Reconstruction from Two Projections with Prohibited Subregion – Algorithm, Switching Graph and Consistency

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Abstract

We consider the two projection tomography problem initiated by Lorentz, Gale and Ryser. Here we assume that a priori knowledge is available that presents a prohibited region for the original figure. We show that a modification of Ryser's reconstruction algorithm gives a solution. We then study the relation of the switching graph for the solution sets with and without the prohibited region. We apply our idea to get a better reconstruction figure imposing prohibited region artificially. Finally, we discuss the condition for a set to be prohibited region.

1 Introduction

We consider the reconstruction problem of a discrete plain figure F contained in a rectangle I from its two projections $f_y(x)$ and $f_x(y)$ along the y- and x-axis, respectively. This is equivalent to the problem of finding a binary matrix (that is, a matrix with only 0 or 1 in its entries) from its column- and row-sums, as employed by many references. But we prefer the geometric notation keeping in mind the connection with continuous tomography. In this report we assume that we know a priori that F has no building cell (that is, filled with the element 0 in the notation of binary matrix) in a subregion J of I, and consider the reconstruction problem satisfying this constraint. J may be any subset of I which satisfies the uniqueness criterion of Lorentz-Ryser. The simplest example is a sub-rectangle placed anywhere in I.

We show that a modification of Ryser's algorithm for the case without such constraint, works to obtain a solution of this reconstruction problem. Then we study the structure of the solution set of this reconstruction problem with constraint. In our former work [8] we studied the structure of the solution set by assigning a graph. We here extend it to the solution set of the reconstruction problem with constraint, and examine the relation of thus obtained graphs.

We apply our reconstruction technique with constraint to obtain a better solution of the reconstruction problem from the two projections which originally has no constraint. If we succeed in the reconstruction by setting a prohibited region in which we do not wish to have cells, we may obtain a reconstructed figure better fitted to our purpose. This idea works faster than the strategy of successive improvement adopted so far.

In the final section we discuss the condition for the prohibited region.

The reconstruction problem with rectangular constraint was considered by Brualdi and Dahl [3]. In [4] an equivalent result for unique figure J is announced without proof. We hope that our reconstruction algorithm is simple and practical. As further related works, Fulkerson [5] considered reconstruction of binary square matrices with zero diagonal. Notice that the diagonal is a figure farthest from unique ones. Also, Kuba [12] studies problem of reconstruction with prescribed 1's, which is intimately related with the present problem, but not equivalent (see Remark 2.3 (2) of §2). The work of Anstee [1] can also be understood of reconstruction with prescribed 1's.

This is a full version with detailed proofs of our report [9]. We thank the referee for that report who kindly informed us many related references cited above concerning this problem.

2 Setting of the problem and reconstruction algorithm

Let I denote the rectangular region $a \le x < b$, $c \le y < d$ in the first quadrant of $\boldsymbol{R}^2$. We only treat discrete figures, so the corner coordinates are all integers. Let F be a subregion of I which is the union

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of integer cells $C_{ij} := [i, i+1) \times [j, j+1)$. Practically, I would be the superscribing rectangle of F which is uniquely defined from the projection data. But to prove something by induction, we sometimes need I bigger than that. Therefore we do not assume this.

We let F denote its characteristic function at the same time. Thus its y-projection $f_y(x)$, or x-projection $f_x(y)$ is defined as

$$f_y(x) = \int_c^d F(x,y)dy, \qquad f_x(y) = \int_a^b F(x,y)dx. \tag{1}$$

In this case these also take integer values, representing the number of cells in the assigned column resp. row. The reconstruction problem from the two projections is to find F from the projection data $f = \{f_y(x), f_x(y)\}$. Define the arrangements by

$$f_{xy}(x) = \text{meas}\{y; f_x(y) \geq x\}, \quad f_{yx}(y) = \text{meas}\{x; f_y(x) \geq y\},$$
$$f_{yxy}(x) = \text{meas}\{y; f_{yx}(y) \geq x\},$$

where meas denotes the one-dimensional length. In the discrete case, this is equivalent (modulo measure 0) to the permutation of the columns or rows in decreasing order and finally view all from the x-axis. Then the consistency condition, that is, the condition for the existence of a solution, given by Lorentz, Gale and Ryser is

$$\forall x \quad \int_0^x f_{xy}(t)dt \geq \int_0^x f_{yxy}(t)dt, \qquad \text{and} \qquad \int_0^\infty f_{xy}(t)dt = \int_0^\infty f_{yxy}(t)dt. \tag{2}$$

According to Lorentz and Ryser, the uniqueness of the solution is assured if and only if the equality holds for all x in the first inequality above. For further information about this problem see the survey article [13]. We here recall only materials needed in the sequel.

First we review Ryser's reconstruction algorithm in a form given in [7]. We shall call this hereafter the Ryser-Kaori algorithm without constraint.

1. Choose the tallest column from $f_y(x)$. In case of tie, choose the leftmost one.
2. Remove this column from $f_y(x)$ and at the same time, remove the same number of cells from the top of $f_x(y)$ one for each row by the strategy of the longest row first.
3. Place the cells at the corresponding column and rows as the part of the reconstructed figure.
4. Return to 1 if there still remain columns in thus modified $f_y(x)$.

Now we have a subregion $J \subset I$ where we should not place any cell in reconstruction. In what follows we assume that J constitutes a unique figure, that is, there is no other figure having the same projection data as J. Note the following

Lemma 2.1 *A figure J is unique if and only if by a permutation of columns and rows it is brought to the form of Lorentz's rearrangement of the projections, that is, to the form of union of horizontally adjacent, height-decreasing subrectangles J_i, $i = 1, \dots, r$ with the lower edges on a common line (see Figure 1) :*

$$J = J_1 \cup J_2 \cup \cdots \cup J_r. \tag{3}$$

In fact, the sufficiency is obvious. To show the necessity, we can first find a permutation bringing the y-projection $f_y(x)$ to the monotone decreasing form. Then by the assumption of uniqueness, the y-projection $f_{xy}(x)$ of the x-projection $f_x(y)$ agrees with this rearranged $f_y(x)$. Since this y-projection $f_x \mapsto f_{xy}$ is achieved by a permutation of rows, after these two permutations the y- and x-projection of J agree with this monotone figure. In view of the uniqueness, this implies that J itself has the same form.

Thus, we shall assume henceforth without loss of generality that J has the above form (3). We can further assume that as in Fig. 1, there is no projection data to the left of the first column of J and below the first row of J. In fact, if there were such, via permutation of columns and rows, these data are brought to the right and to the above of the other data without touching the rearranged J.

Now we explain the reconstruction algorithm. We shall say that a pair of projection data $f = \{f_y(x), f_x(y)\}$ is *J-consistent* if there exists a reconstruction F with J as prohibited region.

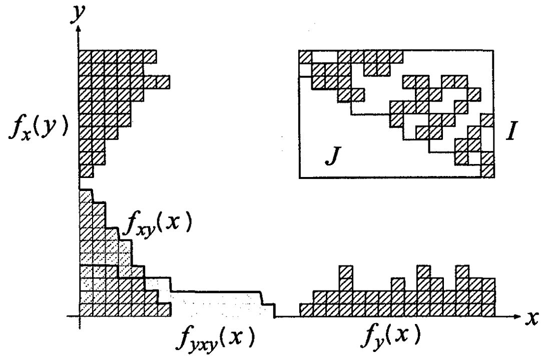


Figure 1: A figure with prohibited region and its projections.

1. For $i = 1$ to r do
2. Choose the tallest column from $f_y(x)$ among those above J_i. In case of tie, choose the leftmost one.
3. Remove this column from $f_y(x)$ and at the same time, remove the same number of cells from the top of $f_x(y)$ one for each row by the strategy of the longest row first, but among the rows not touching J_i.
4. Place the cells at the corresponding column and rows as the part of the reconstruction figure.
5. Return to 2 if there still remain columns above J_i.
6. End for.

Theorem 2.2 *The above algorithm for rearranged J successfully gives a reconstruction figure which does not contain any cell in J, provided that the projection data are J-consistent.*

Proof. We proceed by the induction of the total number of cells in the figure. The case of one cell is trivial. Assume that the assertion is true for any J and for any J-consistent projection data up to n cells, and consider a problem with $n+1$ cells. By the assumption of J-consistency, there exists a solution figure F which we may not know concretely. Following the above algorithm, we first choose the tallest column among those above J_1. For each cell in this column, we pick up a cell from $f_x(y)$ at the longest row not touching J_1. If the chosen cell exists in the figure F, we are correctly diminishing the data. If we chose a cell, say P, from a row where there was no cell in that column of the figure F, then there should exist a cell of F, say P$'$, in the column of P which was not chosen by the reconstruction algorithm. On the other hand, P comes from a cell of F, say Q, in another column by x-projection, There are several candidates of such Q, but we claim that among them there is at least one such that P$'$ and Q constitute a switching component in F, that is, the place R in Figure 2 is vacant. In fact, if all the counterparts in the row of P$'$ are occupied by the cells of F, then the x-projection at the row of P$'$ will have length greater than that at the row of P. This violates the rule of algorithm that we should pick up the cell from the longer rows of $f_x(y)$ first. This argument applies to all cells chosen in relation to this column. Thus after removing the column from $f_y(x)$ and the corresponding cells from $f_x(y)$ there remain projection data which come from a true figure F' obtained by several switching as mentioned above from F. This means that the remaining projection data are J-consistent, and by the induction hypothesis, we can obtain a solution of reconstruction with the constraint. By adding the first treated column to this solution, we obtain a solution for the given size. QED

The converse is obvious: if our algorithm ends up using all the cells in the projection, we obtain a reconstruction with the given constraint J. Thus our algorithm presents a practical criterion for the J-consistency.

Remark 2.3 1). The direct application of Ryser-Kaori algorithm, that is, processing from the tallest of all columns ignoring J, does not work. Figure 3 is such an example.

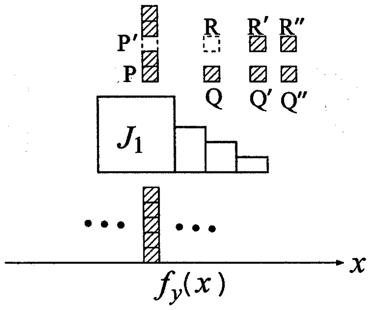


Figure 2: Proof of justification of reconstruction algorithm.

2). One may think of an alternative algorithm such as filling the prohibited region by cells and applying Ryser-Kaori algorithm without constraint, then removing the cells in the prohibited region will give a desired solution. But in general it is not easy to obtain a reconstruction of which the prohibited region is filled with cells. See, however, §5 in this respect.

3) In the above argument, we assumed that the building blocks $J_1, \ldots, J_r$ of J strictly diminish their height. The above reconstruction algorithm works, however, without this restriction. In the extreme case, we can assume that all the heights of J_i are equal to 0, namely, $J = \emptyset$. In this case the above algorithm gives a reconstruction with assigned column order. The above proof then shows that we can reconstruct a figure in any order of choice of the columns, irrespective of their heights. Recall that the reason why we took the strategy of tallest column first in [7] was to minimize the reconstruction error for the discrete approximation of continuous figures.

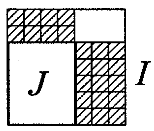


Figure 3: Necessity of modification of reconstruction order.

3 Switching graph

It is well known that a discrete figure is non-unique, namely, there is another figure with the same projection data, if and only if it contains a switching component. For a non-unique figure we can relate a graph to the solution set by regarding each solution as a vertex, and connecting a pair of solutions by an edge if and only if they are transformed by one switching operation. This graph seems to have been first introduced by Brualdi [2] under the name of interchange graph. Later, [11] re-introduced it and called Ryser graph. Unaware of these, we called it the switching graph and studied its properties with many examples. Further we gave a direction to each edge showing the type modification from type 2 to type 1 (see Figure 4), thus producing the switching digraph ([8]). Since we employ in the sequel permutation of columns and rows which may change the direction of engaged edges, we only consider the switching graph in this paper.

We shall denote by G_J the switching graph for the solution set with constraint J, and simply call it the *J-constraint switching graph.* If distinction is preferable, we shall call the switching graph G of all the solutions without constraint the *full switching graph* and further add the projection data like $G[f]$

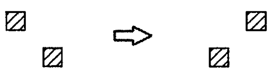


Figure 4: Switching components and switching operation: type 2 (left) and type 1 (right).

Figure 5: Examples of G_J for non-unique J's.

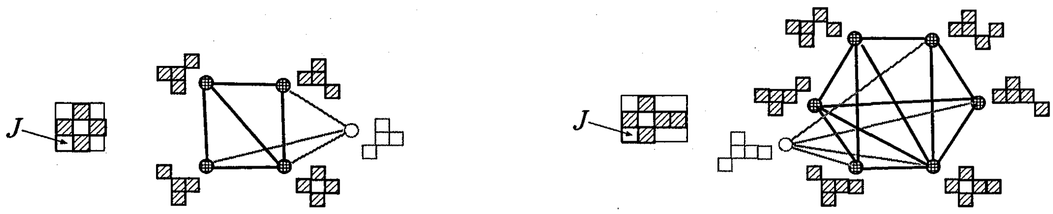


Figure 6: Examples of G_J for unique J's.

or $G_J[f]$. It is obvious from the definition that for any J (not necessarily unique) G_J becomes a full subgraph of G.

In this section, we shall study properties of G_J, especially taking in mind which properties for G hold as well for G_J.

First consider a few examples: Here G_J is denoted by shadowed vertices and thick edges embedded in G.

From these examples we can see that the situation is much different according to whether J is unique or not. Even the connectedness of G_J is not guaranteed if J is not unique. Moreover, the graphs in Fig. 6 corresponding to unique J's imply that G_J is not necessarily obtained as the full switching graph of another set of projection data. (Cf. the list of 4- and 6-vertex graphs in [8].) Thus in spite of the connectedness of G assured by Ryser's theorem, the connectedness of G_J is not obvious, and our first task is to prove this.

Theorem 3.1 *Let J be a prohibited region which is a unique figure. Then the J-constraint switching graph G_J is a connected full subgraph of the full switching graph G.*

Proof. Instead of giving the details of the proof outlined in [9] (which can be found in [10]), we here present a new shorter one. As remarked after the proof of Lemma 2.1, we assume that J is in the rearranged form (3) and that the cells of the reconstructed figures all lie above or to the right of J. We then proceed by induction on the number of columns of such rearranged J. The first step of induction is the case where $J = \emptyset$, that is, 0 column, and is assured by the connectedness of G by Ryser's theorem. Assume that the theorem is true for any J of up to k columns, and consider any J with $k+1$ columns. Let us consider any projection data which is J-consistent. Assume that the J-constraint switching graph G_J for these data is not connected. Then there exists a pair of reconstructed figures which cannot be passed from one to the other via switching operations not utilizing the region J. Let us take F, F' among them of which the Hamming distance, or the L_1 norm of the difference of their characteristic functions, is minimal. We observe the first column. If F, F' have all the cells at the same places in this column, we can apply the induction hypothesis and connect F and F' by a series of switching operations without using J and without moving cells of the first column. Thus let P be a cell of F in the first column of which the place is vacant in F'. Then, in order to make the y projection data of the first column identical, F' must have a cell, say P$'$ in this column which is vacant in F. Now observe the rows of P and P$'$. In order to make the x-projection equal for these rows, F must contain a cell Q in the row of P$'$ which is vacant for F'. If the place R of row P and column Q is vacant in F, then the admissible switching of P-Q would diminish the Hamming distance of F with F' at least by 2, contradicting the assumption of minimality of the Hamming distance. Thus R should be occupied by a cell of F. By the same reason, the place R of F' should be vacant.

Thus to compensate, F' must have at least two cells in the row of P which are vacant in F. Then by the same reason as above, the corresponding place in the row of P$'$ for these columns should be occupied in F' and vacant in F to disturb the switching operation diminishing the Hamming distance. This brings two more cells of F' than of F in the row of P′, and requires other two cells in F vacant in F'. This procedure must continue forever to equate the x-projection of these rows. Since the number of cells is finite this cannot happen, This contradiction proves that there is no pair which cannot be connected by switching operations. QED

Note that the above proof does not apply for non-unique J, because for such J, instead of R being a cell of F, it may be a cell of J to prevent the switching of P-Q.

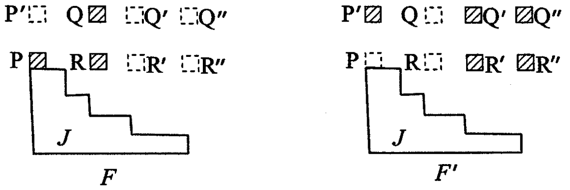


Figure 7: Figure for the proof of Theorem 3.1.

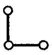
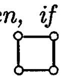
We list up a few more facts about G_J. First the following is obvious.

Proposition 3.2 *Let F be a figure contained in the rectangle I of the integral lattice, with the prohibited region J not necessarily unique. Then its J-complementary figure $\complement_J F := (I \setminus J) \setminus F$ has the same J-constraint switching graph as F.*

In fact, just by the same reasoning as for the full switching graphs, the switching of a pair of cells disjoint with J is equivalent to the switching of the associated holes, which is also disjoint with J.

Proposition 3.3 *Assume that J is unique. Then G_J does not have end-vertex except for the segment graph* o—o.

Fig. 5 shows that this does not necessarily hold for non-unique J. Therefore we present the proof of the above theorem anyway, although it is almost the same as for the full switching graph, This is done by checking the validity of the following:

Lemma 3.4 (triangle-square lemma) *Let J be unique. Then, if a J-constraint switching graph contains a path of the form*  *, then it contains either*  *or*  *completing the original path.*

Proof. If independent pairs P, Q and R, S switch in the two successive edges, then we can obviously obtain a square, as in Fig. 8 A. If the switching occurs among three cells P, Q, and P′, R, where P′ is the new position of P, then we can also obtain a square, as in Fig. 8 B, C, according to whether either of the cells S, T switching with Q, R′, respectively, is occupied by a cell or not. Therefore, in order to disturb all these, S, T must both belong to J. This means that J is non-unique.

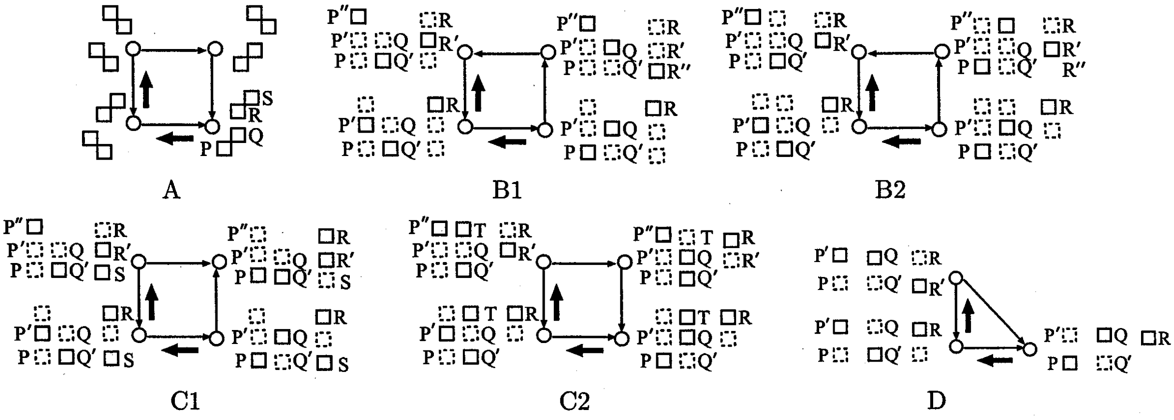


Figure 8: Square-Triangle subgraphs.

The remaining case where the third cell R is in the same column or row as one of the first switching component, say, Q, is just the same as the full switching graph and gives the triangle as shown in Fig. 8 D. QED

From the above lemma follows as well the following:

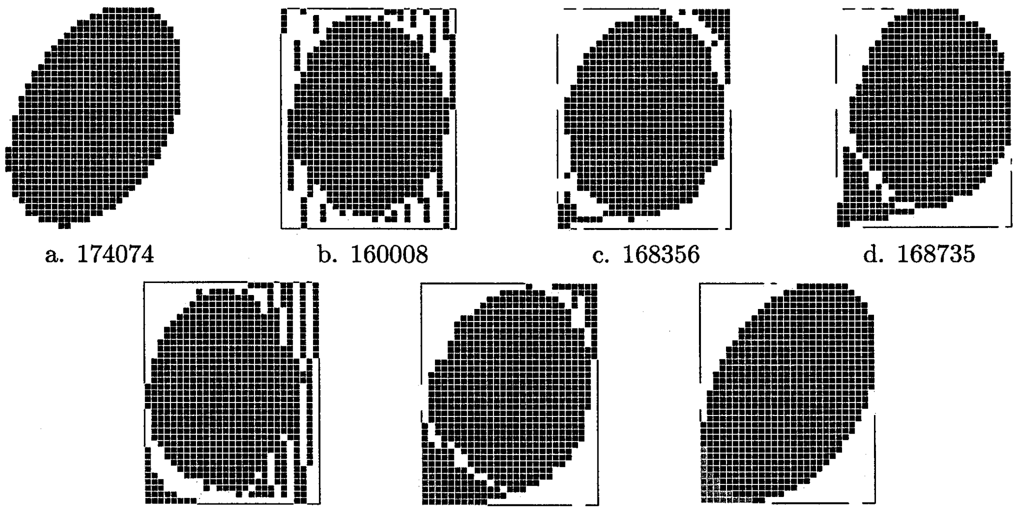
Proposition 3.5 *If J is unique, then there is no primitive cycle of length ≥ 5 in G_J. That is, every cycle has a shortcut of length ≤ 4*

4 Experiments of the reconstruction

We apply our strategy to the discretized slant ellipse, our favorite figure. Figure 9a is the original figure. Figure 9b is the Ryser-Kaori reconstruction without constraint, and Figure 9c the type 2 to 1 modification. The value indicated in each figure denotes that of the standard weight function introduced in [7], which increases by the type modification:

$$w(F) := \sum_{C_{ij} \subset F} ij, \qquad \text{where } C_{ij} = [i, i+1) \times [j, j+1).$$

If we apply the combination of randomized regression and type 2 to 1 modification (or Metropolis algorithm according to [11]), we soon fall in a strong local maximum as in Figure 9d hard to get rid of. On the other hand, if we apply our algorithm with prohibited region J as in Figure 9g, we obtain Figure 9e. The type 2 to 1 modification with J-constraint gives Figure 9f. This time, a number of regression and type 2 to 1 modification with J-constraint easily regains the original figure as shown in Figure 9g.



a. 174074 b. 160008 c. 168356 d. 168735

e. 159944 f. 170293 g. 174074

Figure 9: Reconstruction of slant ellipse.

5 Condition for prohibited region

In the above experiment, we have set a prohibited region from the known original figure. It is, however, desirable that we can set such a region only on the knowledge of the projection data. We therefore examine here a necessary and sufficient condition for J which allows at least one reconstruction for the given projection data. In this section we treat general subsets as prohibited region and do not necessarily require that they are unique figures.

Lemma 5.1 *Let J be unique, and let $f = \{f_y(x), f_x(y)\}$, $f' = \{f'_y(x), f'_x(y)\}$ be two pairs of J-consistent projection data. Assume that*

$$\forall x \;\; f_y(x) \le f'_y(x), \quad \forall y \;\; f_x(y) \le f'_x(y), \quad \|f'_y - f_y\|_{L^1} = 1, \quad \|f'_x - f_x\|_{L^1} = 1,$$

that is, they are different only by one cell. Then there exist reconstruction F from f and F' from f', each with J-constraint, such that the Hamming distance of F, F' is equal to 1, that is, differing only by one cell.

Proof. We shall assume that the columns and rows are so arranged that J has the rearranged form. We proceed by induction on the size n of f. For $n = 0$, the assertion is trivial. Let it be true for any J and for any J-consistent projection data of up to $n-1$ cells, and consider the case of n cells. We apply the modified Ryser-Kaori algorithm for reconstruction with J constraint. Let us take the tallest column from $f_y(x)$ in the range of J_1. By permutation of these columns, we can assume without loss of generality that this column is leftmost and that it is the tallest of $f'_y(x)$ among J_1, too. In fact, if this is not the case, it means that the cell was added to another column of $f_y(x)$ of the same height in the range of J_1. Then we could take this latter column from the beginning. Now process the lowest cell of this column by the modified Ryser-Kaori algorithm and stop. This should produce the same cell P outside J to the reconstruction figure. By the permutation of rows above J_1, we can assume without loss of generality that P is just the next cell above the leftmost column of J_1, hence $J \cup \{P\}$ is still a unique figure. Since this algorithm can be continued to finally produce a legitimate solution with J constraint for each projection data, the remaining projection data should be $J \cup \{P\}$-consistent, and have $n-1$, resp. n cells. Thus by the induction hypothesis, these should have at least one reconstruction F_0, F'_0 each, with $J \cup \{P\}$-constraint and of Hamming distance 1. Then $F = F_0 \cup \{P\}$, $F' = F'_0 \cup \{P\}$ will be a desired pair of reconstructions with J-constraint. QED

Lemma 5.2 *Let $f = \{f_y(x), f_x(y)\}$ be a consistent pair of projection data. Then for a cell P, the projection data augmented by P, $f + \mathrm{proj}(P)$, is consistent if and only if there exists a reconstruction F for f for which the place P is vacant.*

Proof. The sufficiency is obvious. In fact, if f admits a reconstruction F for which the place P is vacant, then, the augmented projection data will be those for a valid figure $F \cup \{P\}$. Let us prove the necessity. Assume that the augmented projection data is consistent. Then there are reconstructions for the augmented data. If there is one F' among them for which the place P is filled, then, $F = F' \setminus \{P\}$ will be a reconstruction for the original data with the place P vacant. Thus assume that the place P is always vacant in any reconstruction F' for the augmented projection data and nevertheless that P is always present in any reconstruction F for the original data. Then the Hamming distance of F and F' gains one at P, and at least 2 outside P, because $|F \setminus \{P\}| = n - 1$ and $|F' \setminus \{P\}| = |F'| = n + 1$. Thus it is ≥ 3, contradicting Lemma 5.1 (applied with $J = \emptyset$). QED

Employing Lemma 5.2 repeatedly, we can add a cell at any place as well as the corresponding augmentation of the projection data keeps the consistency of Lorentz-Ryser. Thus we can finally reach a unique figure. But this does not imply that we can adopt the place of thus added cells as the constraint set. (See Remark 5.4 below.) We have, however, the following criterion for prohibited region which is verifiable only from the projection data:

Theorem 5.3 *Let $f = \{f_y(x), f_x(y)\}$ be a consistent projection data. A region J (not necessarily unique) can be set as a prohibited region to the data f if the augmented projection data $f + \mathrm{proj}(J)$ satisfies the consistency condition of Lorentz-Ryser, and if J is incrementally constructed cell by cell in such a way that each column is filled until it is maximal satisfying the consistency. Any (not necessarily unique) subset of a set constructed in this way is again an admissible prohibited region. Especially, we can construct a subregion J such that $I \setminus J$ is a reconstruction from f.*

Thus a subregion of such J consisting of rectangles in the rearranged form, or more generally a unique subset, can serve as a prohibited region for our discussion hitherto.

Proof. Note that if we verify the existence of a reconstruction with the constraint each time we add a cell to J, then the assertion is trivial and the assumption of the maximality of the already processed columns is not needed. But here we are judging only by the consistency of the augmented projection data $f + \mathrm{proj}(J)$.

We only have to discuss the case when J consists of only one column. In fact, when we pass to the next column, the columns hitherto processed are filled in the enveloping rectangle by the original cells and the augmented cells, and they are no more concerned in further switching (although the form of the remaining x-projection depends on where we have set the augmented cells).

Thus we observe what happens by adding a cell one by one to a column, which we can assume to be the leftmost one. Since the order of adding cells is inessential, we can assume without loss of generality, that we add cells starting from the lowest one and proceed upward, verifying the consistency of the augmented projection data. In view of Lemma 5.2, the first cell, which we can assume to be $[0,1)\times[0,1]$ modulo a translation of the origin, can be added keeping the consistency of the projection data if and only if f admits a reconstruction with this cell vacant. Hence in that case this cell certainly constitute a part of J. Otherwise, this cell appears in all the reconstruction figure, and we pass to the next cell. Assume that we have successfully filled up to $I_k := \bigcup_{j=0}^{k-1} \mathrm{C}_j$, where $\mathrm{C}_j = [0,1)\times[j,j+1)$, having a reconstruction from $f+\mathrm{proj}(J_k)$ with these cells filled. Here $J_k \subset I_k$ denotes (a little differently from the former usage) the subset of by now added cells. The next cell $\mathrm{C} := \mathrm{C}_k$, again in view of Lemma 5.2, can be added keeping the consistency of the projection data if and only if there exists a reconstruction $\widetilde{F}$ from $f+\mathrm{proj}(J_k)$ with this cell vacant. If this is not the case, we go upward, simply setting $J_{k+1} = J_k$. But even if this is, the figure $\widetilde{F}$ thus found may have also vacant place among J_k. We claim that nevertheless there is a reconstruction $\widetilde{F}$ with J_k filled and with C vacant. This is not obvious, but can be proved by an elementary argument as follows: Assume, on the contrary, that there is no such $\widetilde{F}$. Then any reconstruction $\widetilde{F}$ from $f+\mathrm{proj}(J_k)$ with the cell C vacant should also have vacant cells in J_k. We choose $\widetilde{F}$ among them with minimum number of vacant cells in J_k. On the other hand, we have a reconstruction $\widetilde{F_0}$ with J_k filled, hence, necessarily I_k and C filled, too. By Ryser's theorem $\widetilde{F}$ can be connected with $\widetilde{F_0}$ by a path Γ in $G(f+\mathrm{proj}(J_k))$. Let us choose a shortest Γ including the choice of $\widetilde{F}$, $\widetilde{F_0}$ with the above mentioned constraints. In Figure 10 the cells of $\widetilde{F}$ vacant in J_k are drawn to be C_j, $l \le j \le k-1$ to make the explanation easier, but this is not indispensable. Now the first edge of Γ will fill C by a switching, say P-Q to C-Q′, since if not, we can omit the first vertex and shorten Γ by replacing $\widetilde{F}$ with the figure for the second vertex. By the same reason, we can assume that C is hereafter occupied until the end-vertex of Γ and moreover, there is no possibility of switching C with another cell from the third vertex on. In fact, if it can, we can take one step back to find another $\widetilde{F}$ and shorten Γ. Also, a switching independent of the cells touched before cannot appear as an edge of Γ, because it might be done before the first switching P-Q, thus shortening Γ.

Hence the next edge is either (i) a switching of the form Q′-R, or (ii) a switching using the vacant place Q. We shall first discuss the case (i) in detail. The vacant cell, e.g. the top one C_{k-1} of J_k in Fig. 10, should eventually be filled by a switching. Therefore we must have a cell, say S, in the k-th row and a cell, say T, in the first column from the beginning, to realize the switching at some stage. But for any such S, T, the places U, V, W of its column at the row of C, P, T, should be occupied, because if not, C_{k-1} would have been filled with vacant C, producing another $\widetilde{F}$ with less vacant cells in J_k, contradicting its minimality. Thus R should not lie in the columns of S's. On the other hand, it should lie in a row of T's

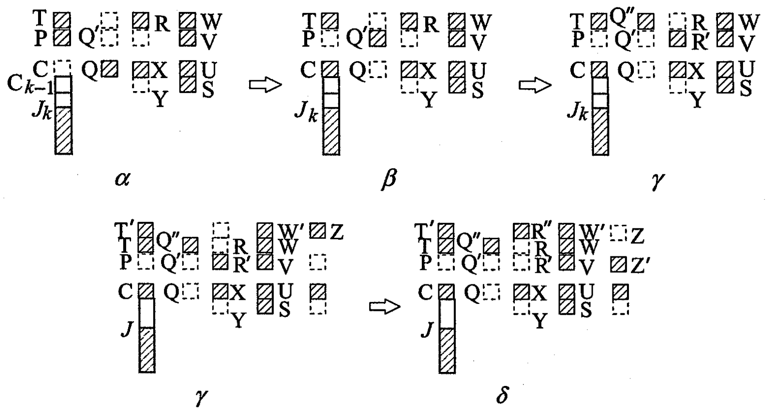


Figure 10: A figure explaining the proof of Theorem 5.3, case (i).

or of the filled cells of J_k. In fact, if the cell at the first column in the row of R is vacant, then we could first switch P-R, and then the new P (i.e. T) with Q, shortening Γ. Similarly, the place X in the row of C and of the column of R should be occupied, since otherwise the switching Q-R, P-X would shorten Γ. Also, the rows corresponding to the cells which are vacant in J_k has also void cells in the column of R, as the one denoted by Y in Figure 10, because if not, the switching of P-Y would violate the minimality of the vacant cells of J_k.

Now consider the next edge γ to δ and on. By the same reason as above, a new cell, say Z, playing the role of R in further switching always produces a cell in the first column T′, because if this latter is vacant, the switching back T-R″, C-Q″ will produce a shorter Γ with another $\widetilde{F}$. Also, the place W′ above S in this row should be occupied, because if not, we can switch back T′-S, T-R″, C-Q″ to obtain another $\widetilde{F}$ with less vacant cells in J_k, violating the minimality assumption. Incidentally, the cell of this column in the row of C_{k-1} is vacant and the one in that of C is occupied by the already mentioned reason. Since the number of cells in the first column is finite, this cannot continue forever. If e.g. R′ switches with a new cell Z in the row of T, then the switching Q′-Z not touching R is a shorter path, hence impossible.

Continuing this way, we can never touch S's and cannot fill the vacant cells of J_k forever. (S may switch with an indifferent cell or with a cell in J_k, but it can be preprocessed before the switching P-Q, hence it is not admitted.) This is only a typical case of all the possibilities, but the situation is similar for the other cases.

We briefly sketch the case (ii). Assume that e.g. R and Z in Fig. 11switches employing the position of Q. Then the cell T in the first column of the row of R should be occupied in this case, too, since otherwise the switching back C-Z′ will shorten Γ. In this case, the story about the cells S in the row of C_j holds as well. Similar observation applies also for this type of switching in the midway of case (i).

The obtained contradiction shows the existence of a desired reconstruction with both J_k and C filled. QED

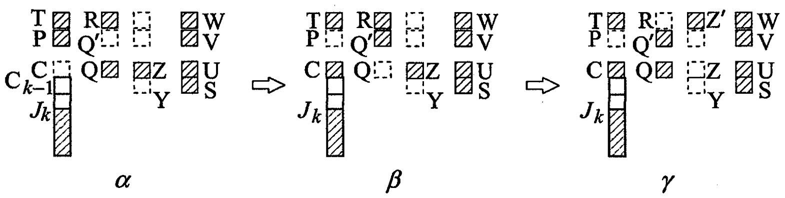


Figure 11: A figure explaning the proof of Theorem 5.3, case (ii).

The order of handling cells may be arbitrary, but to obtain a good figure, a well arranged ordering is preferable. Below we show two examples against the projection data of Figure 9a applied (i) in the order left to right and downward, (ii) from the higher column of $f_y(x)$ upward. The black region shows the maximal J found by the above algorithm and the complement in white is a reconstruction canonical in some sense.

Figure 12: Reconstructions of slant ellipse with maximal prohibited regions;
left: from leftmost column; right: from the highest column.

Remark 5.4 We cannot omit the assumption of column-wise maximality of J in the above theorem. Actually we have the following counter-example. This shows in the same time that the assertion "if f, f' are two pairs of consistent projection data and $f \cup f'$ is also consistent, then there exist reconstructions F, F' of f, f' such that $F \cap F' = \emptyset$" is false even if all projections are unique.

In general, it is difficult to find the relation between switching graphs $G[f]$ and $G[f \cup \mathrm{proj(J)}]$. We now see, however, that they are connected through the common connected full subgraph $G_J[f]$.

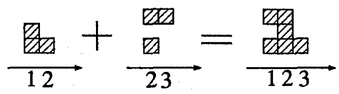


Figure 13: A counter example without the assumption of maximality.

Errata in our former articles: (i) The formulas in page 34, line 12, 14 of [8] lack the integral sign. It should be as in (2) of the present article, and similarly for the uniqueness condition. (ii) The place of J in Fig. 9 of [9] was wrong, The correct J is that of the first one in Fig. 5 of this article.

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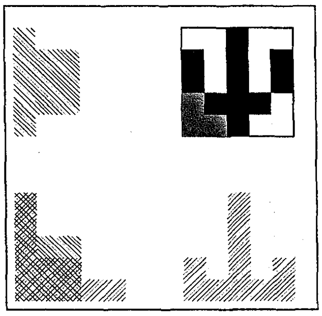
Appendix. Some examples of switching graphs

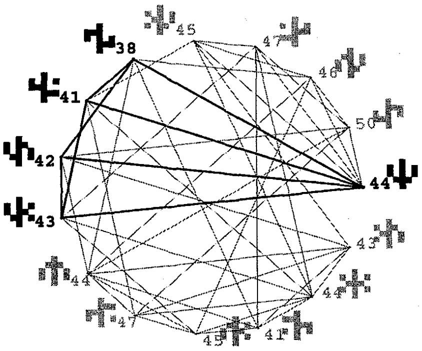
According to the suggestion of the referee, we here add some examples of switching graphs generated by our computer program. The algorithm is the same as for the full switching graph used in [8]: Starting from a reconstructed fugure, we search for all the possible switchings, thereby adding vertices and edges to the graph, Then we pass to the adjacent vertices and repeat the same thing, skipping the already registered vertices and edges. When a constraint J is set, we omit those vertices corresponding to the figures containing cells in J. Alternatively, we can output G_J embedded in the full switching graph G by drawing the prohibited vertices and edges with a faded color and thin lines. This is useful to see how small the G_J is. A rough estimate shows that our algorithm has complexity of $O(n^3k^2)$, where n is the number of cells and k is the size of the switching graph. It is naive and slow for big n. Also, we need a better way of presentation of big graphs as in Figure 15. We have a magnifier to see the partial details, but it serves little to grasp the overall structure. The graph data are output in text files, too. Hence any graph presentation tool is available.



Candle: 11 cells (in black).
J consists of faded 3 cells to the lower-left.

Data for the full switching graph G:
15 vertices, 48 edges, 6 – 8 degrees,
9 weight levels.

Data for the J-constraint switching graph G_J:
5 vertices, 8 edges, 3 – 4 degrees,
5 weight levels,



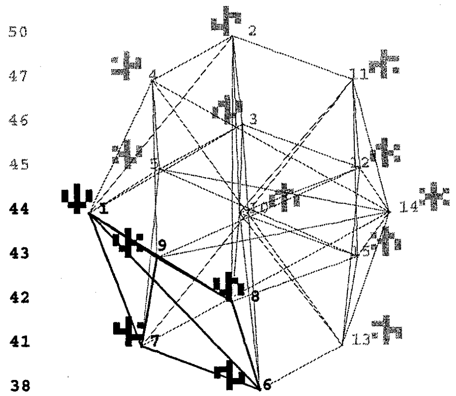
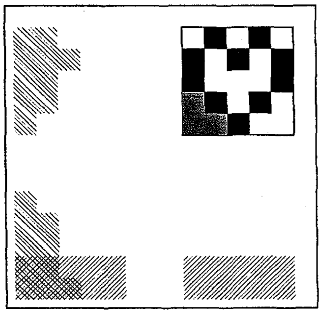


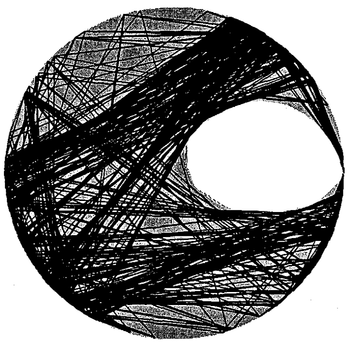
Figure 14 a) Original figure (candle) and prohibited region with the projection data.
b) Auto-generated switching graph. **c)** Auto-arrangement to poset.



Heart: 10 cells.
J consists of faded 3 cells to the lower-left.

Data for the full switching graph G:
1170 vertices, 13410 edges, 22 – 26 degrees,
31 weight levels, ranging in 30 – 62, lacking 31, 61.

Data for the J-constraint switching graph G_J:
387 vertices, 3195 edges, 14 – 20 degrees,
23 weight levels, ranging in 30 – 54, lacking 31, 53.



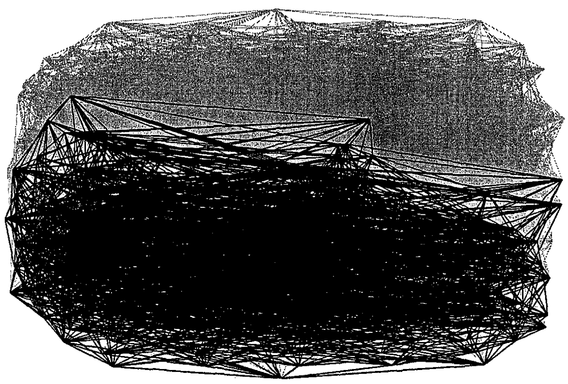


Figure 15 a) Original figure (heart) and prohibited region with the projection data.
b) Auto-generated switching graph. **c)** Auto-arrangement to poset.