お茶の水女子大学自然科学報告　第57巻　第2号
Natural Science Report, Ochanomizu University, Vol. 57, No. 2 (2006)

Critcal Exponent for Logarithmic Nonlinearity of Nonlinear Heat Equation

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(Received September 30, 2006)

Abstract

Our aim is to extend the blowup result by Levine-Meier for nonlinear heat equation of Fujita type on conical domains to generalized paraboloidal domains by replacing fractional type nonlinearity to logarithmic one. In this first note we give an abstract result assuring the existence of the critical exponent for such nonlinearity.

1 Introduction

In [1], Fujita showed for the first time the so called Fujita phenomenon for the initial value problem of the heat equation on $\boldsymbol{R}^N$

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \qquad u(0,x) = u_0(x).$$

Namely, there exists a critical value $p^* = 1 + 2/N$ such that for $1 < p < p^*$ the solution for any non-negative non-zero initial data blows up in finite time, whereas for $p > p^*$ the solution is time-global for sufficiently small non-negative initial data. This phenomenon was extended to many directions. Among others we mention Levine-Meier [2], who showed that in a conical domain with spherical profile Ω the Fujita-type phenomenon occurs for the solutions of the homogeneous Dirichlet boundary condition with the critical exponent $p^* = 1 + 1/(N+\gamma)$, where γ is the positive root of the quadratic equation $\gamma^2 + (N-2)\gamma - \omega_1 = 0$ and ω_1 is the first Dirichlet eigenvalue of the (positive) Laplace-Beltrami operator of the spherical domain Ω. Our aim is to consider the unbounded domains where $p^* = 1$ in their theory. More concretely, we wish to distinguish cylindrical domains and general paraboloidal domains of the form

$$x_n < |x'|^q + C,$$

with some $q > 1$. For this purpose, we introduce a logarithmic nonlinearity, and consider on such domains D the homogeneous Dirichlet mixed problem for the equation

$$\frac{\partial u}{\partial t} = \Delta u + u(\operatorname{logg} u)^p, \qquad u(0,x) = u_0(x), \qquad u|_{\partial D} = 0, \tag{P}$$

where the new function symbol $\operatorname{logg} u$ in the nonlinear term stands for the following:

$$\operatorname{logg} u = \begin{cases} \log u + 1 & \text{for } u \geq 1, \\ \dfrac{1}{1 - \log u} & \text{for } 0 < u < 1. \end{cases}$$

We make the convention that $\operatorname{logg} 0 = 0$. This function was so chosen as to balance the growth rate at $u = \infty$ with the shrink rate at $u = 0$ as in the case of fractional nonlinearity. The graph of this function is as follows:

[1]Partially supported by Grant-in-Aid for Scientific Research No. 16540140

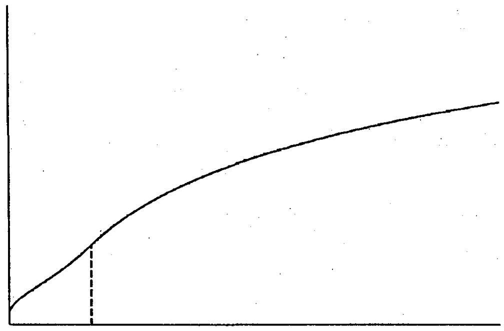

Fig.1. graph of $\operatorname{logg} x$;

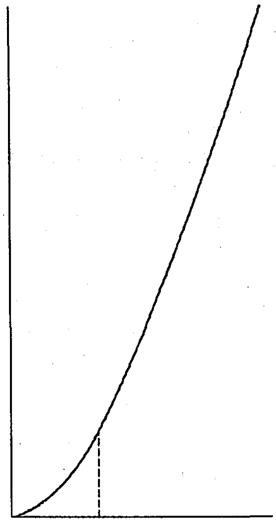

Fig.2. graph of $x \operatorname{logg} x$

This is a monotone increasing function. Notice that $u \operatorname{logg} u$ as well as $u(\operatorname{logg} u)^p$ for any $p > 0$ becomes convex, as is easily verified by a simple calculation: For $u \geq 1$

$$\begin{aligned}
\frac{d^2}{du^2} u(\operatorname{logg} u)^p &= \frac{d^2}{du^2} u(\log u + 1)^p = \frac{d}{du}\{(\log u + 1)^p + p(\log u + 1)^{p-1}\} \\
&= \frac{p}{u}(\log u + 1)^{p-1} + \frac{p(p-1)}{u}(\log u + 1)^{p-2} \\
&= \frac{p}{u}(\log u + 1)^{p-2}(\log u + p) > 0.
\end{aligned}$$

For $0 < u < 1$ we have

$$\begin{aligned}
\frac{d^2}{du^2} u(\operatorname{logg} u)^p &= \frac{d^2}{du^2} \frac{u}{(1-\log u)^p} = \frac{d}{du}\left\{\frac{1}{(1-\log u)^p} + \frac{p}{(1-\log u)^{p+1}}\right\} \\
&= \frac{p}{u(1-\log u)^{p+1}} + \frac{p(p+1)}{u(1-\log u)^{p+2}} > 0.
\end{aligned}$$

In this article we present an abstract result which assures the existence of critical exponent for such nonlinearity. It will be applied in our forthcoming paper to determine the critical exponent to the mixed problems on generalized parabolic regions as mentioned above.

When the first author's work [4] with his former student S. Ohta was reported as the master thesis of the latter at the University of Tokyo, Prof. Y. Tsutsumi suggested to generalize the fractional nonlinearity to treat more general domains. Our work is based on his proposal, and we are very much indebted to this valuable suggestion.

2 Critical exponent for logarithmic nonlinearity

We modify Meyer's abstract criterion for finding critical exponent expressed in terms of the decay rate of the solution of the linear heat equation and adapt it to our logarithmic type case. In the sequel, a (super- or sub-) solution will always imply a non-negative one unless otherwise mentioned.

Consider the corresponding linear problem:

$$\frac{\partial u}{\partial t} = \Delta u, \qquad u(0,x) = u_0(x), \qquad u|_{\partial D} = 0. \tag{P$_0$}$$

The following is our main result modifying the corresponding lemma by Meier [3] for the case of fractional nonlinearity to our logarithmic one.

Theorem 2.1 1) *Assume that there exists a non-trivial super-solution W of* (P_0) *which satisfies, for some* $\varepsilon > 0$ *and* $C > 0$,

$$t^{1+\varepsilon}\,(\mathrm{logg}\,\|W(t,\cdot)\|_\infty)^{p+1} \le C. \tag{1}$$

Then a global solution of (P) *exists.*

2) *Assume that for some non-trivial sub-solution W of* (P_0) *we have*

$$\overline{\lim_{t\to\infty}}\, t\,(\mathrm{logg}\,\|W(t,\cdot)\|_\infty)^{p+1} = \infty. \tag{2}$$

Then every solution of (P) *blows up in finite time.*

Proof 1) Let $\beta(t)$ be the solution of the ordinary differential equation

$$\beta'(t) = \beta(t)\{\mathrm{logg}\,(\beta(t)\|W(t,\cdot)\|_\infty)\}^p. \tag{3}$$

Then

$$\overline{u}(t,x) := \beta(t)W(t,x)$$

becomes a supersolution of the solution u of (P). In fact, in view of the assumption $W_t - \Delta W \ge 0$ and the obvious fact $\beta \ge 0$, we have

$$\begin{aligned}\overline{u}_t - \Delta\overline{u} &= \beta'(t)W(t,x) + \beta(t)W'(t,x) - \beta(t)\Delta W(t,x)\\ &\ge \beta(t)W(t,x)\{\mathrm{logg}\,(\beta(t)\|W(t,\cdot)\|_\infty))\}^p\\ &\ge \beta(t)W(t,x)\{\mathrm{logg}\,(\beta(t)W(t,x))\}^p = \overline{u}(\mathrm{logg}\,\overline{u})^p.\end{aligned}$$

Now we seek for a global solution of the equation (3). Notice that the condition (1) implies that $\|W(t,x)\|_\infty \to 0$. Hence, adjusting the initial value to be $w < 1$ by a constant factor, we can assume without loss of generality that $\|W(t,x)\|_\infty < 1$ and it is monotone decreasing to 0. Hence the assumption (1) is rewritten as

$$(1 - \log\|W(t,\cdot)\|_\infty)^{p+1} \ge C^{-1}t^{1+\varepsilon}, \qquad \text{or} \qquad \log\|W(t,\cdot)\|_\infty \le 1 - C't^{(1+\varepsilon)/(p+1)}.$$

In view of the above made assumption $\|W(t,\cdot)\|_\infty < 1$, the last inequality may be rewritten as

$$\log\|W(t,\cdot)\|_\infty \le -c(1 + t^{(1+\varepsilon)/(p+1)})$$

with some constant $c > 0$. Note also that if $\beta(t)\|W(t,x)\|_\infty < 1$ at some t, then, $\beta(t) \le \|W(t,x)\|_\infty^{-1}$ and the solution obviously exists up to this time. Thus we may assume that $\beta(t)\|W(t,x)\|_\infty \ge 1$ from some time t on, and we shall shift the time so that this inequality holds for $t \ge 0$. Then equation (3) reads as

$$\begin{aligned}\frac{d}{dt}\log\beta = \frac{\beta'(t)}{\beta(t)} &= \{\mathrm{logg}\,(\beta(t)\|W(t,\cdot)\|_\infty)\}^p = \{\log(\beta(t)\|W(t,\cdot)\|_\infty) + 1\}^p\\ &= \{\log\beta(t) + 1 + \log\|W(t,\cdot)\|_\infty\}^p \le \{\log\beta(t) + 1 - c - ct^{(1+\varepsilon)/(p+1)}\}^p.\end{aligned}$$

Thus the solution of the equation

$$\frac{d}{dt}\log\beta = \{\log\beta(t) + 1 - c - ct^{(1+\varepsilon)/(p+1)}\}^p$$

becomes a supersolution of the equation (3). Hence it suffices to show that this one has a global solution for sufficiently small $\beta(0) \ge e^c$. Rewriting $y(t)$ for $\log\beta(t) - c$, and x for t, we obtain from above the following nonlinear ordinary differential equation:

$$y' = (y + 1 - cx^{(1+\varepsilon)/(p+1)})^p. \tag{4}$$

Lemma 2.2 *Equation* (4) *has a global solution for some non-negative initial data* $y(0)$, *provided c is chosen large enough.*

Since this is rather an independent subject, we give the proof after the end of the proof of the present theorem. Now we have a globally defined $\beta(t)$, and hence a global solution of (P).

2) Consider the following initial value problem for ordinary differential equation:

$$\frac{dz}{dt} = z(\operatorname{logg} z)^p, \qquad z(0) = w. \tag{5}$$

Let $z(t; w)$ denote the solution. We assert that $\underline{u} := z(t; W(t,x))$ becomes a subsolution to the nonlinear problem P. In fact, we have

$$\underline{u}_t - \Delta\underline{u} = z(t;W)(\operatorname{logg} z(t;W))^p + z_w W_t - z_w \Delta W - z_{ww}|\nabla W|^2 = \underline{u}(\operatorname{logg}\underline{u})^p - z_{ww}|\nabla W|^2.$$

Hence, to prove $\underline{u}_t - \Delta\underline{u} - \underline{u}(\operatorname{logg}\underline{u})^p \leq 0$, it suffices to show that $z_w \geq 0$ and $z_{ww} \geq 0$. The equation (5) is integrated in view of the following elementary lemma of which the proof we omit:

Lemma 2.3 *We have*

$$\int \frac{dx}{x(\operatorname{logg} x)^p} = \begin{cases} -\dfrac{1}{p-1}\dfrac{1}{(\operatorname{logg} x)^{p-1}}, & \text{for } x \geq 1, \\ -\dfrac{1}{p+1}\dfrac{1}{(\operatorname{logg} x)^{p+1}}, & \text{for } 0 < x < 1. \end{cases}$$

Employing this we can explicitly solve for z. To obtain a subsolution, we can always assume without loss of generality that $w < 1$.

First, starting from such w, during the time z remains below 1, we have

$$t = \int_w^z \frac{dz}{z(\operatorname{logg} z)^p} = \frac{1}{p+1}\Big\{(1-\log w)^{p+1} - (1-\log z)^{p+1}\Big\},$$

hence,

$$z = \exp\Big\{1 - \Big((1-\log w)^{p+1} - (p+1)t\Big)^{1/(p+1)}\Big\}. \tag{6}$$

Second, when $z \geq 1$ from some time on, we have

$$\begin{aligned} t &= \int_w^z \frac{dz}{z(\operatorname{logg} z)^p} = \int_w^1 \frac{dz}{z(\operatorname{logg} z)^p} + \int_1^z \frac{dz}{z(\operatorname{logg} z)^p} \\ &= -\frac{1}{p+1} + \frac{1}{p+1}(1-\log w)^{p+1} + \frac{1}{p-1} - \frac{1}{p-1}\frac{1}{(\log z+1)^{p-1}}. \end{aligned}$$

Thus

$$z = \frac{1}{e}\exp\Big[\Big(\frac{p+1}{p-1}\Big)^{1/(p-1)}\Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{-1/(p-1)}\Big]. \tag{7}$$

Now we can differentiate these with respect to w twice. In the first case (6), we obtain

$$\begin{aligned} \frac{dz}{dw} &= \exp\Big[1 - \Big\{(1-\log w)^{p+1} - (p+1)t\Big\}^{1/(p+1)}\Big] \\ &\quad\times \Big\{(1-\log w)^{p+1} - (p+1)t\Big\}^{-p/(p+1)}\frac{(1-\log w)^p}{w}, \\ \frac{d^2z}{dw^2} &= \exp\Big[1 - \Big\{(1-\log w)^{p+1} - (p+1)t\Big\}^{1/(p+1)}\Big] \\ &\quad\times \Big\{(1-\log w)^{p+1} - (p+1)t\Big\}^{-(2p+1)/(p+1)}\frac{(1-\log w)^{p-1}}{w^2} \\ &\quad\times \Big[(1-\log w)^{p+1}\Big\{(1-\log w)^{p+1} - (p+1)t\Big\}^{1/(p+1)} + p(1-\log w)^{p+1} \\ &\qquad - p\Big\{(1-\log w)^{p+1} - (p+1)t\Big\} - (1-\log w)\Big\{(1-\log w)^{p+1} - (p+1)t\Big\}\Big]. \end{aligned}$$

This is easily seen to be nonnegative. In the second case, we have

$$\frac{dz}{dw} = \frac{1}{e}\Big(\frac{p+1}{p-1}\Big)^{p/(p-1)} \exp\Big[\Big(\frac{p+1}{p-1}\Big)^{1/(p-1)}\Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{-1/(p-1)}\Big]$$
$$\times \Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{-p/(p-1)} \frac{(1-\log w)^p}{w},$$

$$\frac{d^2z}{dw^2} = \frac{1}{e}\Big(\frac{p+1}{p-1}\Big)^{p/(p-1)} \exp\Big[\Big(\frac{p+1}{p-1}\Big)^{1/(p-1)}\Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{-1/(p-1)}\Big]$$
$$\times \Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{-2p/(p-1)} \frac{(1-\log w)^{p-1}}{w^2}$$
$$\times \Big[\Big(\frac{p+1}{p-1}\Big)^{p/(p-1)}(1-\log w)^{p+1}$$
$$+ \frac{p(p+1)}{p-1}(1-\log w)^{p+1}\Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{1/(p-1)}$$
$$- p\Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{p/(p-1)}$$
$$- (1-\log w)\Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{p/(p-1)}\Big]$$
$$= \frac{1}{e}\Big(\frac{p+1}{p-1}\Big)^{p/(p-1)} \exp\Big[\Big(\frac{p+1}{p-1}\Big)^{1/(p-1)}\Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{-1/(p-1)}\Big]$$
$$\times \Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{-2p/(p-1)} \frac{(1-\log w)^{p-1}}{w^2}$$
$$\times \Big[\Big(\frac{p+1}{p-1}\Big)^{p/(p-1)}(1-\log w)^{p+1}$$
$$- \Big\{\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t\Big\}^{p/(p-1)}(1-\log w)$$
$$+ p\Big\{\frac{2}{p-1}\{(1-\log w)^{p+1}-1\}+(p+1)t\Big\}\Big\{\frac{2}{p-1}+(1-\log w)^{p+1}-(p+1)t\Big\}^{1/(p-1)}\Big].$$

The positivity of this is not at all evident, and even formally false. Note, however, that this formula is valid only if $z \geq 1$ in (7), hence under the condition

$$\frac{2}{p-1} + (1-\log w)^{p+1} - (p+1)t \leq \frac{p+1}{p-1}.$$

Then the positivity of the above quantity is obvious. The positivity of z_w is obvious in either case. Thus we have established that $\underline{u}$ is the subsolution.

Now we examine the lifespan of the subsolution $z(t; W(t,w))$ for $w < 1$. Obviously the first case (6) cannot last forever, hence we have finally $z \geq 1$, hence the case (7). If $(1-\log W)^{p+1}/t \to 0$, or equivalently, if $t(\operatorname{logg} W)^{p+1} \to \infty$, as $t \to \infty$, then (7) also becomes illegal in finite time. Thus in view of the assumption the solution cannot exist globally. This ends the proof of Theorem 2.1. QED

Proof of Lemma 2.2 For general power p the equation is only meaningful for $y + 1 - cx^{(1+\varepsilon)/(p+1)} \geq 0$, that is, for $y \geq cx^{(1+\varepsilon)/(p+1)} - 1$. In the interior of this region the inclination of the solution curve is positive, and if it touches the boundary curve C_1 (the loci of maxima), it does horizontally. Differentiating the equation (4) once, we obtain

$$y'' = p(y+1-cx^{(1+\varepsilon)/(p+1)})^{p-1}\Big(y' - c\frac{1+\varepsilon}{p+1}x^{-(p-\varepsilon)/(p+1)}\Big)$$
$$= p(y+1-cx^{(1+\varepsilon)/(p+1)})^{p-1}\Big\{(y+1-cx^{(1+\varepsilon)/(p+1)})^p - c\frac{1+\varepsilon}{p+1}x^{-(p-\varepsilon)/(p+1)}\Big\}.$$

Thus the solution curve is convex in the region $(y+1-cx^{(1+\varepsilon)/(p+1)})^p - c\frac{1+\varepsilon}{p+1}x^{-(p-\varepsilon)/(p+1)} \geq 0$, namely on $y \geq cx^{(1+\varepsilon)/(p+1)} - 1 + c^{1/p}\big(\frac{1+\varepsilon}{p+1}\big)^{1/p}x^{-(p-\varepsilon)/p(p+1)}$. This region is inside that of positive inclination. Its boundary curve C_2 (loci of inflection points) has obviously a minimum, say m_0. The solution curve intersects C_2 with an inclination bigger than that of C_2, because on C_2 wh have

$$\begin{aligned} y &= cx^{(1+\varepsilon)/(p+1)} - 1 + c^{1/p}\Big(\frac{1+\varepsilon}{p+1}\Big)^{1/p} x^{-(p-\varepsilon)/p(p+1)}, \\ y' &= c\frac{1+\varepsilon}{p+1}x^{-(p-\varepsilon)/(p+1)} - c^{1/p}\Big(\frac{1+\varepsilon}{p+1}\Big)^{1/p}\frac{p-\varepsilon}{p(p+1)}x^{-(p-\varepsilon)/p(p+1)-1} \\ &< c\frac{1+\varepsilon}{p+1}x^{-(p-\varepsilon)/(p+1)} = (y+1-cx^{(1+\varepsilon)/(p+1)})^p. \end{aligned}$$

Thus a solution curve starting with the initial value $y(0) \geq m_0$ should touch C_2 and then remain above forever. Hence, if we trace back a solution curve of our differential equation, starting from any point Q on C_1 and in the backward direction of x, it should come below C_2 and should finally shoot the y-axis below m_0. When we let Q tend to infinity along C_1, the solution curve therefore converges to a global solution with $y(0) \leq m_0$. Thus to prove the lemma, it suffices to assure that $y(0) \geq 0$ for this limit solution, or equivalently that the solution starting from $y(0) = 0$ does not hit the curve C_2. Recall that we can choose c as large as we wish. The intersection of C_1 with the x-axis is $x = c^{-(p+1)/(1+\varepsilon)}$, which approaches 0 as $c \to \infty$. Also C_1 approaches the part $y \geq -1$ of the y-axis locally uniformly. On the other hand, the solution with the initial value $y(0) = 0$ satisfies

$$y' \leq (y+1)^p, \qquad \text{i.e.} \qquad y \leq \frac{1}{(1-(p-1)x)^{1/(p-1)}} - 1$$

as long as it remains above C_1. Thus for sufficiently large c, it hits C_1, and the proof is complete. QED

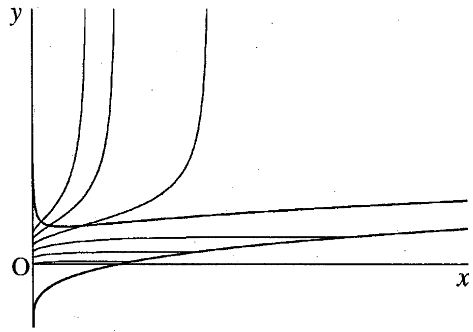

Fig.3. Thick lines: loci of maxima and loci of inflection points ($p = 3$, $c = 1$, $\varepsilon = 0.1$); Thin lines: solution curves of the initial values $0.0 \sim 0.5$ by step 0.1. The drawn range is $0 \leq x \leq 5$, $-1 \leq y \leq 4$.

Theorem 2.1 assures the existence of critical exponent p^* for the case of logarithmic nonlinearity. Notice that the assumption $p > 1$ is not a restriction, because, for $p < 1$ the solution of (P) is always global. The case $p = 1$ is delicate. But we are, for the moment, interested in the value of the critical exponent itself. What happens when p is equal to the critical exponent may depend on the concrete problem.

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