# Numerical simulation of vehicular traffic with a junction on a circular road.

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**Abstract**

In the present work the fluid dynamic traffic models have been examined and computed under several initial traffic conditions with or without a junction. The model used in this study is assumed to be a finite and circular road. MacCormack's scheme was applied for the numerical scheme. Different initial conditions are investigated. Furthermore, a source term is introduced into the continuity equation in order to handle an interchange on the road. The congestion has been formed by an initial perturbation or an on-ramp. The results have indicated that once a traffic jam has been formed, the spatial and temporal profiles become to be similar regardless of initial conditions or existence of a junction. The traffic jam moves backward in the process of formation a traffic jam, on the other hand, moves forward in the process of clearing a traffic jam.

## 1. INTRODUCTION

The traffic congestion is becoming very urgent problem due to increased population and automobiles. Over the past decades, traffic models have been developed to optimize traffic flow. There are four main methodologies of modeling dynamic traffic flow: car-following model, kinetic model, Boltzmann-like model and cellular automation.

The kinetic model is the simplest of the methodologies and the advantages of having analyzable equations and low computation cost. Lighthill and Whitham[1] and Richards[2] firstly proposed a macroscopic kinetic traffic flow model(LWR model), which is based on a continuity equation. The model is derived from the conservation of vehicle numbers. Kerner and Konhäuser[3] applied the Navier Stokes equations for a unidirectional n-lane road in 1993. Furthermore, Helbing[4] has introduced an additional differential equation for the velocity variances as the improved model.

This paper is intended to report results of the numerical simulation of vehicular traffic on a circular road with the fluid dynamical model of Helbing and to extend the model so as to handle the effect of junction.

The simulations have been conducted under condition of a small initial perturbation of trigonometric function. In order to take a junction into account, a source term is introduced into the conservation of vehicles' equations.

## 2. FLUID DYNAMIC TRAFFIC MODELS

In the fluid flow analogy, the traffic stream is treated as a one-dimensional compressible fluid. This leads to two following assumptions. The first one is that the traffic flow is conserved, which leads to the continuity equation. The second one is a relation between velocity and density. The simplest continuum model consists of the conservation equation and the equation of the relation between velocity and density.

The conservation equation is as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial (\rho V)}{\partial x}=0 \qquad (1)$$

where $\rho$ is the average spatial density per lane and V is the average velocity.

Since V is an unknown quantity, Lighthill and Whitham[1] suggested to assume that the velocity V can be described as a function of the density $\rho$ as follows:

$$V=V_e(\rho) \qquad (2)$$

In this study a unidirectional single-lane circular road is assumed. The length of the road is set to be 10km. For the numerical solution, MacCormack's method was applied. The boundary conditions are periodic, i.e.:

$$\rho(L,t)=\rho(0,t)$$
$$V(L,t)=V(0,t)$$
$$\Theta(L,t)=\Theta(0,t)$$

### 2.1. Kerner and Konhäuser model

The assumption of eq(2) means that the velocity V is always in equilibrium. Therefore, it does not supply a realistic traffic flow such as ramps or bottlenecks where traffic flow is not in equilibrium.

In order to handle V as an independent dynamic quantity, an additional differential equation was suggested by Kerner and konhäuser[3]. The quantity V is assumed as an application of Navier Stokes equation as follows:

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{1}{\tau}(V_e-V) \qquad (3)$$

Where $\tau$ denotes a relaxation time that corresponds to the average acceleration and deceleration time. In this calculation, the value of $\tau$ is set to be 0.5 minutes.

The equilibrium velocity $V_e(\rho)$ is given by the average desired velocity which represents the phenomenon that the car can run at its maximum speed when there is no other car on the road. However, it reduces its speed with increase in the density.

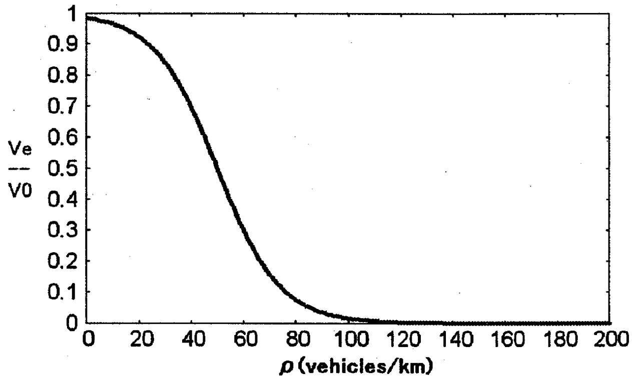


**Fig. 1 Relation between the average velocity V and the density $\rho$ in the stationary and spatially homogeneous case.**

Analogous to gas kinetic models, the pressure of the vehicular gas is as follows:

$$P(x,t) = \rho\Theta - \eta_0 \frac{\partial V}{\partial x} \qquad (4)$$

Where $\Theta$ is the vehicular velocity variance and $\eta_0 (= 600km)$ is a viscosity coefficient. The viscosity term $\eta_0 \frac{\partial V}{\partial x}$ causes sudden velocity changes to be smoothed.

Kerner and Konhäuser[3] simulated Eq.(1),(3) and (4) on the following assumptions:

$$\Theta(L,t) = \Theta_0 (= (45km)^2) \qquad (5)$$

$$V_e(\rho) = V_0 \left[ \frac{1}{1+\exp\left(\frac{\rho/\rho_{max} - 0.25}{0.06}\right)} - 3.72\times 10^{-6} \right] \quad \text{where } V_0 = 120km/h \qquad (6)$$

where $\rho_{max} = 1/l$ is the maximum density(l=5m is the average vehicle length). This relation between the average velocity V and the density $\rho$ in the stationary and spatially homogeneous case is shown in fig.1. In Figure 1, the vertical axis represents $V_e/V_0$. This enables that even for $\rho \approx 0$ a finite velocity V is expected.

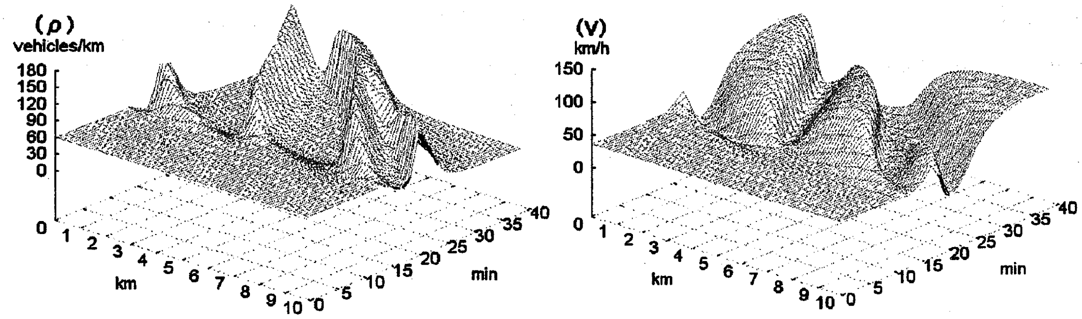


**Fig. 2 Density and velocity profiles according to the model of Kerner and Konhäuser.**

Shown in fig.2 are temporal series of density and velocity with the model of Kerner and Konhäuser[3]. The initial conditions are as follows:

$$\rho(t=0):\rho_e(=60 vehicles/km) \quad (7)$$

$$V(t=0):V_e\{1+0.01\sin(2\pi/L)\} \quad (8)$$

The initial velocity of eq(8) gives a simple perturbation and seems quite natural that some vehicles drive a little faster while others move a little slower than the average. These results show the formation of a cluster on a part of the circular road. The cluster propagates backward and the back of the cluster has a shock structure. It is observed that the velocity decreases as the density increases.

## 2.2. Helbing model

However, the assumption of a constant velocity variance of eq(5) is not realistic. Rather, in equilibrium, the velocity variance $\Theta$ decreases with increase in the traffic density and vanishes if V vanishes.

In order to remove these shortcomings, Helbing[4] introduced an additional partial differential equation for the velocity variance $\Theta$ as follows:

$$\frac{\partial\Theta}{\partial t}+V\frac{\partial\Theta}{\partial r}=-\frac{2P}{\rho}\frac{\partial V}{\partial r}-\frac{1}{\rho}\frac{\partial J}{\partial r}+\frac{2}{\tau}(\Theta_e-\Theta) \qquad (9)$$

Here $\Theta_e(\rho)$ is assumed as follows like eq(6):

$$\Theta_e=\Theta_0\left[\frac{1}{1+\exp\left(\frac{\rho/\rho_{\max}-0.25}{0.06}\right)}-3.72\times10^{-6}\right].$$

Helbing[4] introduced some corrections in order to consider the fact that vehicles are not pointlike objects, but objects that occupy a space of length

$$s(V)=l+V\Delta T \qquad (10)$$

Here $\Delta T(=0.75s)$ is the reaction time to a car forward. When the value of $\Delta T$ is large, the driver's reaction is slow. This causes instabilities of the flow. The value of $V\Delta T$ is a velocity dependent safe distance.

As a consequence of introducing eq(10) into the pressure of the vehicular gas of eq(4) becomes to be

$$P(x,t)=\frac{\rho\Theta}{1-\rho s}-\eta\frac{\partial V}{\partial x} \quad (11)$$

Where

$$\eta=\frac{\eta_0}{1-\rho s} \quad (12).$$

The quantity $J(x,t)=-\kappa\frac{\partial\Theta}{\partial x}$ describes a flux of velocity variance where

$$\kappa(\rho,V)=\frac{\kappa_0}{1-\rho s(V)}, \quad \kappa_0=600km/h.$$

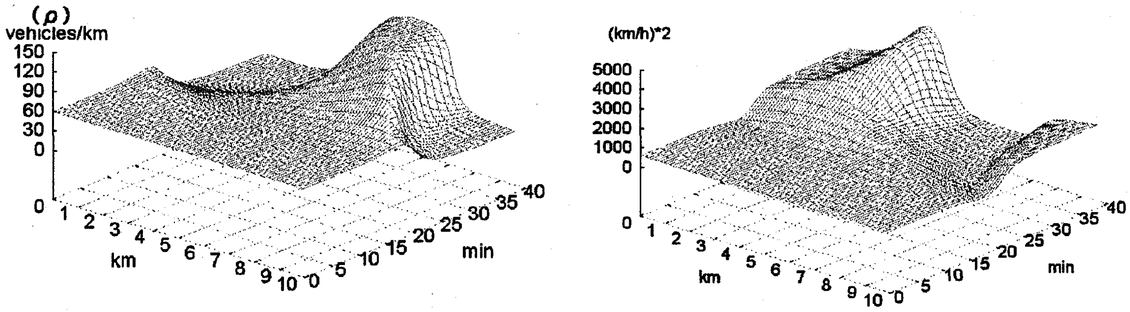


**Fig. 3 Density and velocity profiles according to the model of Helbing.**

Shown in fig.3 are corresponding figures to fig.2 obtained by the Helbing model with the same initial conditions. Althought the shape of propagation is different fom those with the model of Kerner and Konhäuser[3], the cluster has also been observed. The spatial temporal density and velocity changes are smoother than those results using the model of Kerner and Konhäuser[3]. This is due to the introuction of eq.(12) which has smoothing effect for high densities.

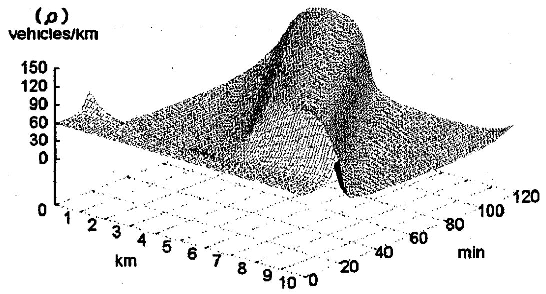


**Fig. 4 Density profile according to the model Helbing for 120minutes.**

The formation of a cluster has also been observed in fig.4. One can observe propagation after a formation of traffic jam. The cluster moves backward with approximately the same speed, that is, they move slowly relative to each other. Because these results coincide with those reported in [3][4] our code is considered to be verified.

Using the code other different initial conditions are investigated because congestion generates with an initial perturbation in periodic system. The first case is where sin in the initial velocity of eq(8) is replaced with cos. The second one is where the amplitude of trigonometric function of nitial velocity is set five times of eq(8). The third case is where the initial perturbation is twice as eq(8). In the last case the amplitude is also doubled in order to identify clusters clearer.

Consequently, the three initial velocity profiles as follows are used:

$$V(t=0): V_e\{1+0.01\cos(2\pi/L)\} \quad (13)$$
$$V(t=0): V_e\{1+0.05\sin(2\pi/L)\} \quad (14)$$
$$V(t=0): V_e\{1+0.02\sin(4\pi/L)\} \quad (15)$$

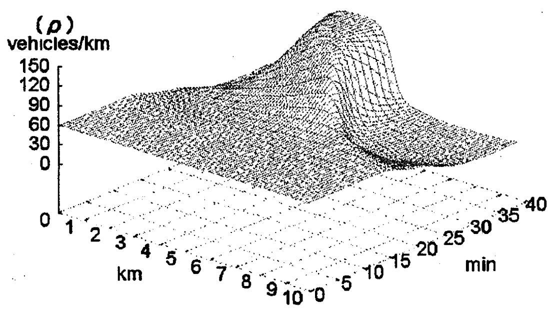


**Fig. 5 Density profile with the initial velocity profile of eq(13)**

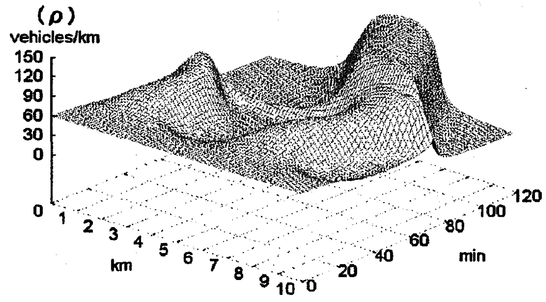


**Fig. 6 Density profile with the initial velocity profile of eq(15).**

Figure 5 is the case with eq(13). In comparison with fig.3 the noticeable difference in position of the cluster is believed to be due to the phase difference of the initial conditions. However, those shapes of profiles are similar to each other. Figure 6 is the case with eq(15). In fig.6 two weak traffic jams appear and move slowly forward before they merge. After they merge the united traffic jam propagates backward in the same way in fig.4.

In order to obtain a quantitative description of how the maximum density $\rho_{\max}$ changes the maximum values of density are plotted in fig.7 with various initial velocity conditions.

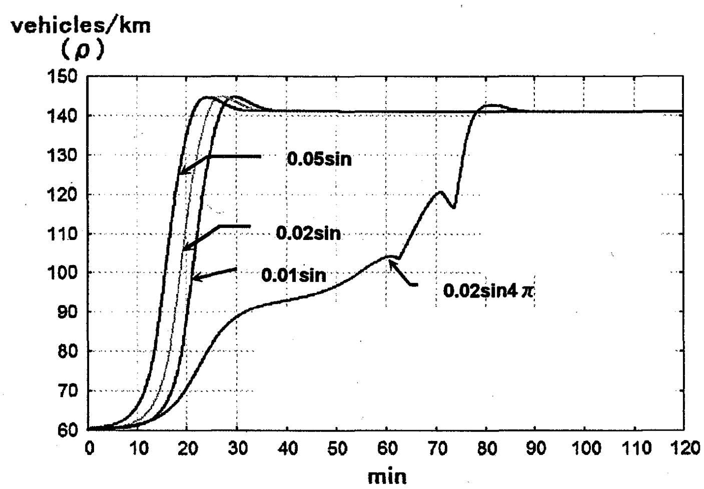


**Fig. 7 Maximum density profiles with various initial conditions for 120minutes.**

It can be seen that traffic jam forms between 20 and 30 minutes in each case except $0.02\sin 4\pi$. From comparison of the amplitude of the initial perturbation, the larger the initial amplitude is, the earlier traffic jam forms. This can be understood in the way that large number of vehicles on the circular road made the traffic flow unstable, which causes a traffic jam easily.

In every case at the point of formation of traffic jam the transition from free flow to congested flow can be found to be seen not simple, rather one can observe an overshoot. With the case of

$0.02 \sin 4\pi$ there are three jam formations with overshoot. Only the last overshoot is followed by stable jam convection.

## 3. INTRODUCTION OF A JUNCTION

Furthermore, in order to introduce a junction, a source/sink term is introduced into eq(1). Physically, the source/sink term could be understood as on-ramps or off-ramps to the highway.

This leads to following relation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial r} = \varphi \qquad (16)$$

In case of off-ramp the branch section has been placed at the point of 2.5km of the circular road. It is placed after 40minutes where a traffic jam has formed fully with initial velocity perturbation. It is possible to vary the amount of outflow by changing the value of source term $\phi$ in eq (16). In this study the value of $\phi$ is varied from -2.0 to -5.0.

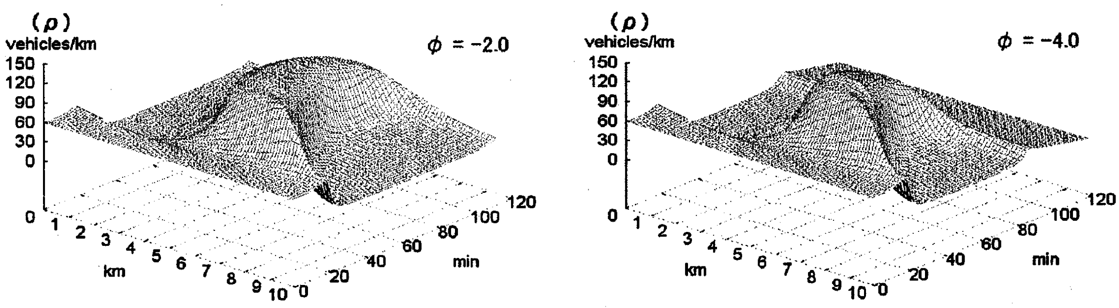


**Fig. 8 Density profile with an off-ramp with $\phi$=-2.0 and -4.0 which is placed after a traffic jam has appeared.**

The density profiles are shown in fig.8($\phi$=-2.0 and -4.0). A remarkable difference between cases with and without an off-ramp is the moving direction of the cluster. That is, after the formation of traffic jam the cluster begins to propagate forward. Moreover, these results indicate that the larger the absolute value of $\phi$ becomes, the shorter time is required to dissolve the congestion.

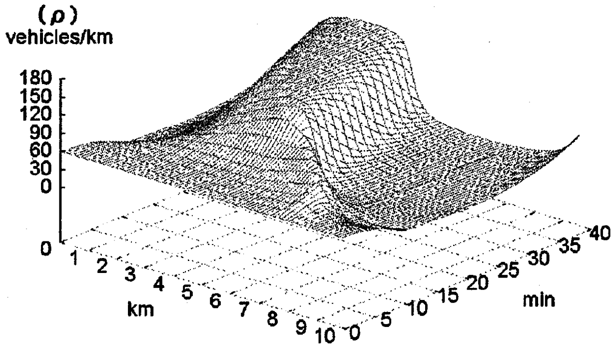


**Fig.9 Density profile with an on-ramp with $\phi$ =2.0 of eq(16).**

Shown in Fig.9 is the density profiles with an on-ramp at the point 7.5km of the circular road. The simulation of merging situation by an on-ramp is conducted with unit initial velocity profile. Compared with the case of off-ramp, the cluster is formed earlier and the shape of the cluster splayed out.

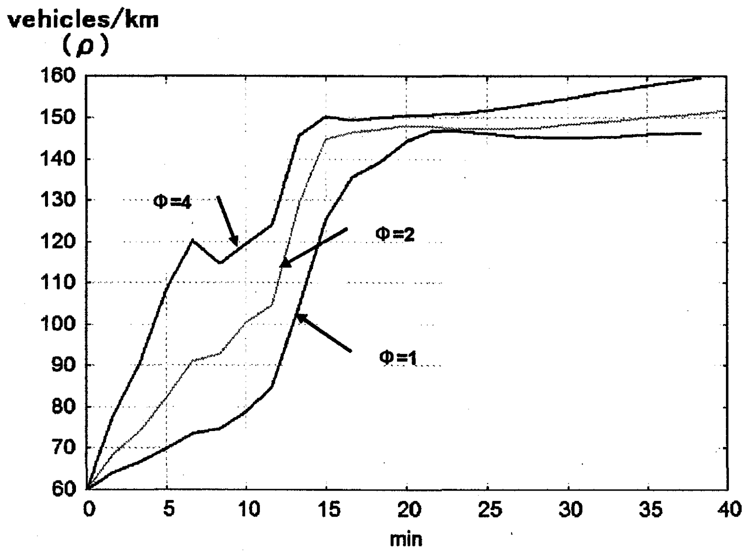


**Fig.10 Maximum density profiles with various $\phi$ of eq(16).**

Shown in fig.10 is the maximum density profiles with various $\phi$ of eq(16) plotted together. The higher the value of $\phi$ becomes, the earlier the formation of cluster begins. In all cases, when the density reaches almost the limit density of the road, the maximum density does not rise. It is also observed in Fig.9 where the cluster splays out after the formation of a traffic jam.

## 4. CONCLUSIONS

In the present work the fluid dynamic traffic models have been examined with MacCormack's scheme and compared with several initial traffic conditions. The road is assumed to be finite, one-lane and circular.

First, the results of Kerner and Konhäuser and those of Helbing were followed with the same parameters of [3][4]. These results coincide with those results depicted in [3][4], which the code is considered to be verified.

Second, in order to examine dependences on initial conditions, three other cases are compared. The first one has a different phase by $\pi/2$, in which the beginning position of formation of the cluster has shifted. The second one has five times amplitude, in which the formation of a cluster starts earlier. The last one has double frequencies, in which two weak traffic jams have appeared and persisted for a long time before they merge. After they merge, the cluster propagated like the other results. From above results, required time to start a formation or a position of cluster is dynamically varying with initial conditions. That is, the larger the amplitude of initial perturbation is, the shorter time is required to be backed up. However, once a traffic jam has been backed up in between 20 and 30 minutes it moves in backward keeping in its formation. The third result indicates that even though some clusters have appeared they will merge and stabilize in one place before long. Because one point influences all other points on the unidirectional single lane circular road.

Third, effect of existence of a junction is examined. In order to handle that interchange on the road, a source term is introduced into the continuity equation. The simulation with an off-ramp has four cases with different value of source term $\phi$ with varying from -5.0 to -2.0. A branch section is introduced after a traffic jam has been formed with initial perturbation of trigonometric function. Those results show that in the process that the number of vehicle on the road diminishes, the position of the cluster moves forward direction. Finally, the traffic jam has disappeared after diminishing continues for some time. The simulation with an on-ramp has three cases with different value of $\phi$ with varying from 1.0 to 4.0. In this case, unit initial velocity profile is used. After the formation of a traffic jam, the cluster splays out because the density of the vehicle reaches limit density at local areas.

Characteristic common to all results in this study is when the density of traffic jam is large the cluster moves against the traffic flow and when it is small the cluster moves same direction with the flow. Furthermore, in both cases with appearance and disappearance a traffic jam an overshoot is observed. Those results indicate that the transition between congested flow and free flow is not monotonically. According to the results with a junction, the introduction of source term representing inflow and outflow is a possible approach to simulate a traffic flow with a junction.

## REFERENCES

[1] M.J. Lighthill and G. B. Whitham, "A theory of traffic flow on long crowded roads.", On kinematic waves. II., Proc. Roy. Soc. London. Ser. A., 229:pp.317-pp.345, 1955.

[2] P. I. Richards, "Shock Waves on the Highway.", Oper. Res., 4:pp.42-pp.51, 1956.

[3] B. S. Kerner and P. Konhauser, "Cluster effect in initially homogeneous traffic flow.", Phys. Rev. E., 48(4):pp.2335-pp.2338, Oct 1993.

[4] Dirk Helbing, "Improved fluid-dynamic model for vehicular traffic.", Phys. Rev. E., 51(4):pp.3164-pp.3169, Apr 1995.

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