int (Mf)(x)^p d\mu(x)\int_{A^{-1}(I_\Phi*f)(x)}^{A(I_\Phi*f)(x)} \frac{1}{t}dt \le c_{13}\|f\|_p^p. \end{aligned}$$

Thus

$$\int_0^{\infty} t^{p-1}\Big(\int_{q_t}^{\infty} \Phi(s)^{p'} h(s)\frac{1}{s}ds\Big)^{-p/p'} dt \le c_{14}\|f\|_p^p.$$

This is the conclusion.

Q.E.D

4. Proof of Theorem 2

In this section we prove Theorem 2. Corresponding to (1.3), we assume that

$$b_1 r^d \leq \mu(B(x,t)) \leq b_2 r^d.$$

Recall that for a μ-measurable nonnegative function f and $t > 0$, the function $I_\alpha * f$ and the set $\Omega_{\alpha,t}$ are defined by (1.9) and (1.8), respectively.

Proof of Theorem 2. We put

$$\Phi(t) = t^{\alpha-d}, \qquad h(t) = t^d$$

and use the results obtained in §2 and §3.

Put $p_t = \left(\mu(\Omega_{\alpha,t})/b_1\right)^{1/d}$. By Lemma 2.5 in [9] we have

$$\begin{aligned}
\mu(\Omega_t) &\leq \frac{c_1}{t}\int f(y)d\mu(y)\int_{B(y,R)}\rho(\xi,y)^{\alpha-d}d\mu(\xi) \\
&\leq \frac{c_2}{t}\Big(\int f(y)^{pd/\eta}d\mu(y)\Big)^{\eta/dp}\mu(X)^{1-\eta/dp}R^\alpha \\
&\leq \frac{c_3}{t}\Big(\int f^p dH_\infty^\eta\Big)^{1/p}\mu(X)^{-\eta/dp}R^\alpha\mu(X).
\end{aligned}$$

Set

$$v' = 2c_3R^\alpha\mu(X)^{-\eta/dp}\Big(\int f^p dH_\infty^\eta\Big)^{1/p}.$$

If $t \geq v'$, then $\mu(\Omega_{\alpha,t}) \leq \mu(X)/2$. Therefore we see that $\Omega_{\alpha,t}^c \neq \emptyset$.

By (3.3) in the proof of Theorem 1 we have, for $t \geq Av'$,

$$\begin{aligned}
t &\leq c_4(Mf)(x)p_t^{\alpha-d}p_t^d + c_4\int_{(\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At})\cap\{\rho(x,y)\geq p_t\}}\rho(x,y)^{\alpha-d}(Mf)(y)d\mu(y) \\
&\leq c_4(Mf)(x)p_t^\alpha \\
&+ c_4\Big(\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^{dp/\eta}d\mu(y)\Big)^{\eta/(dp)}\Big(\int_{\rho(x,y)\geq p_t}\rho(x,y)^{dp(\alpha-d)/(dp-\eta)}d\mu(y)\Big)^{(dp-\eta)/(dp)}.
\end{aligned}$$

Noting that

$$\int_{\rho(x,y)\geq p_t}\rho(x,y)^{dp(\alpha-d)/(dp-\eta)}d\mu(y) \leq c_5\int_{p_t}^\infty (s^{(\alpha-d)dp/(dp-\eta)}s^{d-1}ds = c_6(p_t)^{d(\alpha p-\eta)/(dp-\eta)},$$

we have

$$t \leq c_7(Mf)(x)p_t^{\alpha-\eta/p}p_t^{\eta/p} + c_7\Big(\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^{dp/\eta}d\mu(y)\Big)^{\eta/dp}p_t^{(\alpha p-\eta)/p}.$$

Integrating over $\Omega_{\alpha,t}\setminus\Omega_{\alpha,At}$ and using Lemma 2.3, we have

$$\begin{aligned}
t &\leq c_8(p_t)^{\alpha-\eta/p}\Big(\frac{\mu(\Omega_{\alpha,t})^{\eta/dp}}{\mu(\Omega_{\alpha,t}\setminus\Omega_{\alpha,At})}\int_{\Omega_{\alpha,t}\setminus\Omega_{\alpha,At}}(Mf)(y)d\mu(y) \\
&+ c_8(p_t)^{\alpha-\eta/p}\Big(\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^{dp/\eta}d\mu(y)\Big)^{\eta/(dp)}
\end{aligned}$$

$$
\begin{aligned}
&\leq c_8(p_t)^{\alpha-\eta/p}\frac{\mu(\Omega_{\alpha,t})^{\eta/dp}\mu(\Omega_{\alpha,t}\setminus\Omega_{\alpha,At})^{(1-\eta/dp)}}{\mu(\Omega_{\alpha,t}\setminus\Omega_{\alpha,At})}\Big(\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^{dp/\eta}d\mu(y)\Big)^{\eta/(dp)} \\
&+ c_8(p_t)^{\alpha-\eta/p}\Big(\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^{dp/\eta}d\mu(y)\Big)^{\eta/(dp)} \\
&\leq c_9 p_t^{\alpha-\eta/p}\Big(\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^{dp/\eta}d\mu(y)\Big)^{\eta/(dp)},
\end{aligned}
$$

whence

$$
t^p \leq c_{10}p_t^{\alpha p-\eta}\Big(\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^{dp/\eta}d\mu(y)\Big)^{\eta/d}.
$$

Using Lemma 2.5 in [9], we have

$$
t^p \leq c_{11}p_t^{\alpha p-\eta}\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^p dH^\eta_\infty.
$$

Since $p_t^{\alpha p-\eta} = \mu(\Omega_{\alpha,t}/b_1)^{(\alpha p-\eta)/d}$, we have

$$
\begin{aligned}
\int_{Av'}^\infty \mu(\Omega_{\alpha,t})^{(\eta-\alpha p)/d}t^{p-1}dt &\leq c_{12}\int_{Av'}^\infty \frac{1}{t}dt\int_{\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At}}(Mf)(y)^p dH^\eta_\infty \\
&= c_{12}\int_{Av'}^\infty \frac{1}{t}dt\int_0^\infty H^\eta_\infty(\{y\in\Omega_{\alpha,t/A}\setminus\Omega_{\alpha,At} : (Mf)(y)^p > \tau\})d\tau \\
&\leq c_{12}\int_0^\infty H^\eta_\infty(\{y : (Mf)(y) > \tau\})d\tau\int_{(I_\alpha * f)(y)/A}^{A(I_\alpha * f)(y)}\frac{dt}{t} \\
&\leq c_{13}\int (Mf)^p dH^\eta_\infty.
\end{aligned}
$$

Using Theorem in [10], we have

$$
\int (Mf)^p dH^\eta_\infty \leq c_{14}\int f^p dH^\eta_\infty.
$$

Hence

$$
\int_{Av'}^\infty \mu(\Omega_{\alpha,t})^{(\eta-\alpha p)/d}t^{p-1}dt \leq c_{15}\int f^p dH^\eta_\infty.
$$

Together with

$$
\int_0^{Av'} \mu(\Omega_{\alpha,t})^{(\eta-\alpha p)/d}t^{p-1}dt \leq \frac{1}{p}\mu(X)^{(\eta-\alpha p)/d}(Av')^p \leq c_{16}\int f^p dH^\eta_\infty
$$

we have

$$
\int_0^\infty \mu(\Omega_{\alpha,t})^{(\eta-\alpha p)/d}t^{p-1}dt \leq c_{17}\int f^p dH^\eta_\infty.
$$

Thus we have the conclusion.

Q.E.D

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Hisako Watanabe
Nishikosenba 2-13-2,
Kawagoe-shi, Saitama-ken,
350-0035, Japan
E-mail: hisakowatanabe@nifty.com