A Numerical Method for Calculations of Flow in Long Region with Branch

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Abstract

A new numerical method that is suitable for the calculation of incompressible fluid flow in a very long region with branches such as a river or a blood vessel is proposed. The flow variables are expressed as sum of those of the main flow and their variations. The former is obtained by solving one-dimensional equation analytically. Using this method, two-dimensional flow in a long duct that has sinusoidal wall with and without branches, axis-symmetrical flow in a pipe and a long duct of large curvaure are computed. For all cases, the conservation of mass of fluid is well satisfied that is very difficult to obtain by other method such as the standard MAC method.

1. Introduction

We can find a lot of incompressible fluid flows in very long regions like a river and a blood vessel. The main problem to calculate the flow in such regions is that it is quite difficult to satisfy the equation of continuity precisely. If we use the method based on the stream-function and vorticity, we are free from problem above. However this method works only on the two dimensional or axi-symmetric flows. Moreover, if the region involve a branch, it is very difficult to determine the flow rate after a branch. Owing to these shortcomings involved in stream-function vorticity formulation, the MAC method[1] and its variations are preferred to treat three-dimensional flow and the flow with branch often appearing in a river and a blood vessel. On the other hand, the conservation of mass (i.e. the equation of motion) is only approximately satisfied by the method based on the Poisson equation of pressure appearing in the MAC method that is difficult to converge by using the iterative method. This disadvantage is emphasized in the case of the region of complex geometry where the generalized coordinate system is often used and the region of large aspect ratio such as a flow in a blood vessel especially when the flow is unsteady due to pumping of the heart.

In this paper, we present a new numerical method that is suitable for the calculation of incompressible fluid flow in a very long region. In this method, the flow variables are expressed as sum of those of the main flow and its variation. The former is obtained by solving an one-dimensional equation analytically.

2. Numerical Method

(a) Two-dimensional flow in a rectangular duct

In this case, the flow is assumed to be two-dimensional and the direction of the main flow is x-direction. In the rough approximation, the flow can be treated as one-dimensional time-dependent flow which is governed by the one dimensional Navier-Stokes equation:

$$\frac{\partial u}{\partial x}=0 \qquad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2} \qquad (2)$$

From eq.(1)

$$u=f(t)$$

The function f(t) is determined by the boundary condition at the inlet.

If we substitute this into eq.(2), we obtain the pressure as follows:

$$f'(t)=-\frac{\partial p}{\partial x} \quad \text{i.e.} \quad p=-f'x+c \quad (c \text{ : constant})$$

Therefore, we can define the variation of the velocity and pressure by

$$u=f(t)+\tilde{u}, \quad p=-f'x+c+\tilde{p} \qquad (3)$$

After the substitution of eq.(3) into the Navier-Stokes equation, the following equations are obtained which are the basic equations of the flow:

$$\frac{\partial \tilde{u}}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4}$$

$$\frac{\partial \tilde{u}}{\partial t}+(f+\tilde{u})\frac{\partial \tilde{u}}{\partial x}+v\frac{\partial \tilde{u}}{\partial y}=-\frac{\partial \tilde{p}}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \tilde{u}}{\partial x^2}+\frac{\partial^2 \tilde{u}}{\partial y^2}\right) \tag{5}$$

$$\frac{\partial v}{\partial t}+(f+\tilde{u})\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial \tilde{p}}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{6}$$

These equations are solved by the standard method such as the MAC method.

Note that the boundary condition on the fixed wall becomes

$$\tilde{u}=-f(t)$$

since physical condition on the wall is no-slip(u=0).

(b) Two dimensional flow in a non-uniform duct

In this case, the flow is assumed to be two-dimensional and the direction of the main flow is not far from x-direction. Therefore, we use generalized coordinate system

$$\xi=\xi(x,y), \quad \eta=\eta(x,y) \tag{7}$$

for eqs. (4), (5) and (6) and transforms them so that the independent variables become ξ and η. Flow variables are obtained by solving basic equations in the transformed plane.

(c) Two-dimensional flow in a duct of large curvature.

The method mentioned above does not work when the duct has large curvature since the direction of the main flow largely changes. For example, if it is required to calculate the flow in a duct shown in fig.1, the direction of the main flow near the entrance is x-direction while that near the exit is y-direction.

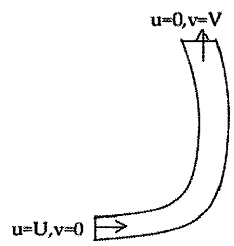

Fig.1 schematic figure of a curved duct

This difficulty can be overcome by using the coordinate system along the streamline. For two-dimensional flow, the simplest example is the polar coordinate system. In this case, eqs.(1) and (2) become

$$\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}=0 \tag{8}$$

$$\frac{\partial V_\theta}{\partial t}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}=-\frac{1}{r}\frac{\partial r}{\partial \theta}+\frac{1}{\mathrm{Re}}\left(\frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2}-\frac{V_\theta}{r^2}\right) \tag{9}$$

From eq.(8)

$$V_\theta = f(t)$$

If we ignore the viscous term in eq.(9) and use above result, we find that the equations corresponding to eq.(3) become

$$V_\theta = f(t)+\tilde{V}_\theta, \quad p=-r\theta f'(t)+c \tag{10}$$

Substituting this into the Navier-Stokes equations expressed by polar coordinate, we obtain

$$\frac{\partial V_r}{\partial r}+\frac{1}{r}\frac{\partial \tilde{V}_\theta}{\partial \theta}+\frac{V_r}{r}=0 \tag{11}$$

$$\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{1}{r}\left(f+\tilde{V}_\theta\right)\frac{\partial \tilde{V}_r}{\partial \theta}-\frac{\left(f+\tilde{V}_\theta\right)^2}{r}=f'\theta-\frac{\partial \tilde{p}}{\partial r}+\frac{1}{\mathrm{Re}}\left(\Delta V_r-\frac{2}{r^2}\frac{\partial \tilde{V}_\theta}{\partial \theta}-\frac{V_r}{r^2}\right) \tag{12}$$

$$\frac{\partial \tilde{V}_\theta}{\partial t}+V_r\frac{\partial \tilde{V}_\theta}{\partial r}+\frac{1}{r}\left(f+\tilde{V}_\theta\right)\frac{\partial \tilde{V}_\theta}{\partial \theta}+\frac{\left(f+\tilde{V}_\theta\right)V_r}{r}=-\frac{1}{r}\frac{\partial \tilde{p}}{\partial \theta}+\frac{1}{\mathrm{Re}}\left(\Delta \tilde{V}_\theta+\frac{2}{r^2}\frac{\partial V_r}{\partial \theta}-\frac{f+\tilde{V}_\theta}{r^2}\right) \tag{13}$$

where

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

If the shape of the duct is a little different from circle, the generalized coordinate system can be used as we have already done, i.e. we transform equations (11)-(13) by the transformation (7) and solve the basic equations in transformed plane by using standard MAC method.

(d) Three dimensional flow

All the procedures can be easily extended to the three dimensional flows if we use the three-dimensional Navier-Stokes equation which is expressed by the velocity and pressure variations from the one-dimensional main flow.

3. Results

We have done several test calculations in order to confirm effectiveness of the present method. In this study two-dimensional duct of three types of shape are examined; sinusoidal, with branch and large curved.

(a) Two dimensional flow in a non-uniform duct

At first, we show the results of two-dimensional flow in a non-uniform duct. The shape of the wall is wavy which changes sinusoidally in the flow direction.

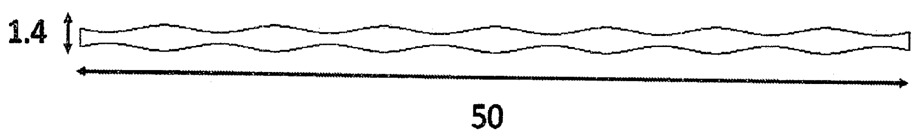

Fig.3(a) The computational domain of sinusoidal duct.

The velocity at the inlet boundary changes temporally as

$$u(t) = 1 + 0.5\sin\omega t$$

where u is the velocity component of x-direction, t is time and ω is angular velocity. Therefore, the inlet velocity fluctuates raging from 0.5 to 1.5 temporally.

Fig.3(b) and Fig.3(c) are the velocity vectors obtained by the standard MAC method and present method respectively. Both vectors are at the same calculation time.

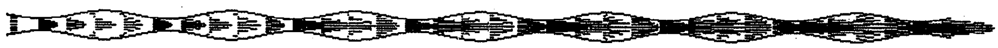

Fig.3(b) velocity vectors obtained by the standard MAC method.

Fig.3(c) velocity vectors obtained by the present method.

Judging from fig.3(b) and fig.3(c), the flow rate obtained by the present method is almost the same in every cross section of the duct. This implies those results obtained by the present method are much better consistent with the assumption of the incompressibility when it is compared with the same number of iterations of the Poisson equation.

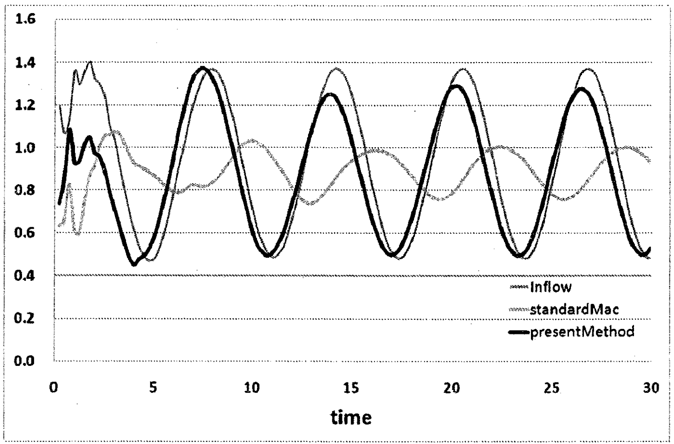

Fig.3(d) The time history of flux at the middle of the channel.

(b) Two dimensional flow in a duct with branches

As an example of the flow in a duct with branches, we choose the region shown in fig.4-a where the duct separate at the center part. The ratio of the width of two branches is 2:1. The shape of each branch is parabola.

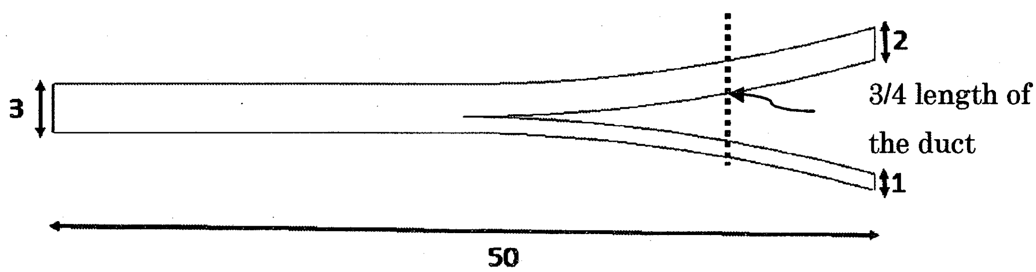

Fig.4(a) The computational domain of a duct with branches.

The inlet velocity is changed temporally as the previous case. Fig.4(b) and fig4(c) are the velocity vectors obtained by the standard MAC method and the present method respectively at non-dimensional time 30.

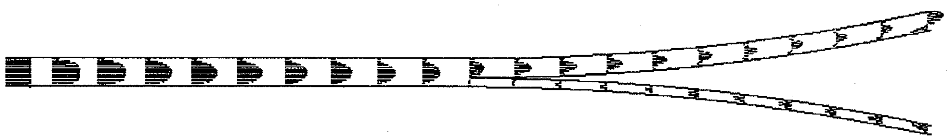

Fig.4(b) Velocity vectors with the standard MAC method

Fig.4(c) Velocity vectors with the present method

Fig.4(b) and fig.4(c) show that the results obtained by the present method synchronize with the pulsatory inflow better than those obtained by the standard MAC method. Furthermore, the flow rate ratio of each duct seems not to be 2:1. This is reasonable since the narrower duct tends to affect the friction of the wall more strongly than the wider one.

Fig.4(d) shows the time history of the flux at inflow and at 3/4 length of the duct with standard MAC method and the present method. It is apparent that when it is compared with 20 times of iteration, the flux obtained by the present method satisfy the equation of continuity more precisely than those obtained by the standard MAC method. This implies that the present method is effective to hold the assumption of incompressibility with smaller number of iterations.

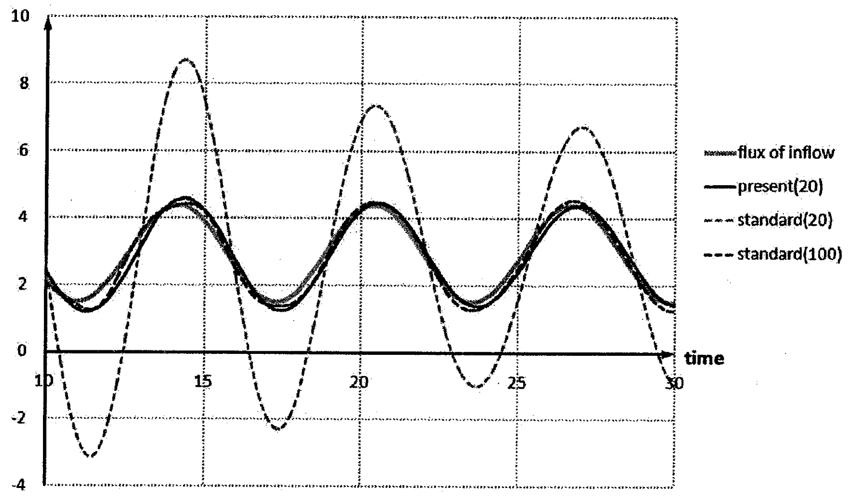

Fig.4(d) The time history of the flux at 3/4 length of the duct with comparison of standard MAC method and the present method. The number noted in bracket represents the iteration number of Poisson equation.

Fig.4(e) shows the flux profile along the duct at non-dimensional time 30. It is apparent that with the standard MAC method with 20 times of iteration the flux becomes smaller and smaller as the flow goes downstream. This is inconsistent with the assumption of the incompressibility. Although results obtained by the standard MAC method with 100 times of iteration are better than that of 20, results obtained by the present method with 20 times of iteration are better.

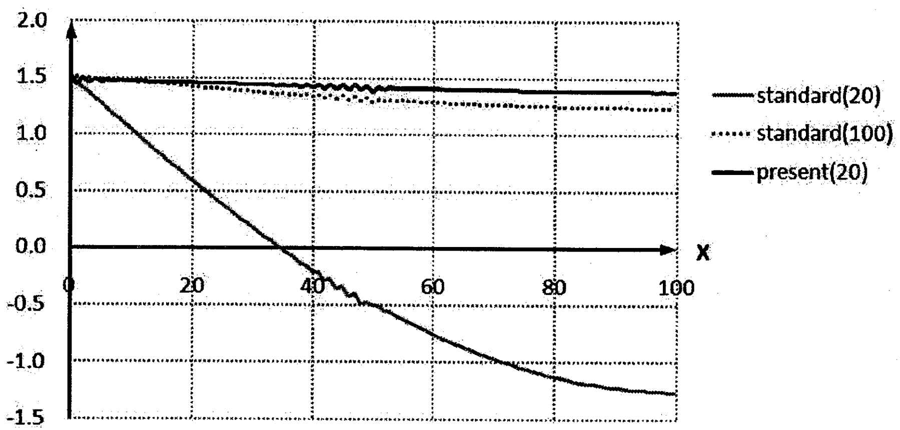

Fig.4(e) The comparison of flux along the duct at time30.

(c) Two-dimensional flow in a duct of large curvature

In many practical applications, it is required to calculate the flow in a duct of large curvature. The shape of the duct is sinusoidal curve superimposed on the circle. In this case, the treatment based on eqs.(1)-(6) does not wok since the computations diverge. Therefore, we choose the method based on eqs.(8)-(13). This method works well as is shown in fig.5(a)-(c) if we compare the standard method.

Fig.5 (a) and (b) are the velocity vectors obtained by the standard MAC method and the present method respectively. The both results are at the time of non-dimensional time 30 and the iteration number of the Poisson equation is 20.

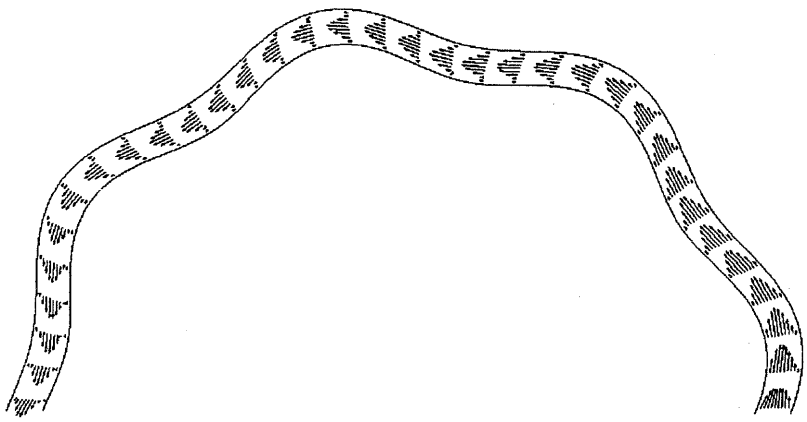

Fig.5 (a) Two-dimensional flow in a duct of large curvature with the standard MAC method

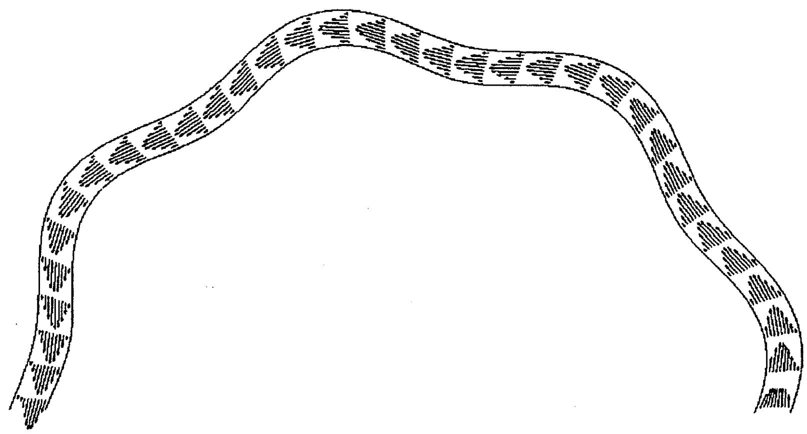

Fig.5 (b) Two-dimensional flow in a duct of large curvature with the present method

Fig.5(c) is the time history of flux in a two-dimensional duct of large curvature. Results obtained by the standard MAC method and the present method are compared by those fluxes at 3/4 length of the duct with that of the inflow.

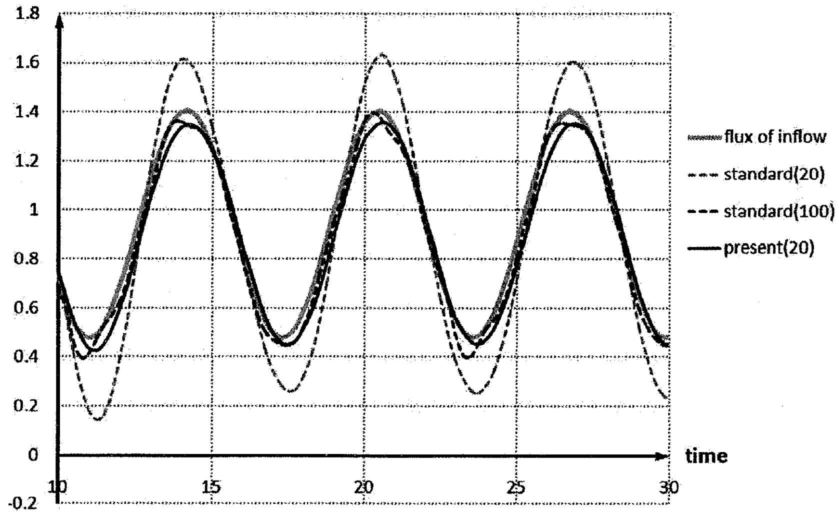

Fig.5(c) Time history of flux of two-dimensional flow in a duct of large curvature

From this figure, it is clear that the present method shows that the conservation of the flow is satisfied more precisely in polar coordinate system as well.

4. Summary

It is well-known that the numerical method for incompressible flow based on solving the Poisson equation of pressure such as the standard MAC method has a weak point that it is difficult to satisfy the equation of continuity precisely. This tendency becomes more serious when the region is very long (i.e. the aspect ratio is large) or it is required to obtain the unsteady solutions. If we use staggered grid system and the Poisson equation is converged sufficiently, this shortcomings are loosen. However it means large computation times and also it is difficult to adopt the generalized coordinate system.

In order to improve this situation, we proposed a new method in this study. The original problem is modified as simple as possible at first. The flow variables are expressed as the sum of those of the simplified problem and their variations, i.e.

(Solution of original problem) = (Solution of simplified problem) + (Variations)

Since we assume that the flow is in very long region, the simplified problem becomes one-dimensional flow. This can be solved analytically. Then the equation for variations of main flow is solved numerically.

By using this concept, we proposed the numerical method to treat two-dimensional flow in a long duct with and without branch and a long duct of large curvature. Then we have done three test calculations in order to confirm effectiveness of the present method. For all cases, the conservation of mass of fluid is well satisfied. From the results in this study, the present method needs less number of iterations in order to solve the Poisson equation than the standard MAC method.

Although we utilize analytical solutions, we can also use numerical solutions for the simplified problem so that the method becomes more general one.

References

[1]Harlow, F. H. and Welch, J. E., Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface, Physics of Fluids, 8-12, 1965, pp.2182-2189.

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