Isotropic Kähler immersions into a complex quadric

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Abstract

We give another definition of a complex conformal structure on a complex quadric Q^n and introduce a (local) tensor field J which satisfies $J^2 = Id$(the identity map). A complex subspace W of the tangent space T_pQ^n is called an isotropic complex subspace if JW is orthogonal to W. A Kähler immersion $\varphi : M^m \to Q^n$ of an m-dimensional Kähler manifold M^m is said to be isotropic if for an arbitrary point $p \in M$, $\varphi_*(T_pM)$ is an isotropic complex subspace in $T_{\varphi(p)}Q^n$. We study the properties of higher fundamental forms of isotropic Kähler immersions and show some reduction theorems. Furthermore we construct isotropic Kähler immersions of Kähler C-spaces using orthogonal representations and study the higher normal spaces and the osculating degrees of isotropic Kähler immersions of Hermitian symmetric spaces.

§1 Introduction

Let $\mathbb{C}P^{n+1}$ be a complex projective space with Fubini-Study metric which has constant holomorphic sectional curvature 4 and Q^n be a complex hyperquadric in $\mathbb{C}P^{n+1}$ defined in terms of the homogeneous coordinate system $z_1, z_2, \cdots, z_{n+2}$ by the following equation

$$z_1^2 + z_2^2 + \cdots + z_{n+2}^2 = 0.$$

We equip Q^n with the induced Kähler metric from the Fubini-Study metric . Then Q^n is a Hermitian symmetric space. Let $\{U_\lambda, \Omega_\lambda\}_{\lambda\in\Lambda}$ be a *complex conformal structure* on Q^n . That is, $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open covering of Q^n and Ω_λ is a holomorphic tensor field of type $(0,2)$ defined on U_λ and at each point p of U_λ , Ω_λ is a non-degenerate symmetric bilinear form $T_pQ^{n+} \times T_pQ^{n+} \to \mathbb{C}$. Moreover on $U_\lambda \cap U_\mu$ there exists a non-zero holomorphic function $f_{\mu\lambda}$ which satisfies

$$\Omega_\mu = f_{\mu\lambda}\Omega_\lambda \qquad \text{on } U_\lambda \cap U_\mu.$$

Let $\varphi : M^m \to Q^n$ be a Kähler immersion of an m-dimensional Kähler manifold M^m into Q^n.We call φ an *isotropic Kähler immersion* if at each point $p \in M$ $\varphi^*\Omega_\lambda = 0$, $\varphi(p) \in U_\lambda$.

Our motivation of studying this class of immersions is the following:

1. Interesting observation relating to Gauss mappings by G.Ishikawa,M.Kimura and R.Miyaoka [6]. We identify Q^n with the Grassmann manifold $G_2(\mathbb{R}^{n+2})$ of oriented 2-dimensional subspaces in $\mathbb{R}^{n+2}$. Let $V_2(\mathbb{R}^{n+2})$ be the Stiefel manifold of orthonormal 2-vectors in $\mathbb{R}^{n+2}$.Then the natural projection $\pi : V_2(\mathbb{R}^{n+2}) \to G_2(\mathbb{R}^{n+2})$ is a principal circle bundle . We denote by η the map of $V_2(\mathbb{R}^{n+2})$ into the unit sphere S^{n+1} defined by

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$\eta(e_1, e_2) = e_1$ for an orthonormal 2-vector (e_1, e_2). Let $\varphi : M^m \to Q^n = G_2(\mathbb{R}^{n+2})$ be a Kähler immersion and $\tilde{M}$ be the total space of the pullback bundle $\varphi^* V_2(\mathbb{R}^{n+2})$.Then we obtain a map $\tilde{\varphi} : \tilde{M} \to S^{n+1}$ defined by $\tilde{\varphi} = \eta \circ \iota$,where $\iota : \tilde{M} \to V_2(\mathbb{R}^{n+2})$ is the bundle mapping. If φ is an isotropic Kähler immersion, then $\tilde{\varphi} : \tilde{M} \to S^{n+1}$ is an immersion and has remarkable properties: its Gauss mapping $\tilde{M} \to G_{2m+2}(\mathbb{R}^{n+2})$ is degenerate and $\tilde{\varphi}$ is *austere* ([6] Proposition 8.2 and Theorem 8.3). Here we say an immersion to be *austere* if for each normal vector ξ, the set of eigenvalues of the shape operator A_ξ is invariant under multiplication of -1.

2. The theory of isotropic Kähler immersions is formally analogous to that of totally complex immersions into a quaternionic Kähler manifold (for the theory of totally complex immersions into a quaternionic Kähler manifold, see [18]).

3. A generalization of the theory of holomorphic curves in Q^n by G.R.Jensen,M.Rigoli and K.Yang ([7]) . In particular, it is known that there exists a one-to-one correspondence between the set of generalized minimal immersions $S^2 \to S^{2m}$ and the set of "totally isotropic" holomorphic curves $S^2 \to Q^{2m-1}$.

In section 2, we give another definition of a complex conformal structure on Q^n and introduce a (local) tensor field J which satisfies $J^2 = Id$(the identity map) (Proposition 2.2). Applying these, we describe the second fundamental form of Q^n regarded as a complex hypersurface of $\mathbb{C}P^{n+1}$ (Proposition 2.4) and express the curvature tensor of Q^n (Proposition 2.5). In section 3, we consider isotropic Kähler immersions with low codimension. In section 4, we study the properties of higher fundamental forms of isotropic Kähler immersions and show some reduction theorems (Theorems 4.4,4.5). In section 5, we construct isotropic Kähler immersions of Kähler C-spaces using orthogonal representations (Theorem 5.3) and study the higher normal spaces and the osculating degrees of isotropic Kähler immersions of Hermitian symmetric spaces (Theorem 5.5).

§2 The complex conformal structure on a complex quadric

In this section, we give another definition of a complex conformal structure on a complex quadric Q^n so that it is convenient to study the Kähler geometry of Q^n.

First we recall the Kähler structure on a complex projective space following Besse ([2]) Chapter 3. Let $\mathbb{C}^{n+2}$ be an $(n+2)$-dimensional complex vector space of complex $(n+2)$-tuples. We define a linear endomorphisim I of $\mathbb{C}^{n+2}$ by $I\boldsymbol{x} = \sqrt{-1}\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{C}^{n+2}$ and call it the complex structure . We often regard $\mathbb{C}^{n+2}$ as a $2(n+2)$-dimensional real vector space with the complex strucure I. The space $\mathbb{C}^{n+2}$ is endowed with its Hermitian inner product

$$(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n+2} \overline{x_i} y_i, \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^{n+2}$$

and its real inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \text{the real part of } (\boldsymbol{x}, \boldsymbol{y}).$$

As usual we identify the tangent space $T_{\boldsymbol{x}}\mathbb{C}^{n+2}$ with $\mathbb{C}^{n+2}$ and define the Riemannian metric on $\mathbb{C}^{n+2}$ induced from the real inner product $\langle , \rangle$, for which we use the same notation $\langle , \rangle$.

We denote by $S\mathbb{C}^{n+2}$ the unit sphere in $\mathbb{C}^{n+2}$ defined by the equation $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1$. The complex projective space $\mathbb{C}P^{n+1}$ is defined as the orbit space by the action of the group $U(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. We denote by $\pi(\boldsymbol{x})$ the orbit of $\boldsymbol{x} \in S\mathbb{C}^{n+2}$. Then $\pi : S\mathbb{C}^{n+2} \to \mathbb{C}P^{n+1}$ is a principal fibre bundle with the structure group $U(1)$. The tangent space $T_{\boldsymbol{x}}S\mathbb{C}^{n+2}$ of $S\mathbb{C}^{n+2}$ at a point $\boldsymbol{x}$ may be identified with the real subspace of $\mathbb{C}^{n+2}$ as follows:

$$T_{\boldsymbol{x}}S\mathbb{C}^{n+2} = \{\boldsymbol{u} \in \mathbb{C}^{n+2} \mid \langle \boldsymbol{x}, \boldsymbol{u}\rangle = 0\}.$$

The subspace tangent to the fibre at $\boldsymbol{x}$ in the principal fibre bundle $\pi : S\mathbb{C}^{n+2} \to \mathbb{C}P^{n+1}$ is then identified with $\{\sqrt{-1}\lambda\boldsymbol{x} \mid \lambda \in \mathbb{R}\}$, which will be denoted by $V_{\boldsymbol{x}}S\mathbb{C}^{n+2}$. Put $H_{\boldsymbol{x}}S\mathbb{C}^{n+2} = \{\boldsymbol{u} \in \mathbb{C}^{n+2} \mid (\boldsymbol{x}, \boldsymbol{u}) = 0\}$. Then we have the decomposition orthogonal with respect to $\langle , \rangle$:

$$T_{\boldsymbol{x}}S\mathbb{C}^{n+2} = V_{\boldsymbol{x}}S\mathbb{C}^{n+2} \oplus H_{\boldsymbol{x}}S\mathbb{C}^{n+2}.$$

Moreover the distribution $\{H_{\boldsymbol{x}}S\mathbb{C}^{n+2} \mid \boldsymbol{x} \in S\mathbb{C}^{n+2}\}$ is invariant by the $U(1)$-action and hence defines a connection on the principal fibre bundle $\pi : S\mathbb{C}^{n+2} \to \mathbb{C}P^{n+1}$.

We shall give the description of the projective space $\mathbb{C}P^{n+1}$ as a Riemannian symmetric homogeneous space. We denote by $\widehat{G} = SU(n+2)$ the group of complex linear transformations of $\mathbb{C}^{n+2}$ with determinant 1 which leave the Hermitian inner product $(,)$ invariant. Note that $\widehat{G}$ acts as automorphisms of the principal fibre bundle $S\mathbb{C}^{n+2}(\mathbb{C}P^{n+1}, U(1))$ which preserve the connection and that $\widehat{G}$ acts transitively on $S\mathbb{C}^{n+2}$ and hence transitively on $\mathbb{C}P^{n+1}$. Let $\{e_1, e_2, \cdots, e_{n+2}\}$ be the canonical basis of $\mathbb{C}^{n+2}$ (i.e.,e_i is the vector of $\mathbb{C}^{n+2}$ whose i-th component is one and the other components are zero) and $\widehat{K}$ be the subgroup of $\widehat{G}$ keeping the point $\pi(e_1)$ fixed. Then $\mathbb{C}P^{n+1}$ may be identified with $\widehat{G}/\widehat{K}$ by the diffeomorphism $\phi : \widehat{G}/\widehat{K} \to \mathbb{C}P^{n+1}$ by $\phi(A\widehat{K}) = \pi(Ae_1)$. Every element $A \in \widehat{K}$ has the following form:

$$A = \begin{pmatrix} 1/\det B & 0 \\ 0 & B \end{pmatrix} \qquad \text{with} \quad B \in U(n+1).$$

Hence $\widehat{K}$ is isomorphic to $S(U(1) \times U(n+1))$. It is well-known that $(\widehat{G}, \widehat{K})$ is a Riemannian symmetric pair. The Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{su}(n+2)$ of $\widehat{G} = SU(n+2)$ is the set of $X \in M_{n+2}(\mathbb{C})$ such that ${}^t\overline{X} + X = 0$ and trace $X = 0$, where $M_{n+2}(\mathbb{C})$ denotes the set of all matrices of degree $n+2$ with coefficients in $\mathbb{C}$. The Lie subalgebra $\widehat{\mathfrak{k}}$ corresponding to $\widehat{K} = S(U(1) \times U(n+1))$ is given as follows:

$$\widehat{\mathfrak{k}} = \left\{ \begin{pmatrix} -\mathrm{tr}Y & 0 \\ 0 & Y \end{pmatrix} \mid Y \in \mathfrak{u}(n+1) \right\}.$$

Let $\widehat{\mathfrak{p}}$ be the subspace of $\widehat{\mathfrak{g}}$ defined as follows:

$$\widehat{\mathfrak{p}} = \left\{ \begin{pmatrix} 0 & -{}^t\overline{\boldsymbol{z}} \\ \boldsymbol{z} & 0 \end{pmatrix} \mid \boldsymbol{z} \in \mathbb{C}^{n+1} \right\}.$$

Then we have the canonical decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{k}} + \widehat{\mathfrak{p}}$ of $\widehat{\mathfrak{g}}$ for the symmetric pair $(\widehat{G}, \widehat{K})$. We identify $\widehat{\mathfrak{p}}$ with $\mathbb{C}^{n+1}$ by the real linear isomorphism

$$\mathbb{C}^{n+1} \ni \boldsymbol{x} \mapsto \begin{pmatrix} 0 & -{}^t\overline{\boldsymbol{x}} \\ \boldsymbol{x} & 0 \end{pmatrix} \in \widehat{\mathfrak{p}}.$$

Then with this identification the adjoint representations of $\widehat{K}$ and $\widehat{\mathfrak{k}}$ on $\widehat{\mathfrak{p}}$, denoted by $\mathrm{Ad}_{\widehat{\mathfrak{p}}}$ and $\mathrm{ad}_{\widehat{\mathfrak{p}}}$ respectively, are written as follows:

$$\mathrm{Ad}_{\widehat{\mathfrak{p}}} \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} (\boldsymbol{z}) = \overline{\lambda}(B\boldsymbol{z}) \qquad \text{for} \quad \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} \in \widehat{K}$$

and

$$\mathrm{ad}_{\widehat{\mathfrak{p}}}\begin{pmatrix} \lambda & 0 \\ 0 & Y \end{pmatrix}(z) = Yz - \lambda z \qquad \text{for} \quad \begin{pmatrix} \lambda & 0 \\ 0 & Y \end{pmatrix} \in \widehat{\mathfrak{k}}.$$

The vector space $\widehat{\mathfrak{p}}$ is canonically identified with the tangent space of $\mathbb{C}P^{n+1} = \widehat{G}/\widehat{K}$ at the point $o = \pi(e_1)$. The standard real inner product $\langle , \rangle$ on $\widehat{\mathfrak{p}} \simeq \mathbb{C}^{n+1}$ is $\mathrm{Ad}_{\widehat{\mathfrak{p}}}\widehat{K}$- invariant and so defines a $\widehat{G}$-invariant Riemannian metric on the homogeneous space $\widehat{G}/\widehat{K}$ by which $\mathbb{C}P^{n+1}$ is a Riemannian symmetric space. Moreover the fibration $\pi : S\mathbb{C}^{n+2} \to \mathbb{C}P^{n+1}$ is a Riemannian submersion of the sphere $S\mathbb{C}^{n+2}$ with the Riemannian metric induced from the real inner product $\langle , \rangle$ in $\mathbb{C}^{n+2}$ and $H_{\boldsymbol{x}}S\mathbb{C}^{n+2}$ $\boldsymbol{x} \in S\mathbb{C}^{n+2}$ are horizontal subspaces with respect to this submersion. To see this, define the mapping $\widehat{q} : \widehat{G} \to S\mathbb{C}^{n+2}$ by $\widehat{q}(A) = Ae_1$, $A \in \widehat{G}$. We use the same notation $\widehat{q}$ for the differential of $\widehat{q}$ at the identity of $\widehat{G}$, i.e., $\widehat{q}(X) = Xe_1$, $X \in \widehat{\mathfrak{g}}$. Then we have

$$\widehat{q}(\begin{pmatrix} 0 & -{}^t\overline{z} \\ z & 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ z \end{pmatrix} \qquad \text{for} \quad \begin{pmatrix} 0 & -{}^t\overline{z} \\ z & 0 \end{pmatrix} \in \widehat{\mathfrak{p}}.$$

Hence $\widehat{q}$ defines a linear isometry of $\widehat{\mathfrak{p}}$ onto $H_{e_1}S\mathbb{C}^{n+2}$. Since $\pi \circ \widehat{q}$ is the projection of $\widehat{G}$ onto $\mathbb{C}P^{n+1}$, the differential of π at e_1 is then a linear isometry of $H_{e_1}S\mathbb{C}^{n+2}$ onto $T_o\mathbb{C}P^{n+1}$. Since the Riemannian metrics on $S\mathbb{C}^{n+2}$ and $\mathbb{C}P^{n+1}$ and the distribution $\{H_{\boldsymbol{x}}S\mathbb{C}^{n+2} \mid \boldsymbol{x} \in S\mathbb{C}^{n+2}\}$ are all $\widehat{G}$-invariant, we see that $\pi : S\mathbb{C}^{n+2} \to \mathbb{C}P^{n+1}$ is a Riemannian submersion.

We define a complex structure I on $\widehat{\mathfrak{p}}$ by

$$I(\begin{pmatrix} 0 & -{}^t\overline{z} \\ z & 0 \end{pmatrix}) = \begin{pmatrix} 0 & \sqrt{-1}\,{}^t\overline{z} \\ \sqrt{-1}z & 0 \end{pmatrix}.$$

Note that $\widehat{q}(IX) = \sqrt{-1}\widehat{q}(X)$ for $X \in \widehat{\mathfrak{p}}$, i.e., $\widehat{q}$ is a complex linear isomorphism of $\widehat{\mathfrak{p}}$ onto $H_{e_1}S\mathbb{C}^{n+2}$. This complex structure I on $\widehat{\mathfrak{p}}$ is invariant by the adjoint representation of $\widehat{K}$ and so defines a $\widehat{G}$-invariant almost complex structure on $\mathbb{C}P^{n+1}$. This almost complex structure and the Riemannian metric defined above gives a Kähler structure on $\mathbb{C}P^{n+1}$.

Next we describe the Kähler structure and the complex conformal structure on a complex quadric Q^n . We define a complex symmetric bilinear form $\widetilde{\Omega}$ on $\mathbb{C}^{n+2}$ by

$$\widetilde{\Omega}(\boldsymbol{x}, \boldsymbol{y}) = x_1y_2 + x_2y_1 + \sum_{i=3}^{n+2} x_iy_i, \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^{n+2}. \tag{2.1}$$

Let $\widetilde{S}$ be the symmetric matrix of degree $n+2$ corresponding to $\widetilde{\Omega}$:

$$\widetilde{S} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & E_n \end{pmatrix},$$

where E_n denotes the unit matrix of degree n. Then we have $\widetilde{\Omega}(\boldsymbol{x}, \boldsymbol{y}) = {}^t\boldsymbol{x}\widetilde{S}\boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^{n+2}$. We define a real linear endomorphism $\widetilde{J}$ by $\widetilde{J}\boldsymbol{x} = \widetilde{S}\overline{\boldsymbol{x}}$ for $\boldsymbol{x} \in \mathbb{C}^{n+2}$. Then we have

(i) $\widetilde{J}I = -I\widetilde{J}$,

(ii) $\widetilde{J}^2 = id$, where id denotes the identity transformation,

(iii) $(\widetilde{J}\boldsymbol{x}, \boldsymbol{y}) = \widetilde{\Omega}(\boldsymbol{x}, \boldsymbol{y})$, where $(,)$ denotes the Hermitian inner product on $\mathbb{C}^{n+2}$.

Let $\widetilde{G}$ be the closed subgroup of $\widehat{G} = SU(n+2)$ defined by

$$\widetilde{G} = \{ A \in SU(n+2) \mid \widetilde{\Omega}(A\boldsymbol{x}, A\boldsymbol{y}) = \widetilde{\Omega}(\boldsymbol{x}, \boldsymbol{y}) \} \tag{2.2}$$

equivalently by

$$\widetilde{G} = \{ A \in M_{n+2}(\mathbb{C}) \mid {}^t A \widetilde{S} A = \widetilde{S},\ {}^t \overline{A} A = E_{n+2},\ \det A = 1 \}$$

and $\widetilde{\mathfrak{g}}$ be its Lie algebra defined by

$$\widetilde{\mathfrak{g}} = \{ X \in \mathfrak{su}(n+2) \mid \widetilde{\Omega}(X\boldsymbol{x}, \boldsymbol{y}) + \widetilde{\Omega}(\boldsymbol{x}, X\boldsymbol{y}) = 0 \} \tag{2.3}$$

equivalently by

$$\widetilde{\mathfrak{g}} = \{ X \in M_{n+2}(\mathbb{C}) \mid {}^t X \widetilde{S} + \widetilde{S} X = 0,\ {}^t \overline{X} + X = 0 \}.$$

Then every element X of $\widetilde{\mathfrak{g}}$ has the following form:

$$X = \begin{pmatrix} \sqrt{-1}\lambda & 0 & -{}^t\overline{\boldsymbol{x}} \\ 0 & -\sqrt{-1}\lambda & -{}^t\boldsymbol{x} \\ \boldsymbol{x} & \overline{\boldsymbol{x}} & Y \end{pmatrix}, \qquad \text{with } \lambda \in \mathbb{R},\ \boldsymbol{x} \in \mathbb{C}^n, Y \in \mathfrak{so}(n, \mathbb{R}).$$

Here we define a Lie group $SO(n, \mathbb{R})$ and its Lie algebra $\mathfrak{so}(n, \mathbb{R})$ by

$$\begin{aligned} SO(n, \mathbb{R}) &= \{ A \in M_n(\mathbb{R}) \mid {}^t A A = E_n,\ \det A = 1 \} \\ \mathfrak{so}(n, \mathbb{R}) &= \{ X \in M_n(\mathbb{R}) \mid {}^t X + X = 0 \}. \end{aligned}$$

It is easily seen that $\widetilde{G}$ is isomorphic to $SO(n+2, \mathbb{R})$ and that $\widetilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{so}(n+2, \mathbb{R})$.

We consider a submanifold $\widetilde{Q}$ of $S\mathbb{C}^{n+2}$ with real codimension 2 and a complex hypersurface Q^n of $\mathbb{C}P^{n+1}$ defined as follows:

$$\begin{aligned} \widetilde{Q} &= \{ \boldsymbol{z} \in S\mathbb{C}^{n+2} \mid \widetilde{\Omega}(\boldsymbol{z}, \boldsymbol{z}) = 0 \} \\ Q^n &= \{ \pi(\boldsymbol{z}) \in \mathbb{C}P^{n+1} \mid \boldsymbol{z} \in \widetilde{Q} \}. \end{aligned}$$

Then Q^n is given in terms of the homogeneous coordinate system $z_1, z_2, \cdots, z_{n+2}$ by the following equation

$$2z_1 z_2 + z_3^2 + \cdots + z_{n+2}^2 = 0.$$

We call it *a complex quadric*. The submanifold $\widetilde{Q}$ is invariant by the $U(1)$-action. Restricting the fibration $\pi : S\mathbb{C}^{n+2} \to \mathbb{C}P^{n+1}$ to $\widetilde{Q}$, we obtain the principal subbundle $\widetilde{Q}(Q^n, U(1))$. That is, we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{Q} & \longrightarrow & S\mathbb{C}^{n+2} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Q^n & \longrightarrow & CP^{n+1} \end{array}$$

The tangent space $T_{\boldsymbol{x}}\widetilde{Q}$ at a point $\boldsymbol{x} \in \widetilde{Q}$ has the orthogonal decomposition

$$T_{\boldsymbol{x}}\widetilde{Q} = V_{\boldsymbol{x}}\widetilde{Q} \oplus H_{\boldsymbol{x}}\widetilde{Q},$$

where $V_{\boldsymbol{x}}\widetilde{Q}$ and $H_{\boldsymbol{x}}\widetilde{Q}$ are given by

$$V_{\boldsymbol{x}}\widetilde{Q} = \{\sqrt{-1}\lambda \boldsymbol{x} \mid \lambda \in \mathbb{R}\ \} \quad \text{and} \quad H_{\boldsymbol{x}}\widetilde{Q} = \{\boldsymbol{u} \in \mathbb{C}^{n+2} \mid (\boldsymbol{x}, \boldsymbol{u}) = 0,\ \widetilde{\Omega}(\boldsymbol{x}, \boldsymbol{u}) = 0\ \},$$

respectively. The distribution $\{H_{\boldsymbol{x}}\widetilde{Q} \mid \boldsymbol{x} \in \widetilde{Q}\ \}$ is invariant by the $U(1)$-action and hence defines a connection on the principal fibre bundle $\widetilde{Q}(Q^n, U(1))$. The subspace $H_{\boldsymbol{x}}\widetilde{Q}$ is a complex subspace of $T_{\boldsymbol{x}}\mathbb{C}^{n+2}$ and $H_{\boldsymbol{x}}S\mathbb{C}^{n+2}$ at $\boldsymbol{x} \in \widetilde{Q}$ has the orthogonal decomposition

$$H_{\boldsymbol{x}}S\mathbb{C}^{n+2} = H_{\boldsymbol{x}}\widetilde{Q} \oplus \mathbb{C}(\widetilde{J}\boldsymbol{x}).$$

Since $\pi_* : H_{\boldsymbol{x}}S\mathbb{C}^{n+2} \to T_{\pi(\boldsymbol{x})}\mathbb{C}P^{n+1}$ is a linear isometry and $\pi_*(H_{\boldsymbol{x}}\widetilde{Q}) = T_{\pi(\boldsymbol{x})}Q^n$, the vector $\pi_*(\widetilde{J}\boldsymbol{x})$ is a unit vector normal to Q^n at $\pi(\boldsymbol{x})$.

We will describe the complex quadric Q^n as a Riemannian symmetric homogeneous space. The group $\widetilde{G}$ defined by (2.2) acts as automorphisms of the principal fibre bundle $\widetilde{Q}(Q^n, U(1))$ which preserve the connection. Moreover $\widetilde{G}$ acts transitively on $\widetilde{Q}$ and hence transitively on Q^n. Let $\{e_1, e_2, \cdots, e_{n+2}\}$ be the canonical basis of $\mathbb{C}^{n+2}$. Then we have $e_1 \in \widetilde{Q}$. Let $\widetilde{K}$ be the subgroup of $\widetilde{G}$ keeping the point $\pi(e_1)$ fixed. Then Q^n may be identified with the homogeneous space $\widetilde{G}/\widetilde{K}$ by the diffeomorphism $\phi : \widetilde{G}/\widetilde{K} \to Q^n$ which is given by $\phi(A\widetilde{K}) = \pi(Ae_1)$. Every element $A \in \widetilde{K}$ has the following form :

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda} & 0 \\ 0 & 0 & B \end{pmatrix} \qquad \text{with} \quad \lambda \in U(1) \quad \text{and } B \in SO(n, \mathbb{R}).$$

On $\widetilde{G}$ we define an involutive automorphism θ by

$$\theta(A) = sAs^{-1}, \quad \text{where} \quad s = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & E_n \end{pmatrix} \in \widetilde{G}.$$

We denote by $\widetilde{G}^\theta$ the subgroup of $\widetilde{G}$ which consists of the fixed elements by θ,i.e., $\widetilde{G}^\theta = \{A \in \widetilde{G} \mid \theta(A) = A\ \}$. Then we have

$$\widetilde{G}^\theta = \widetilde{K} \cup \{ \begin{pmatrix} 0 & \overline{\lambda} & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & B \end{pmatrix} \mid \lambda \in U(1),\ B \in O(n, \mathbb{R}),\ \det B = -1\ \}.$$

Therefore $(\widetilde{G}, \widetilde{K})$ is a symmetric pair with the involution θ . The Lie subalgebra $\widetilde{\mathfrak{k}}$ corresponding to $\widetilde{K}$ is given as follows:

$$\widetilde{\mathfrak{k}} = \{ \begin{pmatrix} \sqrt{-1}\lambda & 0 & 0 \\ 0 & -\sqrt{-1}\lambda & 0 \\ 0 & 0 & Y \end{pmatrix} \mid \lambda \in \mathbb{R},\ Y \in \mathfrak{so}(n, \mathbb{R})\ \} \cong \mathbb{R} \oplus \mathfrak{so}(n, \mathbb{R}).$$

We define the subspace $\widetilde{\mathfrak{p}}$ of $\widetilde{\mathfrak{g}}$ as follows:

$$\widetilde{\mathfrak{p}} = \{ \begin{pmatrix} 0 & 0 & -{}^t\overline{\boldsymbol{z}} \\ 0 & 0 & -{}^t\boldsymbol{z} \\ \boldsymbol{z} & \overline{\boldsymbol{z}} & 0 \end{pmatrix} \mid \boldsymbol{z} \in \mathbb{C}^n\ \}.$$

Then we have the canonical decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ for the symmetric pair $(\widetilde{G}, \widetilde{K}, \theta)$. We identify $\widetilde{\mathfrak{p}}$ with $\mathbb{C}^n$ by the real linear endomorphism

$$\mathbb{C}^n \ni \boldsymbol{x} \mapsto \begin{pmatrix} 0 & 0 & -{}^t\overline{\boldsymbol{x}} \\ 0 & 0 & -{}^t\boldsymbol{x} \\ \boldsymbol{x} & \overline{\boldsymbol{x}} & 0 \end{pmatrix} \in \widetilde{\mathfrak{p}}. \tag{2.4}$$

Then with this identification the adjoint representations $\mathrm{Ad}_{\widetilde{\mathfrak{p}}}$ and $\mathrm{ad}_{\widetilde{\mathfrak{p}}}$ of $\widetilde{K}$ and $\widetilde{\mathfrak{k}}$ on $\widetilde{\mathfrak{p}}$ are written as follows:

$$\mathrm{Ad}_{\widetilde{\mathfrak{p}}} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda} & 0 \\ 0 & 0 & B \end{pmatrix} (z) = \overline{\lambda}(Bz) \qquad \text{with} \quad \lambda \in U(1) \quad \text{and } B \in SO(n, \mathbb{R})$$

and

$$\mathrm{ad}_{\widetilde{\mathfrak{p}}} \begin{pmatrix} \sqrt{-1}\lambda & 0 & 0 \\ 0 & -\sqrt{-1}\lambda & 0 \\ 0 & 0 & Y \end{pmatrix} (z) = Yz - \sqrt{-1}\lambda z \qquad \text{with} \quad \lambda \in \mathbb{R} \quad \text{and } Y \in \mathfrak{so}(n, \mathbb{R}).$$

The standard real inner product $\langle , \rangle$ on $\widetilde{\mathfrak{p}} \simeq \mathbb{C}^n$ is $\mathrm{Ad}_{\widetilde{\mathfrak{p}}}\widetilde{K}$- invariant and so defines a $\widetilde{G}$-invariant Riemannian metric on the homogeneous space $\widetilde{G}/\widetilde{K} = Q^n$.

We define a complex structure I on $\widetilde{\mathfrak{p}}$ by

$$I(\begin{pmatrix} 0 & 0 & -{}^t\overline{z} \\ 0 & 0 & -{}^tz \\ z & \overline{z} & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & \sqrt{-1}{}^t\overline{z} \\ 0 & 0 & -\sqrt{-1}{}^tz \\ \sqrt{-1}z & -\sqrt{-1}\overline{z} & 0 \end{pmatrix}.$$

This complex structure I on $\widetilde{\mathfrak{p}}$ is invariant by the adjoint representation of $\widetilde{K}$ and so defines a $\widetilde{G}$-invariant almost complex structure on Q^n. This almost complex structure and the Riemannian metric defined above gives a Kähler structure on Q^n. Moreover the inclusion map of Q^n into $\mathbb{C}P^{n+1}$ is a holomorphically isometric imbedding,i.e., a Kähler imbedding. We will prove this. Since the inclusion map is $\widetilde{G}$-equivariant, it is sufficient to see this at the point $o = \pi(e_1)$. Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{k}} + \widehat{\mathfrak{p}}$ be the canonical decomposition of $\widehat{\mathfrak{g}} = \mathfrak{su}(n+2)$ which corresponds to $\mathbb{C}P^{n+1}$ and $j : \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{p}}$ be the projection with respect to this decomposition. Then we have $\widetilde{\mathfrak{k}} = \widetilde{\mathfrak{g}} \cap \widehat{\mathfrak{k}}$. Since

$$j(X) = \begin{pmatrix} 0 & 0 & -{}^t\overline{\boldsymbol{x}} \\ 0 & 0 & 0 \\ \boldsymbol{x} & 0 & 0 \end{pmatrix} \qquad \text{for} \quad X = \begin{pmatrix} 0 & 0 & -{}^t\overline{\boldsymbol{x}} \\ 0 & 0 & -{}^t\boldsymbol{x} \\ \boldsymbol{x} & \overline{\boldsymbol{x}} & 0 \end{pmatrix} \in \widetilde{\mathfrak{p}},$$

we have $\langle j(X), j(Y) \rangle = \langle X, Y \rangle$ and $j(IX)) = I(j(X))$ for $X, Y \in \widetilde{\mathfrak{p}}$. Hence it is seen that the differential of the inclusion map at the $o = \pi(e_1)$ is a complex linear isometry.

Now we will construct a complex conformal structure on a complex quadric Q^n. Under the identification of $\widetilde{\mathfrak{p}}$ with $\mathbb{C}^n$,we define a complex symmetric bilinear form Ω by

$$\Omega(\boldsymbol{x}, \boldsymbol{y}) = {}^t\boldsymbol{x}\boldsymbol{y} = \sum_{i=1}^{n} x_i y_i, \qquad \text{for } \boldsymbol{x}, \boldsymbol{y} \in \widetilde{\mathfrak{p}} \cong \mathbb{C}^n \tag{2.5}$$

and define a real linear endomorphism J of $\widetilde{\mathfrak{p}}$ by

$$J(\boldsymbol{x}) = \overline{\boldsymbol{x}} \qquad \text{for } \boldsymbol{x} \in \widetilde{\mathfrak{p}} \cong \mathbb{C}^n. \tag{2.6}$$

Then we have

(i) $JI = -IJ$,

(ii) $J^2 = id$,

(iii) $(J\boldsymbol{x}, \boldsymbol{y}) = \Omega(\boldsymbol{x}, \boldsymbol{y})$, where $(,)$ denotes the Hermitian inner product on $\widetilde{\mathfrak{p}} \cong \mathbb{C}^n$.

The adjoint representation $\mathrm{Ad}_{\widetilde{\mathfrak{p}}}$ of $\widetilde{K}$ acts on Ω and J as follows:

$$\mathrm{Ad}_{\widetilde{\mathfrak{p}}}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda} & 0 \\ 0 & 0 & B \end{pmatrix}\Omega = \lambda^2\Omega, \qquad \mathrm{Ad}_{\widetilde{\mathfrak{p}}}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \overline{\lambda} & 0 \\ 0 & 0 & B \end{pmatrix} J = \overline{\lambda}^2 J,$$

where $\lambda \in U(1), B \in SO(n, \mathbb{R})$. We define the mapping $\widetilde{q} : \widetilde{G} \to \widetilde{Q}$ by $\widetilde{q}(A) = Ae_1$, $A \in \widetilde{G}$. We use the same notation $\widetilde{q}$ for the differential of $\widetilde{q}$ at the identity of $\widetilde{G}$, i.e., $\widetilde{q}(X) = Xe_1$, $X \in \widetilde{\mathfrak{g}}$. Then we have

$$\widetilde{q}(\begin{pmatrix} 0 & 0 & -{}^t\overline{z} \\ 0 & 0 & -{}^t z \\ z & \overline{z} & 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \qquad \text{for} \quad \begin{pmatrix} 0 & 0 & -{}^t\overline{z} \\ 0 & 0 & -{}^t z \\ z & \overline{z} & 0 \end{pmatrix} \in \widetilde{\mathfrak{p}}.$$

Hence $\widetilde{q}$ defines a complex linear isometry of $\widetilde{\mathfrak{p}}$ onto $H_{e_1}\widetilde{Q}$. Moreover we have

$$\begin{aligned} \widetilde{\Omega}(\widetilde{q}(X), \widetilde{q}(Y)) &= \Omega(X, Y) \qquad X, Y \in \widetilde{\mathfrak{p}} \\ \widetilde{q}(JX) &= \widetilde{J}\widetilde{q}(X) \qquad X \in \widetilde{\mathfrak{p}}. \end{aligned}$$

We denote by $S^2(\widetilde{\mathfrak{p}})$ the complex vector space of all complex symmetric bilinear forms on $\widetilde{\mathfrak{p}}$. Then we have the following.

Lemma 2.1. *There exists a unique complex 1-dimensional subspace L of $S^2(\widetilde{\mathfrak{p}})$ which satisfies the following conditions:*

(a) *There exists an element Ω in L satisfying the following: the real linear endomorphism J of $\widetilde{\mathfrak{p}}$ defined by the equation*

$$\langle JX, Y \rangle = \textit{the real part of } \; \Omega(X, Y) \quad X, Y \in \widetilde{\mathfrak{p}}$$

satisfies $J^2 = id$(the identity transformation).

(b) *The subspace L is $\mathrm{Ad}_{\widetilde{\mathfrak{p}}}\widetilde{K}$-invariant.*

Proof. Let Ω be the complex symmetric bilinear form on $\widetilde{\mathfrak{p}}$ defined by (2.5) and L be the subspace of $S^2(\widetilde{\mathfrak{p}})$ spanned by Ω. Then L satisfies the conditions (a) and (b) in Lemma 2.1. By straightforward arguments, we can prove the uniqueness of such subspace. □

We view the tangent space T_pQ^n at $p \in Q^n$ with the complex structure I as a complex vector space and denote by $S^2(T_pQ^n)$ the complex vector space of complex symmetric bilinear forms on T_pQ^n. Then $S^2(TQ^n) = \cup_{p \in Q^n} S^2(T_pQ^n)$ is a complex vector bundle over Q^n. Let $(\widetilde{G}, \widetilde{K})$ be the symmetric pair which corresponds to the complex quadric Q^n. Then the bundle $S^2(TQ^n)$ is the vector bundle over $Q^n = \widetilde{G}/\widetilde{K}$ with the standard fibre $S^2(\widetilde{\mathfrak{p}})$ associated with the principal fibre bundle $\widetilde{G}(\widetilde{G}/\widetilde{K}, \widetilde{K})$. Let L be the complex 1-dimensional subspace of $S^2(\widetilde{\mathfrak{p}})$ shown in Lemma 2.1. Since L is an $\mathrm{Ad}_{\widetilde{\mathfrak{p}}}\widetilde{K}$-invariant subspace, it induces a complex line subbundle of $S^2(TQ^n)$, which is denoted by the same notation L. The line bundle L is parallel in $S^2(TQ^n)$ with respect to the canonical connection in $\widetilde{G}(\widetilde{G}/\widetilde{K}, \widetilde{K})$

defined by the canonical decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$, which coincides with the Riemannian connection $\tilde{\nabla}$ on $Q^n = \widetilde{G}/\widetilde{K}$. By these, we have the following:

Proposition 2.2. *There exists a unique complex line subbundle L of $S^2(TQ^n)$ which satisfies the following conditions:*

(a) *For each $p \in Q^n$, there is a neighborhood U of p over which there exists a local section Ω of L satisfying the following: the tensor field J on U defined by the equation*

$$\langle JX, Y\rangle = \textit{the real part of } \ \Omega(X,Y) \quad X, Y \in T_qQ^n, \ \ q \in U$$

satisfies $J^2 = id$(the identity map).

(b) *The complex line bundle L is a parallel subbundle of $S^2(TQ^n)$ with respect to the Riemannian connection $\tilde{\nabla}$.*

We call the complex line bundle L a *complex conformal structure* on Q^n. Let Ω and J be local tensor fields on U in Proposition 2.2 (a). Then we have

$$\langle JX, Y\rangle = \langle X, JY\rangle, \qquad JI = -IJ \tag{2.7}$$

By Proposition 2.2 (b), there exists a $\mathbb{R}$-valued 1-form α on U such that

$$\tilde{\nabla}_X J = \alpha(X) IJ. \tag{2.8}$$

We construct the complex line bundle L and local tensor fields Ω and J in Proposition 2.2 more explicitly. Given $A \in \widetilde{G}$, we denote by $\tau(A)$ the diffeomorphism of $Q^n = \widetilde{G}/\widetilde{K}$ defined by $\tau(A)\pi(Be_1) = \pi(ABe_1)$ for $B \in \widetilde{G}$. At the point $p = \pi(Ae_1) \in Q^n$, we define the complex symmetric bilinear form Ω^A on T_pQ^n and a real linear endomorphism J^A of T_pQ^n by

$$\Omega^A(X,Y) = \Omega(\tau(A)_*^{-1}X, \tau(A)_*^{-1}Y) \qquad \text{for } X, Y \in T_pQ^n,$$

and

$$J^A X = \tau(A)_* J \tau(A)_*^{-1} X \qquad \text{for } X \in T_pQ^n,$$

where Ω and J denote the complex symmetric bilinear form on $T_oQ^n \cong \widetilde{\mathfrak{p}}$ and the real linear endomorphism defined by (2.5) and (2.6), respectively. Evidently it follows that $\langle J^A X, Y\rangle =$ the real part of $\Omega^A(X,Y)$ and $(J^A)^2 = id$. We remark that if A and A' of $\widetilde{G}$ satisfy $\pi(Ae_1) = \pi(A'e_1) = p$, there exists a unitary complex number $\nu = a + \sqrt{-1}b$ such that $\Omega^{A'} = \nu\Omega^A$ and $J^{A'} = aJ^A - bIJ^A$ on T_pQ^n. We denote by L_p the complex subspace of $S^2(T_pQ^n)$ spanned by Ω^A and put $L = \cup_{p\in Q^n} L_p$. Then L is a complex line subbundle which satisfies the conditions in Proposition 2.2.

We denote by the same notation $\widetilde{\Omega}$ the tensor field of type $(0,2)$ on $\mathbb{C}^{n+2}$ induced from the complex symmetric bilinear form $\widetilde{\Omega}$ defined by (2.1). We simply denote by $\widetilde{\Omega}$ the tensor field $\iota^*\widetilde{\Omega}$ on $\widetilde{Q}$ induced by the inclusion map $\iota : \widetilde{Q} \to \mathbb{C}^{n+2}$. At each point $\boldsymbol{x} \in \widetilde{Q}$, $\widetilde{\Omega}$ is degenerate and its nullity space is $V_{\boldsymbol{x}}\widetilde{Q}$, that is,

$$V_{\boldsymbol{x}}\widetilde{Q} = \{X \in T_{\boldsymbol{x}}\widetilde{Q} \mid \widetilde{\Omega}(X,Y) = 0 \quad \text{for any } \ Y \in T_{\boldsymbol{x}}\widetilde{Q} \ \}.$$

For an arbitrary point $p \in Q^n$, there is a neighborhood U of p over which there exist a section s of the fibration $\pi : \widetilde{Q} \to Q^n$ and a section A of the fibration $\phi : \widetilde{G} \to Q^n = \widetilde{G}/\widetilde{K}$

satisfying $A(q)e_1 = s(q)$ at each point $q \in U$. Then by straightforward computation, we have the following.

Lemma 2.3. *At each point of U, we have $s^*\widetilde{\Omega} = \Omega^A$.*

We denote by $\widetilde{\sigma}$ the second fundamental form of Q^n regarded as a complex hypersurface of $\mathbb{C}P^{n+1}$. We will express $\widetilde{\sigma}$ in terms of the complex conformal structure on Q^n.

Proposition 2.4. *Let s be a section of the fibration $\pi : \widetilde{Q} \to Q^n$ defined on a neighborhood of a point $p \in Q^n$. Then we have*

$$\widetilde{\sigma}(X,Y) = -(s^*\widetilde{\Omega})(X,Y)\pi_*\widetilde{J}s(p) \quad X,Y \in T_pQ^n. \tag{2.9}$$

Proof. Since the inclusion map $\iota : Q^n \to \mathbb{C}P^{n+1}$ is $\widetilde{G}$-equivariant, it is sufficient to prove (2.9) at the origin $o = \pi(e_1)$. Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{k}} + \widehat{\mathfrak{p}}$ be the canonical decomposition of $\widehat{\mathfrak{g}} = \mathfrak{su}(n+2)$ which corresponds to $\mathbb{C}P^{n+1}$ and $\mu : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{k}}$ and $j : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{p}}$ the projections onto $\widehat{\mathfrak{k}}$ and $\widehat{\mathfrak{p}}$ with respect to this decomposition,respectively. On the other hand, let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ be the canonical decomposition of $\widetilde{\mathfrak{g}}$ which corresponds to Q^n. Then we have

$$\mu(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -{}^t\boldsymbol{x} \\ 0 & \overline{\boldsymbol{x}} & 0 \end{pmatrix}, \ j(X) = \begin{pmatrix} 0 & 0 & -{}^t\overline{\boldsymbol{x}} \\ 0 & 0 & 0 \\ \boldsymbol{x} & 0 & 0 \end{pmatrix} \quad \text{for} \ X = \begin{pmatrix} 0 & 0 & -{}^t\overline{\boldsymbol{x}} \\ 0 & 0 & -{}^t\boldsymbol{x} \\ \boldsymbol{x} & \overline{\boldsymbol{x}} & 0 \end{pmatrix} \in \widetilde{\mathfrak{p}}.$$

The orthogonal complement $j(\widetilde{\mathfrak{p}})^\perp$ in $\widehat{\mathfrak{p}}$ is given by

$$j(\widetilde{\mathfrak{p}})^\perp = \{ \begin{pmatrix} 0 & -\overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid c \in \mathbb{C} \ \}.$$

Now we may regard the second fundamental form $\widetilde{\sigma}$ at the point $o = \pi(e_1)$ as the $j(\widetilde{\mathfrak{p}})^\perp$-valued symmetric bilinear form on $\widetilde{\mathfrak{p}}$. Then we have for $X, Y \in \widetilde{\mathfrak{p}}$

$$\begin{aligned} \widetilde{\sigma}(X,Y) &= ([\mu(X), j(Y)])_{j(\widetilde{\mathfrak{p}})^\perp} \\ &= (\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -{}^t\boldsymbol{x} \\ 0 & \overline{\boldsymbol{x}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -{}^t\overline{\boldsymbol{y}} \\ 0 & 0 & 0 \\ \boldsymbol{y} & 0 & 0 \end{pmatrix} \right])_{j(\widetilde{\mathfrak{p}})^\perp} \\ &= \begin{pmatrix} 0 & {}^t\overline{\boldsymbol{y}}\overline{\boldsymbol{x}} & 0 \\ -{}^t\boldsymbol{x}\boldsymbol{y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

By the map $\widehat{q} : \widehat{\mathfrak{p}} \to H_{e_1}S\mathbb{C}^{n+2}$, we obtain

$$\widehat{q}(\widetilde{\sigma}(X,Y)) = -\widetilde{\Omega}(\widehat{q}(j(X)), \widehat{q}(j(Y)))e_2 = -\widetilde{\Omega}(\widehat{q}(j(X)), \widehat{q}(j(Y)))\widetilde{J}e_1.$$

Therefore we have

$$\widetilde{\sigma}(X,Y) = -\Omega(X,Y)\pi_*\widetilde{J}e_1 \qquad \text{for} \ X,Y \in T_oQ^n.$$

By Lemma 2.3, our assertion holds. □

We will express the curvature tensor $\widetilde{R}$ of the complex quadric Q^n in terms of its complex conformal structure. Let Ω and J be local tensor fields in Proposition 2.2 (a).

Proposition 2.5. *The curvature tensor* $\widetilde{R}$ *of* Q^n *is of the form*

$$\begin{aligned}(2.10)\quad \widetilde{R}(X,Y)Z \;=\; & \langle Y,Z\rangle X-\langle X,Z\rangle Y+\langle IY,Z\rangle IX-\langle IX,Z\rangle IY-2\langle IX,Y\rangle IZ\\ &+\langle JY,Z\rangle JX-\langle JX,Z\rangle JY+\langle IJY,Z\rangle IJX-\langle IJX,Z\rangle IJY,\end{aligned}$$

where X,Y,Z *are tangent vectors .*

Proof. At the point $o=\pi(e_1)$, we identify the tangent space T_oQ^n with $\widetilde{\mathfrak{p}}$. It is well-known that the curvature tensor $\widetilde{R}$ is given as follows:

$$\widetilde{R}(X,Y)Z=-[[X,Y],Z] \qquad \text{for } X,Y,Z\in\widetilde{\mathfrak{p}}.$$

By (2.4), we identify $\widetilde{\mathfrak{p}}$ with $\mathbb{C}^n$. Then for $\boldsymbol{x},\boldsymbol{y}\in\mathbb{C}^n\cong\widetilde{\mathfrak{p}}$, we have

$$[\boldsymbol{x},\boldsymbol{y}]=\begin{pmatrix}-2\sqrt{-1}\lambda & 0 & 0\\ 0 & 2\sqrt{-1}\lambda & 0\\ 0 & 0 & B\end{pmatrix},$$

where $\lambda=$ the imaginary part of $(\boldsymbol{x},\boldsymbol{y})=\langle I\boldsymbol{x},\boldsymbol{y}\rangle$ and $B=-\boldsymbol{x}^t\overline{\boldsymbol{y}}-\overline{\boldsymbol{x}}^t\boldsymbol{y}+\boldsymbol{y}^t\overline{\boldsymbol{x}}+\overline{\boldsymbol{y}}^t\boldsymbol{x}$. From these, it follows that

$$\begin{aligned}\widetilde{R}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z} &= -[[\boldsymbol{x},\boldsymbol{y}],\boldsymbol{z}]\\ &= -(B\boldsymbol{z}+2\sqrt{-1}\lambda\boldsymbol{z})\\ &= (\boldsymbol{y},\boldsymbol{z})\boldsymbol{x}+\Omega(\boldsymbol{y},\boldsymbol{z})\overline{\boldsymbol{x}}-(\boldsymbol{x},\boldsymbol{z})\boldsymbol{y}-\Omega(\boldsymbol{x},\boldsymbol{z})\overline{\boldsymbol{y}}-2\sqrt{-1}\langle I\boldsymbol{x},\boldsymbol{y}\rangle\boldsymbol{z}\\ &= \langle\boldsymbol{y},\boldsymbol{z}\rangle\boldsymbol{x}-\langle\boldsymbol{x},\boldsymbol{z}\rangle\boldsymbol{y}+\langle I\boldsymbol{y},\boldsymbol{z}\rangle I\boldsymbol{x}-\langle I\boldsymbol{x},\boldsymbol{z}\rangle I\boldsymbol{y}-2\langle I\boldsymbol{x},\boldsymbol{y}\rangle I\boldsymbol{z}\\ &\quad+\langle J\boldsymbol{y},\boldsymbol{z}\rangle J\boldsymbol{x}-\langle J\boldsymbol{x},\boldsymbol{z}\rangle J\boldsymbol{y}+\langle IJ\boldsymbol{y},\boldsymbol{z}\rangle IJ\boldsymbol{x}-\langle IJ\boldsymbol{x},\boldsymbol{z}\rangle IJ\boldsymbol{y}.\end{aligned}$$

□

The totally geodesic submanifolds of a complex quadric Q^n were first classified by B.Y.Chen and T.Nagano [4]. Their classification was not complete. S. Klein [8] gave the satisfactory answer to this classification problem. We recall totally geodesic complex submanifolds (i.e., Kähler submanifolds) of Q^n. We explain two examples.

Example 2.6. The m-dimensional complex projective space $\mathbb{C}P^m$ $(2m\leq n)$.

Let $\{e_1,e_2,e_3,\cdots,e_{n+2}\}$ be the canonical basis of $\mathbb{C}^{n+2}$. We put $u_1=e_3+\sqrt{-1}e_4,\cdots$, $u_m=e_{2m+1}+\sqrt{-1}e_{2m+2}$ and denote by V^{m+1} the $m+1$-dimensional complex subspace of $\mathbb{C}^{n+2}$ spanned by $e_1,u_1,\cdots,u_m$. Then we note that $\widetilde{\Omega}(\boldsymbol{x},\boldsymbol{y})=0$ for any $\boldsymbol{x},\boldsymbol{y}\in V^{m+1}$. The Kähler submanifold $\pi(V^{m+1}\cap S\mathbb{C}^{n+2})$ of $\mathbb{C}P^{n+1}$ is totally geodesic in $\mathbb{C}P^{n+1}$ and isometric to $\mathbb{C}P^m$. Since $V^{m+1}\cap S\mathbb{C}^{n+2}=V^{m+1}\cap\widetilde{Q}$, $\pi(V^{m+1}\cap S\mathbb{C}^{n+2})=\mathbb{C}P^m$ is a Kähler submanifold of Q^n. Since $\mathbb{C}P^m$ is totally geodesic in $\mathbb{C}P^{n+1}$, it is also totally geodesic in Q^n.

Example 2.7. The m-dimensional complex quadric Q^m $(m\leq n-1)$.

Let W^{m+2} be the $m+2$-dimensional complex subspace of $\mathbb{C}^{n+2}$ spanned by $e_1,e_2,\cdots$, e_{m+2}. Then the manifold $\pi(W^{m+2}\cap\widetilde{Q})$ is a complex quadric Q^m of $\mathbb{C}P^{m+1}=\pi(W^{m+2}\cap S\mathbb{C}^{n+2})$ and it is a totally geodesic submanifold of Q^n.

Let $\widetilde{M}$ be a Riemannian manifold with the curvature tensor $\widetilde{R}$ and V be a subspace of the tangent space $T_p\widetilde{M}$. Then V is called a *curvature invariant subspace* if $X,Y,Z\in V$ implies $\widetilde{R}(X,Y)Z\in V$. If $\widetilde{M}$ is a Riemannian symmetric space, for a curvature invariant subspace $V \subset T_p\widetilde{M}$ there exists a unique complete totally geodesic submanifold S through p whose tangent space at p is V.

Let Ω and J be local tensor fields defined on a neighborhood of $p \in Q^n$ in Proposition 2.2 (a) and V be a complex subspace (i.e., an I-invariant real subspace) of T_pQ^n.

Defunition 2.8. The subspace V is called an *isotropic complex subspace* if $\Omega(X,Y)=0$ for any $X, Y \in V$ and is called a *J-invariant complex subspace* if $JX \in V$ for $X \in V$.

We remark that the conditions of an isotropic complex subspace and a J-invariant complex subspace do not depend on the choice of local tensor fields Ω and J in Proposition 2.2 and that a complex subspace V is isotropic if and only if JV is orthogonal to V.

Applying the formula (2.10) of the curvature tensor $\widetilde{R}$, we can easily prove the following.

Lemma 2.9. *Let V be a complex subspace of T_pQ^n. The subspace V is curvature invariant if and only if either V is isotropic or J-invariant.*

By the classification of totally geodesic submanifolds of Q^n (Theorem 4.2, Theorem 5.1 and Corollary 5.27 in [8]), we have the following.

Proposition 2.10. *Let V be an m-dimensional isotropic complex subspace of T_pQ^n. Then the complete totally geodesic submanifold which is tangent to V is congruent to $\mathbb{C}P^m$ in Example 2.6 by a holomorphic isometry of Q^n. Let V be an m-dimensional J-invariant complex subspace of T_pQ^n. Then the complete totally geodesic submanifold which is tangent to V is congruent to Q^m in Example 2.7 by a holomorphic isometry of Q^n.*

Remark 2.11. H.Reckziegel ([14]) introduced the concept of *$\mathbb{C}Q$-structures* for the study of complex quadrics and gave quite natural presentations of aspects of the geometry of Q^n. His introduction of this concept is based on the following observation: we regard a complex quadric as a complex hypersurface of $\mathbb{C}P^{n+1}$ and denote by A_η the shape operator of Q^n with respect to the normal vector η . Then $\mathcal{U}(Q^n,p) = \{\ A_\eta|\eta$: unit normal vectors at $p\ \}$ is a " *circle of conjugations*" on the tangent space T_pQ^n, which is called by Reckziegel a *$\mathbb{C}Q$-structure*. In our notation introduced in Proposition 2.2, $\{\ \lambda J | \lambda \in U(1)\ \}$ is a $\mathbb{C}Q$-structure on each tangent space. S.Klein ([8]) developed the theory of Reckziegel and obtained the results of the geometry of complex quadrics. For example, he classified the totally geodesic submanifolds of Q^n and investigated certain congruence families of totally geodesic submanifolds in Q^n.

§3 Isotropic Kähler immersions into Q^n with low codimension

We use the same notations as those in section 2 for the geometry of a complex quadric Q^n . Let Ω and J be local tensor fields defined on a neighborhood of each point of Q^n in Proposition 2.2.

Definition 3.1. Let $\varphi : M^m \to Q^n$ be a Kähler immersion (i.e.,a holomorphically isometric immersion) of an m-dimensional Kähler manifold M^m into a complex quadric Q^n. Then φ is said to be an *isotropic Kähler immersion* if for an arbitrary point $p \in M^m$ $\varphi_*(T_pM)$ is an isotropic complex subspace in $T_{\varphi(p)}Q^n$.

Let $Gr_{2m}(TQ^n)$ be the Grassmann bundle over Q^n of all real $2m$-dimensional subspaces of the tangent spaces of Q^n and $\mathcal{O}_m$ be the subset of $Gr_{2m}(TQ^n)$ of all complex m-dimensional isotropic subspaces. The group $\widetilde{G}$ defined by (2.2) acts holomorphically and isometrically on Q^n and acts on $Gr_{2m}(TQ^n)$ through the differentials of isometries. If

$2m < n$, $\mathcal{O}_m$ is a $\widetilde{G}$-orbit. If $2m = n$, $\mathcal{O}_m$ has two connected components and each component is a $\widetilde{G}$-orbit. So the collection of isotropic Kähler immersions constitutes a geometry in the framework of Grassmann geometries that were introduced by R. Harvey and H.B. Lawson [5] (see also H.Naitoh [9]). When $2m = n$, for any $V \in \mathcal{O}_m$, its orthogonal complement $V^\perp = JV$ is also an isotropic complex subspace and in particular both V and $V^\perp$ are curvature-invariant. This case is contained in Grassmann geometries which H.Naitoh studied in a series of papers [9],[10],[11],[12].

Let $\varphi : M^m \to Q^n$ be an isotropic Kähler immersion. Then we have the following orthogonal decomposition:

$$\varphi^* TQ^n = TM + T^\perp M,$$

where $T^\perp M$ denotes the normal bundle. By the assumption, it follows that $JT_pM \subset T_p^\perp M$. We denote by $\widetilde{\nabla}$ the induced connection of $\varphi^* TQ^n$ from the Riemannian connection of Q^n and by ∇ and $\nabla^\perp$ the Riemannian connection of M and the normal connection on the normal bundle $T^\perp M$,respectively and by σ and A the second fundamental form and the shape operator of the immersion φ ,respectively.

Lemma 3.2. *Let $\varphi : M^m \to Q^n$ be an isotropic Kähler immersion. Then for tangent vectors $X, Y, Z, V \in T_pM$ at any point $p \in M$, we have*

(1) $\langle \sigma(X,Y), JZ \rangle = 0$,

(2) $\langle (\overline{\nabla}_X \sigma)(Y,Z), JV \rangle = -\langle \sigma(Y,Z), J\sigma(X,V) \rangle$,

where $\overline{\nabla}$ denotes the connection in $Hom(\otimes^2 TM, T^\perp M)$ induced from ∇ and $\nabla^\perp$.

Proof. On a neighborhood U of p in M, there exists a $\mathbb{R}$-valued 1-form α on U such that

$$\widetilde{\nabla}_X J = \alpha(X) IJ \qquad \text{for a tangent vector field } X.$$

(1) Let X, Y, and Z be tangent vector fields on U. Differentiating the equation $\langle JY, Z \rangle = 0$ by a vector field X, we have

$$\begin{aligned} 0 &= \langle \widetilde{\nabla}_X (JY), Z \rangle + \langle JY, \widetilde{\nabla}_X Z \rangle \\ &= \langle (\widetilde{\nabla}_X J)(Y), Z \rangle + \langle J\widetilde{\nabla}_X Y, Z \rangle + \langle JY, \sigma(X,Z) \rangle \\ &= \alpha(X) \langle IJY, Z \rangle + \langle \widetilde{\nabla}_X Y, JZ \rangle + \langle JY, \sigma(X,Z) \rangle \\ &= \langle \sigma(X,Y), JZ \rangle + \langle \sigma(X,Z), JY \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \langle \sigma(X,Y), JZ \rangle &= -\langle \sigma(X,Z), JY \rangle = -\langle \sigma(Z,X), JY \rangle \\ &= \langle \sigma(Z,Y), JX \rangle = -\langle \sigma(X,Y), JZ \rangle \end{aligned}$$

and hence we obtain $\langle \sigma(X,Y), JZ \rangle = 0$.

(2) Differentiating the equation $\langle \sigma(Y,Z), JV \rangle = 0$ by a vector field X, we have

$$0 = \langle \widetilde{\nabla}_X \sigma(Y,Z), JV \rangle + \langle \sigma(Y,Z), \widetilde{\nabla}_X (JV) \rangle.$$

Computing the right hand side, we obtain

$$\begin{aligned} \langle \widetilde{\nabla}_X \sigma(Y,Z), JV \rangle &= \langle \nabla_X^\perp \sigma(Y,Z) - A_{\sigma(Y,Z)} X, JV \rangle \\ &= \langle (\overline{\nabla}_X \sigma)(Y,Z) + \sigma(\nabla_X Y, Z) + \sigma(Y, \nabla_X Z), JV \rangle \\ &= \langle (\overline{\nabla}_X \sigma)(Y,Z), JV \rangle \end{aligned}$$

and

$$\begin{aligned}\langle\sigma(Y,Z),\widetilde{\nabla}_X(JV)\rangle &= \langle\sigma(Y,Z),\alpha(X)IJV+J(\widetilde{\nabla}_XV)\rangle\\ &= \langle\sigma(Y,Z),J\{\sigma(X,V)+\nabla_XV\}\rangle\\ &= \langle\sigma(Y,Z),J\sigma(X,V)\rangle.\end{aligned}$$

This implies $\langle(\overline{\nabla}_X\sigma)(Y,Z),JV\rangle+\langle\sigma(Y,Z),J\sigma(X,V)\rangle=0$. □

Lemma 3.2 (1) implies the following.

Corollary 3.3. *Let $\varphi : M^m \to Q^{2m}$ be an isotropic Kähler immersion. Then φ is totally geodesic and M^m is isometric to an open set of $\mathbb{C}P^m$.*

We remark that the result above has been already shown as a part of the remarkable result obtained by H.Naitoh [9].

For an arbitrary point $p \in M$, we define the subspace RN_p by

$$RN_p=\{X\in T_pM \mid \sigma(X,Y)=0 \quad \text{for all } Y\in T_pM\},$$

which is called the *relative nullity space* of an immersion φ at p. In our case, RN_p is a complex subspace of T_pM.

Proposition 3.4. *Let $\varphi : M^m \to Q^{2m+1}$ be an isotropic Kähler immersion. Then we have* $\dim_{\mathbb{C}} RN_p \geq m-1$ *at any point $p\in M^m$.*

Proof. By the equation of Codazzi, we have

$$(\overline{\nabla}_X\sigma)(Y,Z)-(\overline{\nabla}_Y\sigma)(X,Z)=\left\{\widetilde{R}(X,Y)Z\right\}^{\perp}=0,$$

where $\left\{\widetilde{R}(X,Y)Z\right\}^{\perp}$ is the normal component of $\widetilde{R}(X,Y)Z$. By Lemma 3.2(2), it follows that

$$\langle\sigma(Y,Z),J\sigma(X,V)\rangle=\langle\sigma(X,Z),J\sigma(Y,V)\rangle. \tag{3.1}$$

In particular $\langle\sigma(X,Y),J\sigma(Z,V)\rangle$ is symmetric with respect to X,Y,Z,V.

Suppose that $\sigma\neq 0$ at a point $p\in M$. We denote by N_p^2 the subspace of $T_p^{\perp}M$ spanned by $\sigma(X,Y)$,$X,Y\in T_pM$. Then by Lemma 3.2 (1), the subspace N_p^2 is orthogonal to T_pM+JT_pM. This implies that $\dim_{\mathbb{C}} N_p^2=1$ and that N_p^2 is J-invariant. Let ξ be a unit vector of N_p^2 which satisfies $J\xi=\xi$. We put

$$\sigma(X,Y)=\sigma_+(X,Y)\xi+\sigma_-(X,Y)I\xi.$$

Then σ_+ and σ_- are $\mathbb{R}$-valued symmetric bilinear forms of T_pM and satisfy

$$\sigma_+(IX,Y)=-\sigma_-(X,Y),\quad \sigma_-(IX,Y)=\sigma_+(X,Y).$$

By (3.1),

$$\sigma_+(Y,Z)\sigma_+(X,V)-\sigma_-(Y,Z)\sigma_-(X,V)=\sigma_+(X,Z)\sigma_+(Y,V)-\sigma_-(X,Z)\sigma_-(Y,V). \tag{3.2}$$

Let $\{e_1,\cdots,e_m,Ie_1,\cdots,Ie_m\}$ be an orthonormal basis of T_pM which diagonalizes σ_+ ,i.e.,

$$\sigma_+(e_i,e_j)=-\sigma_+(Ie_i,Ie_j)=\lambda_i\delta_{ij}\ (i,j=1,\cdots,m),\quad \sigma_+(e_i,Ie_j)=0\ (i,j=1,\cdots,m).$$

Putting $Y = Z = e_i, X = V = e_j$ $(i \neq j)$ in (3.2), we obtain $\lambda_i\lambda_j = 0$. If $\lambda_1 \neq 0$, then $\lambda_j = 0$ $(j = 2, \cdots, m)$. This implies that $\dim_{\mathbb{C}} RN_p = m - 1$. □

Let $\iota : Q^n \to \mathbb{C}P^{n+1}$ be an inclusion map. We consider the composition $\widetilde{\varphi} = \iota \circ \varphi$ of the inclusion ι and an isotropic Kähler immersion $\varphi : M \to Q^n$. Let $\widetilde{\sigma}$ and σ be the second fundamental forms of $\widetilde{\varphi}$ and φ ,respectively. Then by Proposition 2.4, we have $\widetilde{\sigma} = \sigma$. In particular the relative nullity spaces of $\widetilde{\varphi}$ and φ coincide.

Applying Corollary 5 in K.Abe [1], we obtain the following:

Corollary 3.5. *Let $\varphi : M^m \to Q^{2m+1}$ be an isotropic Kähler immersion of an $m(\geq 2)$-dimensional complete Kähler manifold M^m . Then φ is totally geodesic and M^m is isometric to $\mathbb{C}P^m$.*

In addition to the assumption of Proposition 3.4, suppose that the second fundamental form σ does not vanish on M . At each point p , we put

$$\mathcal{D}_p = \{X \in T_pM \mid \sigma(X, Y) = 0 \quad \text{for all } Y \in T_pM\}.$$

Then $\dim_{\mathbb{C}} \mathcal{D}_p = m - 1$. The distribution $\mathcal{D}$ is completely integrable and each leaf of $\mathcal{D}$ is a totally geodesic Kähler submanifold of M and also totally geodesic in Q^{2m+1} and $\mathbb{C}P^{2m+2}$. It is a so-called ruled submanifold of Q^{2m+1} and $\mathbb{C}P^{2m+2}$.

Example 3.6. We construct isotropic Kähler immersions $M^m \to Q^{2m+1}$ which are ruled.

We denote by $\widetilde{\Omega}$ the complex symmetric bilinear form defined by (2.1) on $\mathbb{C}^{2(m+1)+1}$ and by $\pi : \mathbb{C}^{2(m+1)+1} - \{0\} \to \mathbb{C}P^{2(m+1)}$ a natural projection. Let $\Xi : U \to \mathbb{C}P^{2(m+1)}$ be an m-th order isotropic curve of a domain U of $\mathbb{C}$ (cf. [7]),i.e., if $\Xi = \pi \circ \xi$,where $\xi : U \to \mathbb{C}^{2(m+1)+1}$ is a holomorphic map, ξ satisfies

- $\xi, \xi', \xi'', \cdots, \xi^m, \xi^{m+1}$ are linearly independent in $\mathbb{C}^{2(m+1)+1}$,
- $\widetilde{\Omega}(\xi, \xi) = \widetilde{\Omega}(\xi', \xi') = \widetilde{\Omega}(\xi'', \xi'') = \cdots = \widetilde{\Omega}(\xi^m, \xi^m) = 0$, $\widetilde{\Omega}(\xi^{m+1}, \xi^{m+1}) \neq 0$,

where $\xi^i = \partial^i\xi/\partial z^i$. Then we note that $\widetilde{\Omega}(\xi^i, \xi^j) = 0$ for $i + j \leq 2m + 1$. We define a holomorphic map $\phi : M = U \times \mathbb{C} \times \cdots \times \mathbb{C} \times (\mathbb{C} - \{0\}) \to \mathbb{C}^{2(m+1)+1} - \{0\}$ by

$$\phi(z, t_1, \cdots, t_{m-2}, t_{m-1}) = \xi(z) + t_1\xi'(z) + \cdots + t_{m-2}\xi^{m-2}(z) + t_{m-1}\xi^{m-1}(z).$$

Then we obtain a holomorphic immersion $\Phi = \pi \circ \phi : M \to \mathbb{C}P^{2(m+1)}$. Moreover the image of Φ is contained in Q^{2m+1} and Φ is an isotropic Kähler immersion into Q^{2m+1}.

§4 Higher fundamental forms and reduction theorems

In this section we study properties of higher fundamental forms of an isotropic Kähler immersion.

First we define higher fundamental forms of Kähler immersions inductively. Let M and $\widetilde{M}$ be Kähler manifolds and $\varphi : M \to \widetilde{M}$ be a Kähler immersion of M into $\widetilde{M}$. Then we have the following orthogonal decomposition:

$$\varphi^*T\widetilde{M} = TM + T^{\perp}M.$$

We denote by $\widetilde{\nabla}$ the induced connection of $\varphi^*T\tilde{M}$ from the Riemannian connection of $\widetilde{M}$. For $j \geq 2$, we define the *j-th fundamental forms* σ_j , *j-th normal spaces* N^j ,*j-th osculating spaces* $\mathcal{O}^j$, and the sets of *j-regular points* $\mathcal{R}_j$, inductively. In particular, σ_j is defined as a smooth section of the vector bundle $\mathrm{Hom}(\otimes^j TM, T^{\perp}M)$ which satisfies the following identity:

$$\sigma_j(X_1, \cdots, IX_{j-1}, X_j) = \sigma_j(X_1, \cdots, X_{j-1}, IX_j) = I\sigma_j(X_1, \cdots, X_{j-1}, X_j), \tag{4.1}$$

where I denotes the complex structure on $\varphi^*T\widetilde{M}$. Let σ be the second fundamental form of φ. It is well-known that σ satisfies (4.1). We put $\sigma_2 = \sigma$. At a point $p \in M$, we define N_p^2 by the subspace of $T_p^\perp M$ spanned by $\sigma_2(X,Y)$, $X, Y \in T_pM$. By (4.1), N_p^2 is a complex subspace of $T_p^\perp M$. We put

$$\mathcal{O}_p^2 = T_pM + N_p^2,$$

which is called *the second osculating space.* We define

$$\mathcal{R}_2 = \left\{p \in M \mid \dim_{\mathbb{C}} N_p^2 = \max\{\dim_{\mathbb{C}} N_{p'}^2 | p' \in M\}\right\},$$

which is called the set of 2-*regular points*. Then $\mathcal{R}_2$ is open in M and $\mathcal{O}^2$ is the subbundle of $\varphi^*T\widetilde{M}$ on $\mathcal{R}_2$. At $p \in \mathcal{R}_2$, we define *the third fundamental form* σ_3 by

$$\sigma_3(X_1, X_2, X_3) = \left(\widetilde{\nabla}_{X_1}\sigma_2(X_2, X_3)\right)_{(\mathcal{O}_p^2)^\perp},$$

for $X_1, X_2, X_3 \in T_pM$ arbitrarily extended to the vector fields on $\mathcal{R}_2$, where $(\mathcal{O}_p^2)^\perp$ denotes the orthogonal complement of $\mathcal{O}_p^2$ in $T_{\varphi(p)}\widetilde{M}$ and $(*)_{(\mathcal{O}_p^2)^\perp}$ denotes the $(\mathcal{O}_p^2)^\perp$-component of $*$ with respect to the orthogonal decomposition

$$T_{\varphi(p)}\widetilde{M} = \mathcal{O}_p^2 + (\mathcal{O}_p^2)^\perp.$$

Then σ_3 is a smooth section of $\mathrm{Hom}(\otimes^3TM, T^\perp M)$ defined on $\mathcal{R}_2$ and σ_3 satisfies (4.1). At a point $p \in \mathcal{R}_2$ we define N_p^3 by the subspace of $T^\perp M$ spanned by $\sigma_3(X_1, X_2, X_3)$,$X_1, X_2, X_3 \in T_pM$. It is a complex subspace of $T^\perp M$. We put $\mathcal{O}_p^3 = T_pM + N_p^2 + N_p^3$ and define

$$\mathcal{R}_3 = \left\{p \in \mathcal{R}_2 \mid \dim_{\mathbb{C}} N_p^3 = \max\{\dim_{\mathbb{C}} N_{p'}^3 | p' \in \mathcal{R}_2\}\right\}.$$

We define σ_j ,N^j and $\mathcal{O}^j$ for $j = 4, 5, \cdots$ inductively on the open set $\mathcal{R}_{j-1}$ of M. Then σ_j satisfies (4.1) and N_p^j is a complex subspace of $T_p^\perp M$, $p \in \mathcal{R}_{j-1}$. Clearly there exists a unique integer d such that $\sigma_d \neq 0$ but $\sigma_{d+1} \equiv 0$. The integer d is called *the degree* of φ and denoted by $d(\varphi)$. A Kähler immersion $\varphi : M \to \widetilde{M}$ is said to be *osculating full* if $\dim_{\mathbb{C}} \mathcal{O}_p^{d(\varphi)} = \dim_{\mathbb{C}} \widetilde{M}$ at $p \in \mathcal{R}_{d(\varphi)}$.

Now we assume that the ambient manifold $\widetilde{M}$ is a complex projective space $\mathbb{C}P^n$. Then j-th fundamental forms σ_j are symmetric tensor fields for any j. That is, the following holds:

(4.2)

$$\sigma_j(X_1, \cdots, X_k, \cdots, X_l, \cdots, X_j) = \sigma_j(X_1, \cdots, X_l, \cdots, X_k, \cdots, X_j), \quad \text{for} \quad 1 \leq k < l \leq j.$$

By (4.1) and (4.2), it follows that

$$\text{(4.3)} \qquad \sigma_j(X_1, \cdots, IX_k, \cdots, X_j) = I\sigma_j(X_1, \cdots, X_k, \cdots, X_j), \quad \text{for } k = 1, \cdots, j.$$

The following reduction theorem is known (cf. Theorem 1 in R.Takagi and M.Takeuchi[15]).

Theorem 4.1. *Let $\varphi : M^m \to \mathbb{C}P^n$ be a Kähler immersion of a Kähler manifold M^m with degree $d(\varphi)$. We put $N = \dim_{\mathbb{C}} \mathcal{O}_p^{d(\varphi)}$, $p \in \mathcal{R}_{d(\varphi)}$. Then there exists an N-dimensional totally geodesic Kähler submanifold $\mathbb{C}P^N$ of $\mathbb{C}P^n$ such that $\varphi(M^m)$ is contained in $\mathbb{C}P^N$.*

Let $\varphi : M^m \to Q^n$ be an isotropic Kähler immersion . We also consider the composition $\widetilde{\varphi} = \iota \circ \varphi : M^m \to \mathbb{C}P^{n+1}$ of φ and the inclusion map $\iota : Q^n \to \mathbb{C}P^{n+1}$. Then we have the following decompositions:

$$\begin{aligned} \varphi^* TQ^n &= TM + T^{\perp}M, \\ \widetilde{\varphi}^* T\mathbb{C}P^{n+1} &= TM + \tilde{T}^{\perp}M. \end{aligned}$$

Evidently the normal bundle $T^{\perp}M$ of φ is a subbundle of the normal bundle $\tilde{T}^{\perp}M$ of $\widetilde{\varphi}$. We denote by $\widetilde{\nabla}$ and D the induced connections of φ^*TQ^n and $\widetilde{\varphi}^*T\mathbb{C}P^{n+1}$ from the Riemannian connections of Q^n and $\mathbb{C}P^{n+1}$,respectively. We use the notations σ_j, N^j, $\mathcal{O}^j$, $\mathcal{R}_j$ for the j-th fundamental forms, the j-th normal spaces, et al. of the immersion φ and $\widetilde{\sigma}_j$, $\widetilde{N^j}$, $\widetilde{\mathcal{O}}^j$, $\widetilde{\mathcal{R}}_j$ for those of the immersion $\widetilde{\varphi}$.

Since φ is an isotropic Kähler immersion, by Proposition 2.4 we have

$$D_X Y = \widetilde{\nabla}_X Y \qquad \text{for } X, Y \in \Gamma(TM).$$

Here we denote by $\Gamma(E)$ the space of smooth sections of a vector bundle E. This implies that $\sigma_2(X,Y) = \widetilde{\sigma}_2(X,Y)$ and hence $N_p^2 = \widetilde{N}_p^2$, $\mathcal{O}_p^2 = \widetilde{\mathcal{O}}_p^2$ at any point $p \in M$ and $\mathcal{R}_2 = \widetilde{\mathcal{R}}_2$. By Lemma 3.2 (1), N_p^2 is ortogonal to JT_pM at any point. Generalizing these, we consider the condition $C(j)$ for a positive integer j.

Definition 4.2. Let $\varphi : M^m \to Q^n$ be an isotropic Kähler immersion. We say that φ satisfies the condition $C(j)$ if the followings hold:
(1) For any point $p \in \mathcal{R}_j$, $\mathcal{O}_p^j = T_pM + N_p^2 + \cdots N_p^j$ is an isotropic complex subspace of $T_{\varphi(p)}Q^n$.
(2) For any integer k, $1 \leq k \leq j$, $\mathcal{R}_k = \widetilde{\mathcal{R}}_k$ and

$$\sigma_{k+1}(X_1, \cdots, X_{k+1}) = \widetilde{\sigma}_{k+1}(X_1, \cdots, X_{k+1}) \qquad \text{for } X_1, \cdots, X_{k+1} \in T_pM, p \in \mathcal{R}_k = \widetilde{\mathcal{R}}_k.$$

(3) At any point $p \in \mathcal{R}_j$, N_p^{j+1} is orthogonal to $J\mathcal{O}_p^j$ and equivalently $\Omega(N_p^{j+1}, \mathcal{O}_p^j) = \{0\}$.

In the above we put $\mathcal{R}_1 = \widetilde{\mathcal{R}}_1 = M$. It has been already shown that an isotropic Kähler immersion φ satisfies the condition $C(1)$. Suppose that an isotropic Kähler immersion satisfies the condition $C(j)$. Then by Definition 4.2 (2), we have $N_p^{j+1} = \widetilde{N}_p^{j+1}$, $\mathcal{O}_p^{j+1} = \widetilde{\mathcal{O}}_p^{j+1}$ at any point $p \in \mathcal{R}_j = \widetilde{\mathcal{R}}_j$ and $\mathcal{R}_{j+1} = \widetilde{\mathcal{R}}_{j+1}$. By Definition 4.2 (3), $\Omega(N_p^{j+1}, \mathcal{O}_p^j) = \{0\}$ and in particular $\Omega(N_p^{j+1}, T_pM) = \{0\}$ at any point $p \in \mathcal{R}_{j+1}$. Therefore for vector fields $X, Y_1, \cdots, Y_{j+1}$ on $\mathcal{R}_{j+1}$, we have

$$\widetilde{\nabla}_X(\sigma_{j+1}(Y_1, \cdots, Y_{j+1})) = D_X(\widetilde{\sigma}_{j+1}(Y_1, \cdots, Y_{j+1}))$$

and hence at any point $p \in \mathcal{R}_{j+1}$

$$\sigma_{j+2}(X, Y_1, \cdots, Y_{j+1}) = \widetilde{\sigma}_{j+2}(X, Y_1, \cdots, Y_{j+1}) \qquad \text{for } X, Y_1, \cdots, Y_{j+1} \in T_pM. \tag{4.4}$$

Since $\widetilde{\sigma}_{j+2}$ is a symmetric tensor field, σ_{j+2} is also a symmetric one and it satisfies
(4.5)
$$\sigma_{j+2}(X_1, \cdots, IX_k, \cdots, X_{j+2}) = I\sigma_{j+2}(X_1, \cdots, X_k, \cdots, X_{j+2}), \quad \text{for } k = 1, \cdots, j+2.$$

Proposition 4.3. *Suppose that an isotropic Kähler immersion $\varphi : M^m \to Q^n$ satisfies the condition $C(j)$ and that N_p^{j+1} is an isotropic complex subspace at any point $p \in \mathcal{R}_{j+1}$. Then φ satisfies the condition $C(j+1)$.*

Proof. We are left to prove that N_p^{j+2} is orthogonal to $J\mathcal{O}_p^{j+1}$ at any point $p \in \mathcal{R}_{j+1}$. We note that $\widetilde{\nabla}_X\xi \in \Gamma(\mathcal{O}^{j+1})$ for any vector field X on $\mathcal{R}_{j+1}$ and a smooth section ξ of $\mathcal{O}^j$. For a smooth section ξ of $\mathcal{O}^j$ on $\mathcal{R}_{j+1}$, we have $\langle \sigma_{j+1}(Y_1, \cdots, Y_{j+1}), J\xi\rangle = 0$. Differentiating the above equation by a vector field X on $\mathcal{R}_{j+1}$,

$$\begin{aligned} 0 &= \langle \widetilde{\nabla}_X(\sigma_{j+1}(Y_1, \cdots, Y_{j+1})), J\xi\rangle + \alpha(X)\langle \sigma_{j+1}(Y_1, \cdots, Y_{j+1}), IJ\xi\rangle \\ &\quad + \langle \sigma_{j+1}(Y_1, \cdots, Y_{j+1}), J\widetilde{\nabla}_X\xi\rangle \\ &= \langle \sigma_{j+2}(X, Y_1, \cdots, Y_{j+1}), J\xi\rangle. \end{aligned}$$

Thus we obtain N_p^{j+2} is orthogonal to $J\mathcal{O}_p^j$.

Differentiating $\langle \sigma_{j+1}(Y_1, \cdots, Y_{j+1}), J\sigma_{j+1}(Z_1, \cdots, Z_{j+1})\rangle = 0$ by a vector field X on $\mathcal{R}_{j+1}$, we have

$$\begin{aligned} 0 &= \langle \widetilde{\nabla}_X(\sigma_{j+1}(Y_1, \cdots, Y_{j+1})), J\sigma_{j+1}(Z_1, \cdots, Z_{j+1})\rangle \\ &\quad + \alpha(X)\langle \sigma_{j+1}(Y_1, \cdots, Y_{j+1}), IJ\sigma_{j+1}(Z_1, \cdots, Z_{j+1})\rangle \\ &\quad + \langle \sigma_{j+1}(Y_1, \cdots, Y_{j+1}), J\widetilde{\nabla}_X(\sigma_{j+1}(Z_1, \cdots, Z_{j+1}))\rangle \\ &= \langle \sigma_{j+2}(X, Y_1, \cdots, Y_{j+1}), J\sigma_{j+1}(Z_1, \cdots, Z_{j+1})\rangle \\ &\quad + \langle \sigma_{j+2}(X, Z_1, \cdots, Z_{j+1}), J\sigma_{j+1}(Y_1, \cdots, Y_{j+1})\rangle. \end{aligned}$$

Noticing that the tensor fields σ_{j+1} and σ_{j+2} are symmetric, by the same arguments as the proof of Lemma 3.2 (1), we have

$$\begin{aligned} &\langle \sigma_{j+2}(X, Y_1, \cdots, Y_{j+1}), J\sigma_{j+1}(Z_1, \cdots, Z_{j+1})\rangle \\ &\quad = -\langle \sigma_{j+2}(X, Z_1, \cdots, Z_{j+1}), J\sigma_{j+1}(Y_1, \cdots, Y_{j+1})\rangle \\ &\quad = \langle \sigma_{j+2}(Z_1, Y_1, \cdots, Y_{j+1}), J\sigma_{j+1}(X, Z_2, \cdots, Z_{j+1})\rangle \\ &\quad = -\langle \sigma_{j+2}(X, Y_1, Z_2, \cdots, Z_{j+1}), J\sigma_{j+1}(Z_1, Y_2, \cdots, Y_{j+1})\rangle \\ &\qquad \cdots\cdots \\ &\quad = -\langle \sigma_{j+2}(X, Y_1, \cdots, Y_{j+1}), J\sigma_{j+1}(Z_1, \cdots, Z_{j+1})\rangle \end{aligned}$$

and hence we obtain $\langle \sigma_{j+2}(X, Y_1, \cdots, Y_{j+1}), J\sigma_{j+1}(Z_1, \cdots, Z_{j+1})\rangle = 0$. This implies that N_p^{j+2} is orthogonal to JN_p^{j+1}. Consequently we have shown that N_p^{j+2} is orthogonal to $J\mathcal{O}_p^{j+1}$ any point $p \in \mathcal{R}_{j+1}$. □

Theorem 4.4. *Suppose that an isotropic Kähler immersion $\varphi : M^m \to Q^n$ satisfies the condition $C(j)$ and that σ_{j+1} vanishes everywhere on $\mathcal{R}_j$. We put $N = \dim_{\mathbb{C}} \mathcal{O}_p^j$, $p \in \mathcal{R}_j$. Then there exists an N-dimensional totally geodesic Kähler submanifold $\mathbb{C}P^N$ of Q^n such that $\varphi(M^m)$ is contained in $\mathbb{C}P^N$.*

Proof. We take a connected component of $\mathcal{R}_j$, which is denoted by $\mathcal{R}_j^\circ$. Since $\mathcal{O}_p^j$ at $p \in \mathcal{R}_j^\circ$ is an isotropic complex subspace, there exists a totally geodesic Kähler submanifold $\mathbb{C}P^N$ which is tangent to $\mathcal{O}_p^j$. By the assumption, $\mathcal{O}^j$ is a parallel subbundle of φ^*TQ^n over $\mathcal{R}_j^\circ$ with respect to $\widetilde{\nabla}$. By the well-known reduction theorem, $\varphi(\mathcal{R}_j^\circ)$ is contained in $\mathbb{C}P^N$. Since M is connected and φ is a holomorphic map, $\varphi(M)$ is contained in $\mathbb{C}P^N$. □

Theorem 4.5. *Suppose that an isotropic Kähler immersion $\varphi : M^m \to Q^n$ satisfies the condition $C(j)$ and that the subspace N_p^{j+1} is J-invariant at any point $p \in \mathcal{R}_{j+1}$. We put $N = 2\dim_{\mathbb{C}} \mathcal{O}_p^j + \dim_{\mathbb{C}} N_p^{j+1}$, at some point $p \in \mathcal{R}_{j+1}$. Then there exists an N-dimensional totally geodesic Kähler submanifold Q^N of Q^n such that $\varphi(M^m)$ is contained in Q^N.*

Proof. Similarly to the proof of Theorem 4.4, we take a connected component of $\mathcal{R}_{j+1}$, which is denoted by $\mathcal{R}^\circ_{j+1}$. At some point $p \in \mathcal{R}^\circ_{j+1}$, we put

$$E_p = \mathcal{O}^j_p + J\mathcal{O}^j_p + N^{j+1}_p.$$

Then E_p is a J-invariant complex subspace of $T_{\varphi(p)}Q^n$ and $\dim_{\mathbb{C}} E_p = N$. There exists a totally geodesic Kähler submanifold Q^N which is tangent to E_p. We define a complex subbundle E of φ^*TQ^n over $\mathcal{R}^\circ_{j+1}$ by

$$E = \bigcup_{q \in \mathcal{R}^\circ_{j+1}} \{\mathcal{O}^j_q + J\mathcal{O}^j_q + N^{j+1}_q\}.$$

We denote by $E^\perp$ the subbundle of φ^*TQ^n which consists of orthogonal complements of E. Then at any point $q \in \mathcal{R}^\circ_{j+1}$ E_q and $E^\perp_q$ are J-invariant complex subspaces in $T_{\varphi(q)}Q^n$. If we can prove that E is a parallel subbundle of φ^*TQ^n over $\mathcal{R}^\circ_{j+1}$ with respect to $\widetilde{\nabla}$, similarly to the proof of Theorem 4.4 we see that $\varphi(M^m)$ is contained in Q^N.

We will prove that E is a parallel subbundle of φ^*TQ^n. For a smooth section $\xi \in \Gamma(\mathcal{O}^j)$ and a vector field $X \in \Gamma(TM)$ on $\mathcal{R}^\circ_{j+1}$, we have $\widetilde{\nabla}_X\xi \in \Gamma(\mathcal{O}^j + N^{j+1})$. Around $q \in \mathcal{R}^\circ_{j+1}$, there exists a $\mathbb{R}$-valued 1-form α such that $\widetilde{\nabla}_X J = \alpha(X)IJ$. Therefore

$$\widetilde{\nabla}_X(J\xi) = \alpha(X)IJ\xi + J(\widetilde{\nabla}_X\xi) = -\alpha(X)JI\xi + J(\widetilde{\nabla}_X\xi)$$

and hence $\widetilde{\nabla}_X(J\xi)$ is a local section $J\mathcal{O}^j + N^{j+1}$. Finally we will prove that for any smooth section $\xi \in \Gamma(N^{j+1})$ over $\mathcal{R}^\circ_{j+1}$, $\widetilde{\nabla}_X\xi \in \Gamma(E)$. For this we define a tensor field τ as a section of $Hom(TM \otimes N^{j+1}, E^\perp)$ over $\mathcal{R}^\circ_{j+1}$ as follows:

$$\tau(X,\xi) = (\widetilde{\nabla}_X\xi)_{E^\perp},$$

for $X \in T_qM$ and $\xi \in N^{j+1}_q$ arbitrarily extended to a smooth section of N^{j+1}. Then the following holds.

Lemma 4.6. *We have the following identities:*
(1) $\tau(X, J\xi) = J\tau(X,\xi)$,
(2) $\tau(IX,\xi) = I\tau(X,\xi)$,
for $X \in T_qM$ *and* $\xi \in N^{j+1}_q$, $q \in \mathcal{R}^\circ_{j+1}$.

Proof of Lemma 4.6. (1) Around q, we have

$$\widetilde{\nabla}_X(J\xi) = \alpha(X)IJ\xi + J(\widetilde{\nabla}_X\xi) = -\alpha(X)JI\xi + J(\widetilde{\nabla}_X\xi).$$

Therefore

$$\tau(X, J\xi) = (\widetilde{\nabla}_X J\xi)_{E^\perp} = J(\widetilde{\nabla}_X\xi)_{E^\perp} = J\tau(X,\xi).$$

(2) We put $\xi = \sigma_{j+1}(Y_1,\cdots,Y_{j+1})$ for vector fields $Y_1,\cdots,Y_{j+1}$ around q. Then we have

$$\begin{aligned}
\tau(IX, \sigma_{j+1}(Y_1,\cdots,Y_{j+1})) &= (\widetilde{\nabla}_{IX}\sigma_{j+1}(Y_1,\cdots,Y_{j+1}))_{E^\perp} \\
&= (\sigma_{j+2}(IX,Y_1,\cdots,Y_{j+1}))_{E^\perp} \\
&= (I\sigma_{j+2}(X,Y_1,\cdots,Y_{j+1}))_{E^\perp} \\
&= I(\sigma_{j+2}(X,Y_1,\cdots,Y_{j+1}))_{E^\perp} \\
&= I(\widetilde{\nabla}_X\sigma_{j+1}(Y_1,\cdots,Y_{j+1}))_{E^\perp} \\
&= I\tau(X,\sigma_{j+1}(Y_1,\cdots,Y_{j+1}))
\end{aligned}$$

□

We continue our proof of Theorem 4.5. By Lemma 4.6, τ vanishes identically. In fact,

$$\tau(IX, J\xi) = I\tau(X, J\xi) = IJ\tau(X, \xi),$$

and

$$\tau(IX, J\xi) = J\tau(IX, \xi) = JI\tau(X, \xi) = -IJ\tau(X, \xi).$$

These imply that $\tau(X, \xi) = 0$. Therefore it has been shown that for any smooth section $\xi \in \Gamma(N^{j+1})$ over $\mathcal{R}^{\circ}_{j+1}$, $\widetilde{\nabla}_X \xi \in \Gamma(E)$. □

§5 Isotropic Kähler immersions of Hermitian symmetric spaces

In this section we construct isotropic Kähler immersions of Kähler C-spaces into a compex quadric using orthogonal representations and study the higher normal spaces and the osculating degrees of isotropic Kähler immersions of Hermitian symmetric spaces.

First we recall the construction of Kähler C-spaces M with $\dim H^2(M, \mathbb{R}) = 1$ and the canonical imbeddings into a complex projective space (cf.[13],[15]). Here by a Kähler C-space we mean a compact simply connected homogeneous Kähler manifold. Let $\bar{\mathfrak{g}}$ be a complex simple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$. Put $l = \dim_{\mathbb{C}} \mathfrak{h}$. Then we have a direct sum decomposition:

$$\bar{\mathfrak{g}} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathbb{C} E_\alpha,$$

where E_α is a root vector of a root α. Let B be the Killing form of $\bar{\mathfrak{g}}$. For $\xi \in \mathfrak{h}^*$, let H_ξ be the vector such that $B(H, H_\xi) = \xi(H)$ for all $H \in \mathfrak{h}$. Put $\mathfrak{h}_0 = \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha$. Then $\dim_{\mathbb{R}} \mathfrak{h}_0 = l$ and the dual space $\mathfrak{h}_0^*$ of $\mathfrak{h}_0$ can be regarded as a real subspace of $\mathfrak{h}^*$. We define a bilinear form $(,)$ on $\mathfrak{h}_0^*$ by $(\xi, \eta) = B(H_\xi, H_\eta)$ for any $\xi, \eta \in \mathfrak{h}_0^*$. Let $\{\alpha_1, \cdots, \alpha_l\}$ be a fundamental root system of Δ . We choose a lexicographic order in $\mathfrak{h}_0^*$ with respect to which $\{\alpha_1, \cdots, \alpha_l\}$ is the set of simple roots and denote by Δ^+ and Δ^- the sets of positive and negative roots respectively. Let $\{\Lambda_1, \cdots, \Lambda_l\}$ be the fundamental weight system associated with $\{\alpha_1, \cdots, \alpha_l\}$,which is defined by

$$2(\Lambda_i, \alpha_j) = (\alpha_j, \alpha_j)\delta_{ij} \qquad (i, j = 1, \cdots, l).$$

We can choose root vectors E_α $(\alpha \in \Delta)$ in the following way;

$$\begin{aligned} B(E_\alpha, E_{-\alpha}) &= -1 \\ [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, &\qquad N_{\alpha,\beta} = N_{-\alpha,-\beta} \in \mathbb{R}, \quad \alpha, \beta \in \Delta. \end{aligned}$$

Then $\sqrt{-1}\mathfrak{h}_0 + \sum_{\alpha \in \Delta^-} (\mathbb{R} A_\alpha + \mathbb{R} B_\alpha)$, denoted by $\mathfrak{g}$, is a compact real form of $\bar{\mathfrak{g}}$, where $(\alpha \in \Delta^-)$.

We choose a simple root α_i $(i = 1, \cdots, l)$ and denote it by γ. We define a subset Δ_γ of Δ^- to be the set of roots

$$\alpha = n_1\alpha_1 + \cdots + n_l\alpha_l \in \Delta^-$$

such that the coefficient of γ in α is strictly negative. Define the subalgebra $\mathfrak{k}$ and the subspace $\mathfrak{p}$ of $\mathfrak{g}$ as follows:

$$\begin{aligned}\mathfrak{k} &= \sqrt{-1}\mathfrak{h}_0 + \sum_{\alpha\in\Delta^- - \Delta_\gamma}(\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)\\ \mathfrak{p} &= \sum_{\alpha\in\Delta_\gamma}(\mathbb{R}A_\alpha + \mathbb{R}B_\alpha).\end{aligned}$$

Let G and K be a simply connected Lie group and its connected (closed) Lie subgroup which correspond to $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Then the compact homogeneous space $M = G/K$ is known to be simply connected, and the complex structure I on $\mathfrak{p}$ defined by $IA_\alpha = B_\alpha, IB_\alpha = -A_\alpha, \alpha \in \Delta_\gamma$ gives rise to a G-invariant complex structure on M. It is also known that M admits a G-invariant Kähler metric associated with the complex structure I . Then M is a Kähler C-space with $\dim H^2(M,\mathbb{R}) = 1$. Conversely every Kähler C-space M with $\dim H^2(M,\mathbb{R}) = 1$ can be obtained in this way from the pair $(\bar{\mathfrak{g}},\gamma)$ of a complex simple Lie algebra $\bar{\mathfrak{g}}$ and a simple root γ. We note that the decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ obtained from the pair $(\bar{\mathfrak{g}},\gamma)$ becomes a canonical decomposition of an orthogonal symmetric Lie algebra of Hermitian type if and only if the coefficient of γ in every $\alpha \in \Delta_\gamma$ is equal to -1.

Next we construct holomorphic imbeddings of a Kähler C-space M obtained from the pair $(\bar{\mathfrak{g}},\gamma)$ into a complex projective space. We put $\Lambda_\gamma = \Lambda_i$ if $\gamma = \alpha_i$. Let ρ be an irreducible complex representation of $\bar{\mathfrak{g}}$ with the highest weight $p\Lambda_\gamma$ for a positive integer p. The representation ρ restricted to $\mathfrak{g}$ defines an irreducible representation of G , which will also be denoted by ρ. Since G is compact, we can choose a complex Hermitian inner product and a unitary frame $\{e_1, e_2, \cdots, e_{n+2}\}$ on the representation space such that e_1 is the highest weight vector and that $\rho(G) \subset SU(n+2)$. Then the representation ρ of $\mathfrak{g}$ induces a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{su}(n+2)$. Let $\mathfrak{su}(n+2) = \widehat{\mathfrak{k}} + \widehat{\mathfrak{p}}$ be the canonical decomposition of $\mathfrak{su}(n+2)$ which corresponds to an $n+1$-dimensional complex projective space $\mathbb{C}P^{n+1}$ (cf §2). Recall that $\widehat{\mathfrak{k}}$ is given by

$$\widehat{\mathfrak{k}} = \{A \in \mathfrak{su}(n+2) \mid Ae_1 = \lambda e_1,\ \lambda \in \sqrt{-1}\mathbb{R}\ \}.$$

Note that $\rho(A_\alpha)e_1 = \rho(E_\alpha + E_{-\alpha})e_1 = \rho(E_\alpha)e_1$, $\rho(B_\alpha)e_1 = \rho(\sqrt{-1}(E_\alpha - E_{-\alpha}))e_1 = \sqrt{-1}(\rho(E_\alpha)e_1)$, for $\alpha \in \Delta^-$ and $\rho(\sqrt{-1}H)e_1 = \sqrt{-1}p\Lambda_\gamma(H)e_1$ for $H \in \mathfrak{h}_0$. Since $(\alpha, p\Lambda_\gamma)$ is zero for $\alpha \in \Delta^- - \Delta_\gamma$, $\alpha + p\Lambda_\gamma$ is not a weight and hence we have $\rho(A_\alpha)e_1 = \rho(B_\alpha)e_1 = 0$. Therefore $\rho(X)$ for $X \in \mathfrak{k}$ is contained in $\widehat{\mathfrak{k}}$. Since $(\alpha, p\Lambda_\gamma)$ is not zero for $\alpha \in \Delta_\gamma$, $\alpha + p\Lambda_\gamma$ is a weight and hence we have $\rho(A_\alpha)e_1$ and $\rho(B_\alpha)e_1$ are non-zero vectors in the weight space of weight $\alpha + p\Lambda_\gamma$. Let j be the projection of $\mathfrak{su}(n+2)$ onto $\widehat{\mathfrak{p}}$ and $\widehat{q}$ be a complex linear isomorphim of $\widehat{\mathfrak{p}}$ onto $H_{e_1}S\mathbb{C}^{n+2}$ defined in §2. Then $\widehat{q} \circ j \circ \rho(A_\alpha) = \rho(A_\alpha)e_1$ and $\widehat{q} \circ j \circ \rho(B_\alpha) = \rho(B_\alpha)e_1$ for $\alpha \in \Delta_\gamma$. Hence the linear mapping $\widehat{q} \circ j \circ \rho$ of $\mathfrak{p}$ into $H_{e_1}S\mathbb{C}^{n+2}$ is injective and $j \circ \rho$ is also a linear injection of $\mathfrak{p}$ into $\widehat{\mathfrak{p}}$. Therefore the mapping $x \in G \to \pi(\rho(x)e_1)$ of G into $\mathbb{C}P^{n+1}$ induces an immersion f of M into $\mathbb{C}P^{n+1}$, where π denotes the fibration of $S\mathbb{C}^{n+2}$ onto $\mathbb{C}P^{n+1}$ (cf. §2). Since $\rho(IA_\alpha)e_1 = \rho(B_\alpha)e_1 = \sqrt{-1}\rho(E_\alpha)e_1 = \sqrt{-1}\rho(A_\alpha)e_1$ and $\rho(IB_\alpha)e_1 = -\rho(A_\alpha)e_1 = -\rho(E_\alpha)e_1 = \sqrt{-1}\rho(B_\alpha)e_1$ for $\alpha \in \Delta_\gamma$,the mapping $\widehat{q} \circ j \circ \rho$ is a complex linear mapping of $\mathfrak{p}$ into $H_{e_1}S\mathbb{C}^{n+2}$ and hence $j \circ \rho$ is a complex linear mapping of $\mathfrak{p}$ into $\widehat{\mathfrak{p}}$. Therefore the mapping f is holomorphic. It is known that f is a full imbedding ,i.e.,that it is an imbedding and that $f(M)$ is not contained in any proper totally geodesic Kähler submanifold of $\mathbb{C}P^{n+1}$. The imbedding f

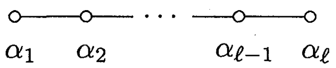
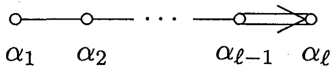
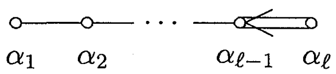
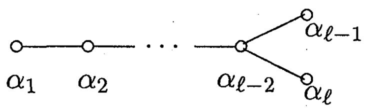
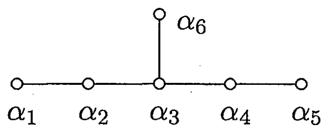
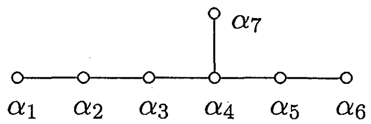
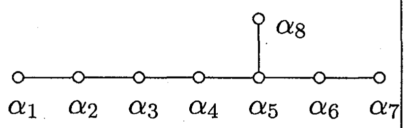
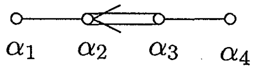
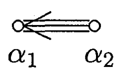
introduces a G-invariant Kähler metric g on M . Thus (M,g) is a homogeneous Kähler manifold. Especially when the pair $(\bar{\mathfrak{g}},\gamma)$ defines an orthogonal symmetric Lie algebra of Hermitian type ,the Kähler C-space (M,g) becomes an irreducible Hermitian symmetric space of compact type. The imbedding f constructed in this way is called the *p-th canonical imbedding* of M .

Now we construct a full Kähler imbedding of the product manifold of some Kähler C-spaces M with $\dim H^2(M,\mathbb{R})=1$ into a complex projective space (cf. [15] and [16]). Let M_k $(1\leq k\leq s)$ be Kähler C-spaces obtained from the pairs $(\bar{\mathfrak{g}}_k,\gamma_k)$ of complex simple Lie algebras $\bar{\mathfrak{g}}_k$ and simple roots γ_k , where γ_k denotes one of simple roots $\alpha_1,\cdots,\alpha_l$ of $\bar{\mathfrak{g}}_k$. Let $f_k:(M_k,g_k)\to \mathbf{C}P^{n_k}$ $(1\leq k\leq s)$ be the p_k-th canonical imbeddings of M_k constructed by the representations ρ_k of $\bar{\mathfrak{g}}_k$. Let $\mathfrak{h}_k$ and $(\mathfrak{h}_k)_0$ be Cartan subalgebras of $\bar{\mathfrak{g}}_k$ and their real parts respectively. Then the direct sum $\mathfrak{h}=\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_s$ is a Cartan subalgebra of the direct sum $\bar{\mathfrak{g}}=\bar{\mathfrak{g}}_1\oplus\cdots\oplus\bar{\mathfrak{g}}_s$. The dual spaces $\mathfrak{h}_k^*$ naturally become the subspaces of $\mathfrak{h}^*$ and the set of all roots of $\bar{\mathfrak{g}}$ coincides with the union of each set of all roots of $\bar{\mathfrak{g}}_k$. The direct sum $\mathfrak{h}_0=(\mathfrak{h}_1)_0\oplus\cdots\oplus(\mathfrak{h}_s)_0$ is a real part of $\mathfrak{h}$. We choose a lexicographic order in $\mathfrak{h}_0^*$ such that each fundamental root of $\bar{\mathfrak{g}}_k$ is a simple root. Let $\mathfrak{g}_k$ be compact real forms of $\bar{\mathfrak{g}}_k$. Then the direct sum $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_s$ is a compact real form of $\bar{\mathfrak{g}}$. We define a subalgebra $\mathfrak{k}$ and a subspace $\mathfrak{p}$ of $\mathfrak{g}$ by $\mathfrak{k}=\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_s$ and $\mathfrak{p}=\mathfrak{p}_1\oplus\cdots\oplus\mathfrak{p}_s$ respectively, where $\mathfrak{g}_k=\mathfrak{k}_k+\mathfrak{p}_k$ $(1\leq k\leq s)$ are the decompositions of $\mathfrak{g}_k$ which correspond to M_k. Let G_k and K_k be compact simply connected Lie groups and their connected Lie subgroups which correspond to $\mathfrak{g}_k$ and $\mathfrak{k}_k$ respectively . Then the product Lie group $G=G_1\times\cdots\times G_s$ and $K=K_1\times\cdots\times K_s$ are a Lie group and its connected Lie subgroup which correspond to $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Since a homogeneous space $G/K=M$ coincides with $M_1\times\cdots\times M_s$, it is a Kähler C-space. Let ρ be the tensor product $\rho_1\otimes\cdots\otimes\rho_s$ of the representations ρ_k of $\bar{\mathfrak{g}}_k$. Let v_k be the highest weight unit vectors with the highest weights $p_k\Lambda_{\gamma_k}$. Then $v_1\otimes\cdots\otimes v_s$ is the highest weight vector of ρ with the highest weight $\sum_{k=1}^{s}p_k\Lambda_{\gamma_k}$. The mapping $x\in G\to\pi(\rho(x)(v_1\otimes\cdots\otimes v_s))$ of G into $\mathbb{C}P^n$ induces a full Kähler imbedding of the product Kähler manifold $(M,g)=(M_1\times\cdots\times M_s,g_1\times\cdots\times g_s)$ into $\mathbb{C}P^n$, where $n=\Pi_{k=1}^{s}(n_k+1)-1$. We call this Kähler imbedding *the tensor product of* $f_1,\cdots,f_s$ and denote it by $f_1\times\cdots\times f_s$.

It is known that any full Kähler immersion into a complex projective space of a product Kähler manifold of some Kähler C-spaces M with $\dim H^2(M,\mathbb{R})=1$ is obtained in this way (cf.[13],[16]).

We construct Kähler imbeddings of Kähler C-spaces into a complex quadric. We recall the notion of orthogonal representations. The representation ρ of a complex semi-simple Lie algebra $\bar{\mathfrak{g}}$ is called *orthogonal* or *symplectic* according as it has an invariant symmetric or skew-symmetric bilinear form. Orthogonal or symplectic representations of a complex simple Lie algebra have been determined in Tits ([17]). Let $\{\Lambda_1,\cdots,\Lambda_l\}$ be the fundamental weight system of a complex simple Lie algebra $\bar{\mathfrak{g}}$ and ρ_j be the irreducible complex representation of $\bar{\mathfrak{g}}$ with the highest weight Λ_j. The members of orthogonal or symplectic representations ρ_j are given in Table 1 due to [17]. If ρ_j is an orthogonal representation , then the irreducible representation ρ with the highest weight $p\Lambda_j$ is also orthogonal. If ρ_j is a symplectic representation , then the irreducible representation ρ with the highest weight $p\Lambda_j$ is orthogonal or symplectic according as p is even or odd. In particular , we give in Table 2 an irreducible Hermitian symmetric space of compact type obtained from the pair $(\bar{\mathfrak{g}},\alpha_j)$ of a complex simple Lie algebra $\bar{\mathfrak{g}}$ and its simple root α_j such that ρ_j is an orthogonal or symplectic representation.

Table 1

	Dynkin diagram	Orthogonal representation	Symplectic representation
A_ℓ $(\ell \geq 1)$	 α_1 α_2 $\alpha_{\ell-1}$ α_ℓ	ρ_{2k+2} when $l = 4k+3$ $(k = 0, 1, \cdots)$	ρ_{2k+1} when $l = 4k+1$ $(k = 0, 1, \cdots)$
B_ℓ $(\ell \geq 2)$	 α_1 α_2 $\alpha_{\ell-1}$ α_ℓ	ρ_i $1 \leq i \leq l-1$ for any l ρ_l when $l = 4k-1$ or $l = 4k$ $(k = 1, 2, \cdots)$	ρ_l when $l = 4k+1$ or $l = 4k-2$ $(k = 1, 2, \cdots)$
C_ℓ $(\ell \geq 3)$	 α_1 α_2 $\alpha_{\ell-1}$ α_ℓ	ρ_{2i} $1 \leq i \leq \left[\frac{l}{2}\right]$	ρ_{2i+1} $0 \leq i \leq \left[\frac{l}{2}\right]$
D_ℓ $(\ell \geq 4)$	 α_1 α_2 $\alpha_{\ell-2}$ $\alpha_{\ell-1}$ α_ℓ	ρ_i $1 \leq i \leq l-2$ for any l ρ_l and ρ_{l-1} when $l = 4k$ $(k = 1, 2, \cdots)$	ρ_l and ρ_{l-1} when $l = 4k+2$ $(k = 1, 2, \cdots)$
E_6	 α_1 α_2 α_3 α_4 α_5 α_6	ρ_3, ρ_6	
E_7	 α_1 α_2 α_3 α_4 α_5 α_6 α_7	ρ_2, ρ_4, ρ_5, ρ_6	ρ_1, ρ_3, ρ_7
E_8	 α_1 α_2 α_3 α_4 α_5 α_6 α_7 α_8	ρ_i $1 \leq i \leq 8$	
F_4	 α_1 α_2 α_3 α_4	ρ_i $1 \leq i \leq 4$	
G_2	 α_1 α_2	ρ_1, ρ_2	

Let $\rho : \bar{\mathfrak{g}} \to \mathfrak{gl}(V)$ be an irreducible orthogonal representation of a complex semi-simple Lie algebra $\bar{\mathfrak{g}}$ on a complex vector space V and $\widetilde{\Omega}$ be an invariant symmetric bilinear form on V . Then $\widetilde{\Omega}$ is non-degenerate. Let $\mathfrak{g}$ be a compact real form of $\bar{\mathfrak{g}}$. We introduce a Hermitian inner product $(,)$ on V such that $\rho(X)$ is skew-Hermitian for any $X \in \mathfrak{g}$.

Lemma 5.1. ([18] Lemma 6.2) *Under the assumption above, we define a real linear endomorphism $\widetilde{J}$ of V such that $\widetilde{\Omega}(u, v) = (\widetilde{J}u, v)$ for $u, v \in V$. Then,*

(1) The real linear endomorphism $\widetilde{J}$ is semi-linear,i.e.,

$$\widetilde{J}(\lambda u) = \bar{\lambda}\widetilde{J}(u) \quad \text{for} \quad u \in V, \lambda \in \mathbb{C}.$$

(2) The real linear endomorphism $\widetilde{J}$ and $\rho(X), X \in \mathfrak{g}$ commute,i.e., $\widetilde{J}\rho(X) = \rho(X)\widetilde{J}$.

(3) By taking a suitable multiple of $\widetilde{\Omega}$ if necessary, we have $\widetilde{J}^2 = id$, where id denotes an identity transformation.

Let $\mathfrak{h}$ and $\mathfrak{h}_0$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$ and its real part respectively. Suppose that the compact real form $\mathfrak{g}$ contains $\sqrt{-1}\mathfrak{h}_0$. Then we have the following.

Lemma 5.2. ([18] Lemma 6.3) *Let $\rho : \bar{\mathfrak{g}} \to \mathfrak{gl}(V)$ be an orthogonal representation of $\bar{\mathfrak{g}}$ and $\widetilde{J}$ be a real linear endomorphism defined in Lemma 5.1. If $\lambda \in \mathfrak{h}_0^*$ is a weight of ρ and V_λ is a weight space with the weight λ, then $-\lambda$ is also a weight and $\widetilde{J}V_\lambda = V_{-\lambda}$.*

By the following theorem, we obtain isotropic Kähler imbeddings of Kähler C-spaces into a complex quadric.

Theorem 5.3. *Let M_k $(1 \le k \le s)$ be Kähler C-spaces obtained from the pairs $(\bar{\mathfrak{g}}_k, \gamma_k)$ and $f_k : M_k \to \mathbb{C}P^{n_k}$ $(1 \le k \le s)$ be the p_k-th canonical imbeddings of M_k constructed by the representations ρ_k of $\bar{\mathfrak{g}}_k$. Let M be the product Kähler manifold of M_k and $f : M \to \mathbb{C}P^{n+1}$ be the tensor product of f_k constructed by the tensor product ρ of ρ_k $(1 \le k \le s)$. We assume that each ρ_k is either orthogonal or symplectic and that the number of symplectic representations is even. Then ρ is an orthogonal representation and $f(M)$ is contained in some complex quadric Q^n in $\mathbb{C}P^{n+1}$. We denote by φ the Kähler imbedding of M into Q^n obtained by this way. Except for a few cases (see Remark 5.4), φ is isotropic.*

Proof. Let V be the representation space of ρ and set $\dim_{\mathbb{C}} V = n + 2$. From the assumption, it follows that $\rho : \bar{\mathfrak{g}} = \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_s \to \mathfrak{gl}(V)$ is an irreducible orthogonal representation. We fix a Hermitian inner product $(,)$, an invariant symmetric bilinear form $\widetilde{\Omega}$ on V and a semi-linear endomorphism $\widetilde{J}$ of V defined in Lemma 5.1. We denote by V_λ a weight space of ρ with a weight λ. For simplicity we use Λ for the highest weight $\sum_{k=1}^{s} p_k \Lambda_{\gamma_k}$. Then $\dim V_\Lambda = 1$ and hence by Lemma 5.2, $\dim V_{-\Lambda} = 1$. Let e_1 be a unit vector of V_Λ and put $e_2 = \widetilde{J}e_1$. Then by Lemma 5.2, it follows that $e_2 \in V_{-\Lambda}$. Moreover we have

$$\begin{aligned} (e_1, e_1) = (e_2, e_2) = 1, &\qquad (e_1, e_2) = 0 \\ \widetilde{\Omega}(e_1, e_1) = \widetilde{\Omega}(e_2, e_2) = 0, &\qquad \widetilde{\Omega}(e_1, e_2) = 1. \end{aligned}$$

Let $\{e_3, \cdots, e_{n+2}\}$ be a unitary basis of the subspace orthogonal to e_1 and e_2 which satisfies that $\widetilde{\Omega}(e_i, e_j) = \delta_{ij}$ $(i, j = 3, \cdots, n + 2)$. From now on, using this unitary basis $\{e_1, e_2, e_3, \cdots, e_{n+2}\}$, we identify V with $\mathbb{C}^{n+2}$. Under this identification, the Hermitian inner product $(,)$, the invariant symmetric bilinear form $\widetilde{\Omega}$ and the semi-linear endomorphism $\widetilde{J}$ on V coincide with those of $\mathbb{C}^{n+2}$ introduced in §2. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be the compact real form of $\bar{\mathfrak{g}}$, where $\mathfrak{g}_k$ denote compact real forms of $\bar{\mathfrak{g}}_k$ $(k = 1, \cdots, s)$.

Then for each $X \in \mathfrak{g}$, $\rho(X)$ is skew-Hermitian with respect to the Hermitian inner product $(,)$ and $\rho(X)$ leaves the bilinear form $\widetilde{\Omega}$ invariant. Therefore $\rho(\mathfrak{g})$ is contained in $\widetilde{\mathfrak{g}}$ and in particular we obtain a Lie algebra homomorphism $\rho : \mathfrak{g} \to \widetilde{\mathfrak{g}}$, where $\widetilde{\mathfrak{g}}$ denotes the Lie algebra defined by (2.3). Let G be a compact simply connected Lie group which corresponds to the compact real form $\mathfrak{g}$. The representation ρ restricted to $\mathfrak{g}$ defines the representation of G on V, which will also be denoted by ρ. Then for each $x \in G$ $\rho(x)$ leaves the Hermitian inner product $\widetilde{(,)}$ and the bilinear form $\widetilde{\Omega}$ invariant. In particular $\rho(G)$ is a Lie subgroup of $\widetilde{G}$, where $\widetilde{G}$ denotes the closed subgroup of $SU(n+2)$ defined by (2.2). Since $\widetilde{\Omega}(\rho(x)e_1, \rho(x)e_1) = \widetilde{\Omega}(e_1, e_1) = 0$ for each $x \in G$, $\rho(x)e_1 \in \widetilde{Q}$, where $\widetilde{Q} = \{z \in S\mathbb{C}^{n+2} \mid \widetilde{\Omega}(z,z) = 0\ \}$. Therefore the image $f(M)$ of the Kähler imbedding $f : M \to \mathbb{C}P^{n+1}$ induced by the mapping $x \in G \mapsto \pi(\rho(x)e_1)$ is contained in a complex quadric Q^n . Hence we obtain the Kähler imbedding of M into Q^n , which is denoted by φ .

Next we shall show that the imbedding φ is isotropic. Let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ be the decomposition of the Lie algebra $\widetilde{\mathfrak{g}}$ which corresponds to a complex quadric Q^n (see §2) and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of the compact real form $\mathfrak{g}$ which corresponds to the Kähler manifold M. Let $\widetilde{j} : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{p}}$ be the projection of $\widetilde{\mathfrak{g}}$ onto $\widetilde{\mathfrak{p}}$ with respect to the decomposition $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ and $\widetilde{q}$ be a complex linear isometry of $\widetilde{\mathfrak{p}}$ onto $H_{e_1}\widetilde{Q}$ defined by $\widetilde{q}(X) = Xe_1$ for $X \in \widetilde{\mathfrak{p}}$ (see §2). Then we have $\widetilde{q} \circ \widetilde{j} \circ \rho(X) = \rho(X)e_1$ for $X \in \mathfrak{g}$. For $\alpha \in \Delta_{\gamma_k}$, we have $\rho(A_\alpha)e_1 = \rho(E_\alpha + E_{-\alpha})e_1 = \rho(E_\alpha)e_1 \in V_{\alpha+\Lambda}$ and $\rho(B_\alpha)e_1 = \rho(\sqrt{-1}(E_\alpha - E_{-\alpha}))e_1 = \sqrt{-1}\rho(E_\alpha)e_1 \in V_{\alpha+\Lambda}$. Here we remark that $\alpha \in \Delta_{\gamma_k}$ is a root whose coefficient of γ_k is strictly negative. From these, it follows that

$$\widetilde{q} \circ \widetilde{j} \circ \rho(\mathfrak{p}) \subset \sum_{k=1}^{s} \sum_{\alpha \in \Delta_{\gamma_k}} V_{\alpha+\Lambda}.$$

By Lemma 5.2, we have

$$\widetilde{J}(\widetilde{q} \circ \widetilde{j} \circ \rho(\mathfrak{p})) \subset \widetilde{J}(\sum_{k=1}^{s} \sum_{\alpha \in \Delta_{\gamma_k}} V_{\alpha+\Lambda}) = \sum_{k=1}^{s} \sum_{\alpha \in \Delta_{\gamma_k}} V_{-\alpha-\Lambda}.$$

Define the set Φ of weights of ρ by

$$\Phi = \{\alpha + \Lambda \mid \alpha \in \Delta_{\gamma_k} \quad (k = 1, \cdots, s)\}.$$

Suppose that $\Phi \cap -\Phi$ is empty. Then $\widetilde{J}(\sum_{k=1}^{s} \sum_{\alpha \in \Delta_{\gamma_k}} V_{\alpha+\Lambda})$ is orthogonal to $\sum_{k=1}^{s} \sum_{\alpha \in \Delta_{\gamma_k}} V_{\alpha+\Lambda}$ with respect to the Hermitian inner product. So $\widetilde{J}(\widetilde{q} \circ \widetilde{j} \circ \rho(\mathfrak{p}))$ is orthogonal to $\widetilde{q} \circ \widetilde{j} \circ \rho(\mathfrak{p})$. This implies that $\widetilde{q} \circ \widetilde{j} \circ \rho(\mathfrak{p})$ is an isotropic complex subspace in $H_{e_1}\widetilde{Q}$ with respect to $\widetilde{\Omega}$ and hence $\widetilde{j} \circ \rho(\mathfrak{p})$ is also isotropic in $\widetilde{\mathfrak{p}}$ with respect to Ω (see §2). Therefore $\varphi_* T_o M$ is an isotropic complex subspace in $T_o Q^n$ at the origin $o \in M$. Since φ is $\rho(G)$-equivariant, φ is an isotropic imbedding.

In the remainder of the proof, we will show that $\Phi \cap -\Phi$ is empty except a few cases. First we consider the irreducible case, i.e., $s = 1$. Let $(\bar{\mathfrak{g}}, \alpha_i)$ be the pair of a complex simple Lie algebra $\bar{\mathfrak{g}}$ and its simple root α_i such that the irreducible representation ρ_i with the highest weight Λ_i is orthogonal (see Table 1). Then the representation with the highest weight $p\Lambda_i$ is orthogonal for any positive integer p. By checking tables in Tits [17] or Bourbaki [3], we can see that for any $\alpha \in \Delta_{\alpha_i}$ the coefficient of α_i in $\alpha + p\Lambda_i$ is positive except a few cases. Therefore $\Phi \cap -\Phi$ is empty. Here exceptional cases are those of $p=1$ in the following pairs:

$$\begin{array}{lllll} (A_3,\alpha_2), & (B_l,\alpha_1)\ (l\geq 2), & (B_l,\alpha_2)\ (l\geq 3), & (B_3,\alpha_3), & (B_4,\alpha_4) \\ (C_l,\alpha_2)\ (l\geq 3), & (D_l,\alpha_1)\ (l\geq 4), & (D_l,\alpha_2)\ (l\geq 4), & (E_6,\alpha_6), & (E_7,\alpha_6), \\ (E_8,\alpha_1), & (F_4,\alpha_1), & (F_4,\alpha_4), & (G_2,\alpha_1), & (G_2,\alpha_2). \end{array}$$

Next we consider the pair $(\bar{\mathfrak{g}},\alpha_i)$ such that the irreducible representation ρ_i with the highest weight Λ_i is symplectic (see Table 1). Then the representation with the highest weight $p\Lambda_i$ is orthogonal for any positive even integer p. Similarly we see that for any $\alpha\in\Delta_{\alpha_i}$ the coefficient of α_i in $\alpha+p\Lambda_i$ is positive except $p=2$ in the following pairs:

$$(A_1,\alpha_1),\quad (B_2,\alpha_2),\quad (C_l,\alpha_1)\ (l\geq 3).$$

When M is reducible and $s\geq 3$, for any $\alpha\in\cup_{k=1}^{s}\Delta_{\gamma_k}$ the number of k $(1\leq k\leq s)$ such that the coefficient of γ_k in $\alpha+\Lambda$ is positive is at least $s-1$. On the other hand, the number of k $(1\leq k\leq s)$ such that the coefficient of γ_k in $-\alpha-\Lambda$ is positive is at most 1. Therefore $\Phi\cap-\Phi$ is empty. Finally we consider the case that M is reducible and $s=2$. In this case ρ_1 and ρ_2 are both orthogonal or both symplectic. Then we see

$$\Phi=\{\alpha+p_1\Lambda_{\gamma_1}+p_2\Lambda_{\gamma_2}\ (\alpha\in\Delta_{\gamma_1}),\quad p_1\Lambda_{\gamma_1}+\beta+p_2\Lambda_{\gamma_2}\ (\beta\in\Delta_{\gamma_2})\}.$$

Suppose that $\beta+p_2\Lambda_{\gamma_2}\neq -p_2\Lambda_{\gamma_2}$ for any $\beta\in\Delta_{\gamma_2}$. Then even if $\alpha+p_1\Lambda_{\gamma_1}=-p_1\Lambda_{\gamma_1}$ for some $\alpha\in\Delta_{\gamma_1}$, it follows that $-\alpha-p_1\Lambda_{\gamma_1}-p_2\Lambda_{\gamma_2}\neq p_1\Lambda_{\gamma_1}+\beta+p_2\Lambda_{\gamma_2}$ and $-p_1\Lambda_{\gamma_1}-\beta-p_2\Lambda_{\gamma_2}\neq\alpha+p_1\Lambda_{\gamma_1}+p_2\Lambda_{\gamma_2}$ for any $\alpha\in\Delta_{\gamma_1}$ and $\beta\in\Delta_{\gamma_2}$. This implies that $\Phi\cap-\Phi$ is empty. Let $(\bar{\mathfrak{g}},\gamma)$ be the pair of a complex simple Lie algebra $\bar{\mathfrak{g}}$ and its simple root γ such that the representation with the highest weight Λ_γ is orthogonal or symplectic. By checking tables in Tits [17] or Bourbaki [3], we see that for any $\alpha\in\Delta_\gamma$, $\alpha+p\Lambda_\gamma\neq-p\Lambda_\gamma$ except $p=1$ of the pairs (A_1,α_1),(B_2,α_2), (C_l,α_1). □

Remark 5.4. We consider excluded cases in the proof of Theorem 5.3. Indeed their canonical imbeddings are not isotropic. We denote by $(\bar{\mathfrak{g}},\gamma,\Lambda)$ the triple such that $\bar{\mathfrak{g}}$ is a complex simple Lie algebra and γ is its simple root and Λ is the highest weight of the representation which induces the canonical imbedding of the Kähler C-space corresponding to the pair $(\bar{\mathfrak{g}},\gamma)$.

(1) $(A_1,\alpha_1,2\Lambda_1)$, (A_3,α_2,Λ_2), (B_l,α_1,Λ_1) $(l\geq 2)$,(B_3,α_3,Λ_3), (D_l,α_1,Λ_1) $(l\geq 4)$, (G_2,α_1,Λ_1).
The corresponding Kähler C-space is holomorphically isometric to a complex quadric Q^n and its canonical imbedding defined by the representation above is a hypersurface imbedding into $\mathbb{C}P^{n+1}$ and hence it is not isotropic.

(2) $(B_2,\alpha_2,2\Lambda_2)$, (B_l,α_2,Λ_2) $(l\geq 3)$, $(C_l,\alpha_1,2\Lambda_1)$, (D_l,α_2,Λ_2), (E_6,α_6,Λ_6),(E_7,α_6,Λ_6), (E_8,α_1,Λ_1),(F_4,α_4,Λ_4),(G_2,α_2,Λ_2).
The corresponding Kähler C-space M is the twistor space of a compact quaternionic symmetric space S and the canonical imbedding f is given by the orbit through the maximal root vector by the adjoint representation of the compact real form $\mathfrak{g}$ on $\bar{\mathfrak{g}}$. The Killing form of $\bar{\mathfrak{g}}$ is an invariant symmetric bilinear form of the representation. The image $f(M)$ is contained in Q^n but the Kähler imbedding φ is not isotropic. In fact we see that on the fibre of the twistor fibration $M\to S$, $\varphi^*\Omega$ does not vanish.

(3) (B_4,α_4,Λ_4), (C_l,α_2,Λ_2),(F_4,α_1,Λ_1).
We see that their canonical imbeddings are not isotropic, checking case by case.

(4) $(C_l, \alpha_1, \Lambda_1) \times (C_m, \alpha_1, \Lambda_1)$.
The corresponding Kähler C-space M is the product of two complex projective spaces $\mathbb{C}P^{2l-1}$ and $\mathbb{C}P^{2m-1}$ and the Kähler imbedding f into $\mathbb{C}P^{n+1}$ $(n = 4lm - 2)$ is the tensor product of two first canonical imbeddings, which is known as the Segre imbedding. It is well-known that the second fundamental form of f is parallel. The image $f(M)$ is contained in Q^n but the Kähler imbedding φ of M into Q^n is not isotropic.

We show interesting properties of those isotropic Kähler imbeddings of Hermitian symmetric spaces constructed in Theorem 5.3.

Theorem 5.5. *In addition to the assumption in Theorem 5.3, we assume that each pair $(\bar{\mathfrak{g}}_k, \gamma_k)$ defines an orthogonal symmetric Lie algebra of Hermitian type. If φ is an isotropic Kähler imbedding of M into Q^n constructed in Theorem 5.3, then φ is osculating full and the degree $d(\varphi)$ of φ is given by*

$$d(\varphi) = d(f) - 1 = \sum_{k=1}^{s} r_k p_k - 1,$$

where r_k denotes the rank of M_k as an Hermitian symmetric space $(1 \leq k \leq s)$. In particular $d(\varphi)$ is odd. Moreover for the normal spaces N^j, we have

$$JN^j = N^{d(\varphi)-j+1} \qquad \text{for} \quad j = 1, \cdots, d(\varphi),$$

where we put $N^1 = TM$.

Proof. We use the notations σ_j and N^j for the j-th fundamental forms and the j-th normal spaces of the imbedding φ into Q^n and $\widetilde{\sigma}_j$ and $\widetilde{N^j}$ for those of the imbedding f into $\mathbb{C}P^{n+1}$. Comparing σ_j and $\widetilde{\sigma}_j$ as in section 4, we prove our theorem.

First we recall the properties of higher fundamental forms of the canonical imbeddings of Hermitian symmetric spaces into $\mathbb{C}P^{n+1}$ following §5 in [18] . Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{k}} + \widehat{\mathfrak{p}}$ be the canonical decomposition of $\widehat{\mathfrak{g}} = \mathfrak{su}(n+2)$ which corresponds to $\mathbb{C}P^{n+1}$ and $\widehat{q} : \widehat{\mathfrak{p}} \to H_{e_1}S\mathbb{C}^{n+2}$ be the complex linear isometry defined by $\widehat{q}(X) = Xe_1$ for $X \in \widehat{\mathfrak{p}}$ (see §2). Let M_k $(1 \leq k \leq s)$ be an irreducible Hermitian symmetric space obtained from the pair $(\bar{\mathfrak{g}}_k, \gamma_k)$ of a complex simple Lie algebra $\bar{\mathfrak{g}}_k$ and a simple root γ_k and $\mathfrak{g}_k = \mathfrak{k}_k + \mathfrak{p}_k$ be the canonical decomposition of the compact real form $\mathfrak{g}_k$ which corresponds to M_k. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the the canonical decomposition of the compact real form $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ which corresponds to the Hermitian symmetric space $M = M_1 \times \cdots \times M_s$. We denote by $\mathfrak{h}_0$ the real part of the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_s$ of the complex semi-simple Lie algebra $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_1 \oplus \cdots \oplus \bar{\mathfrak{g}}_s$. For a linear form $\varepsilon \in \mathfrak{h}_0^*$, we write $\varepsilon = \sum_\gamma c(\varepsilon; \gamma)\gamma$, where γ runs over the fundamental root system and $c(\varepsilon; \gamma)$ denotes the coefficient of γ. We consider the representation ρ of $\bar{\mathfrak{g}}$ in our theorem. For a positive integer j , we denote by Φ_j the set of weights λ of ρ which satisfies $\sum_{k=1}^{s} c(\Lambda - \lambda; \gamma_k) = j$, where $\Lambda = \sum_{k=1}^{s} p_k \Lambda_{\gamma_k}$ is the highest weight of ρ. Identify $\widehat{\mathfrak{p}}$ and $\mathfrak{p}$ with the tangent spaces at the origins of $\mathbb{C}P^{n+1}$ and M , respectively. Then we can view the j-th fundamental form $\widetilde{\sigma}_j$ of the Kähler imbedding f at the origin as an element of $Hom(\otimes^j \mathfrak{p}, \widehat{\mathfrak{p}})$. It was shown in the proof of Theorem 5.2 in [18] that the following holds:

$$(5.1) \quad \widehat{q}(f_* T_o M) = \widehat{q}(j(\rho(\mathfrak{p}))) = \sum_{\lambda \in \Phi_1} V_\lambda, \qquad \widehat{q}(\widetilde{N}_o^j) = \widehat{q}(\widetilde{\sigma}_j(\otimes^j \mathfrak{p})) = \sum_{\lambda \in \Phi_j} V_\lambda \quad (j \geq 2).$$

Here we take the unit vector e_1 in V_Λ as in the proof of Theorem 5.3 when we define $\widehat{q}$. It is also known that f is osculating full and that its degree $d(f)$ is given by

$$d(f) = \sum_{k=1}^{s} r_k p_k, \quad r_k = \text{the rank of } M_k$$

(Theorem 2 in [15] and Theorem 5.2 in [18]). Under the assumption of Theorem 5.5, checking each Hermitian symmetric space in Table 2 one by one , we see that $d(f) = \sum_{k=1}^{s} r_k p_k$ is even and that $d(f) \geq 4$. In our case we have

$$\widehat{q}(\widetilde{N}_o^{d(f)}) = \widehat{q}(\widetilde{\sigma}_{d(f)}(\otimes^{d(f)}\mathfrak{p})) = V_{-\Lambda} = \widetilde{J}(V_\Lambda). \tag{5.2}$$

In fact since $-\Lambda$ is the lowest weight of ρ, we have $\sum_{k=1}^{s} c(\Lambda-(-\Lambda);\gamma_k) = d(f)$. Moreover the representation theory of semi-simple Lie algebras implies that

$$\sum_{k=1}^{s} c(\Lambda - \lambda;\gamma_k) < \sum_{k=1}^{s} c(\Lambda-(-\Lambda);\gamma_k) = d(f)$$

for any weight λ ($\neq -\Lambda$) of ρ. Since $H_{e_1}S\mathbb{C}^{n+2} = H_{e_1}\widetilde{Q} \oplus \mathbb{C}(\widetilde{J}e_1)$, by (5.1) and (5.2) we have

$$H_{e_1}\widetilde{Q} = \widehat{q}(f_*T_oM) + \sum_{2\leq j\leq d(f)-1} \widehat{q}(\widetilde{N}_o^j). \tag{5.3}$$

Evidently we see that

$$\sum_{k=1}^{s} c(\Lambda-\lambda;\gamma_k) + \sum_{k=1}^{s} c(\Lambda-(-\lambda);\gamma_k) = \sum_{k=1}^{s} c(2\Lambda;\gamma_k) = d(f)$$

for a weight λ of ρ. This together with Lemma 5.2 implies that $\Phi_{d(f)-j} = -\Phi_j$. Therefore by (5.1) and Lemma 5.2 we have $\widetilde{J}(\widehat{q}(\widetilde{N}_o^j)) = \widehat{q}(\widetilde{N}_o^{d(f)-j})$ for $1 \leq j \leq d(f)-1$.

Now we consider the isotropic Kähler imbedding φ of M into Q^n. Let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ be the decomposition which corresponds to Q^n and $\widetilde{q}$ be the complex linear isometry of $\widetilde{\mathfrak{p}}$ onto $H_{e_1}\widetilde{Q}$ defined by $\widetilde{q}(X) = Xe_1$ as in the proof of Theorem 5.3. As usual we identify $\widetilde{\mathfrak{p}}$ with the tangent space at the origin of Q^n. As we have shown in section 4, we have $\sigma_2(X_1,X_2) = \widetilde{\sigma}_2(X_1,X_2)$ for $X_1, X_2 \in T_pM$, $p \in M$. In particular $N_o^2 = \widetilde{N}_o^2$ and $\widetilde{q}(N_o^2) = \widehat{q}(\widetilde{N}_o^2)$ in $H_{e_1}S\mathbb{C}^{n+2}$. Noticing (5.3), inductively we calculate the higher fundamental forms σ_j for $2 \leq j \leq d(f)-1$ and obtain

$$\sigma_j(X_1,\cdots,X_j) = \widetilde{\sigma}_j(X_1,\cdots,X_j) \qquad \text{for } X_1,\cdots,X_j \in T_pM, p\in M.$$

In particular $N_o^j = \widetilde{N}_o^j$ and $\widetilde{q}(N_o^j) = \widehat{q}(\widetilde{N}_o^j)$ in $H_{e_1}S\mathbb{C}^{n+2}$ for $2 \leq j \leq d(f)-1$. By (5.3), we see that φ is osculating full and the degree $d(\varphi)$ is equal to $d(f)-1$. Moreover we have $J(N_o^j) = N_o^{d(f)-j} = N_o^{d(\varphi)-j+1}$ $j = 1,\cdots,d(\varphi)$. In fact

$$\widetilde{q}(J(N_o^j)) = \widetilde{J}(\widetilde{q}(N_o^j)) = \widetilde{J}(\widehat{q}(\widetilde{N}_o^j)) = \widehat{q}(\widetilde{N}_o^{d(f)-j}) = \widetilde{q}(N_o^{d(f)-j}).$$

Since φ is $\rho(G)$-equivariant, $J(N_p^j) = N_p^{d(\varphi)-j+1}$ at any point $p \in M$. □

Table 2 Hermitian symmetric spaces with orthogonal representations

	M	$\dim_{\mathbb{C}} M$	rank M
$(A_{4k+3}, \alpha_{2k+2})$ $(k \geq 0)$	$G_{2k+2}(\mathbb{C}^{4k+4}) = SU(4k+4)/S(U(2k+2) \times U(2k+2))$	$(2k+2)^2$	$2k+2$
(B_l, α_1) $(l \geq 2)$	$Q^{2l-1} = SO(2l+1)/SO(2l-1) \times SO(2)$	$2l-1$	2
(C_{2k}, α_{2k}) $(k \geq 2)$	$Sp(2k)/U(2k)$	$k(2k+1)$	$2k$
(D_l, α_1) $(l \geq 4)$	$Q^{2l-2} = SO(2l)/SO(2l-2) \times SO(2)$	$2l-2$	2
(D_{4k}, α_{4k}) $(k \geq 2)$	$SO(8k)/U(4k)$	$2k(4k-1)$	$2k$

Hermitian symmetric spaces with symplectic representations

	M	$\dim_{\mathbb{C}} M$	rank M
(A_1, α)	$\mathbb{C}P^1$	1	1
$(A_{4k+1}, \alpha_{2k+1})$ $(k \geq 1)$	$G_{2k+1}(\mathbb{C}^{4k+2}) = SU(4k+2)/S(U(2k+1) \times U(2k+1))$	$(2k+1)^2$	$2k+1$
$(C_{2k+1}, \alpha_{2k+1})$ $(k \geq 1)$	$Sp(2k+1)/U(2k+1)$	$(2k+1)(k+1)$	$2k+1$
$(D_{4k+2}, \alpha_{4k+2})$ $(k \geq 1)$	$SO(2(4k+2))/U(4k+2)$	$(2k+1)(4k+1)$	$2k+1$
(E_7, α_1)	$E_7/E_6 \cdot T^1$	27	3

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