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A remark on classification of connected palette diagrams without area and moment

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Abstract

In this short paper, the author would like to correct some errors and supplement the classification theorem of palette diagrams without area and moment in the previous paper [2].

1. A review of [2]

A *palette diagram* is defined by a collection of some unit weighted squares

$$\Gamma = \Big\{[S_1, p_1], [S_2, p_2], \dots, [S_n, p_n]\Big\}, \tag{1}$$

where $[S_i, p_i]$ is a pair of a unit open square S_i with its vertices on lattice points and a non zero integer p_i. First the author would like to correct the wrong definition of *connected* palette diagram in [2]. The correct definition is as follows : A palette diagram Γ in (1) is said to be *connected* if the union $\cup_{i=1}^{n} \bar{S}_i$ of closures of S_i is connected.

The following definition of types (A) $\sim$ (D) of palette diagrams is in better expression than in [2]. The meaning the author intended is however the same.

Definition 1 ([2], Definition 1.1). *If a palette diagram Γ has one of the following property (A) or (B) or (C) or (D):*

(A) $Area(\Gamma) = 0$, $G(\Gamma) = (0,0)$, $\det M(\Gamma) \neq 0$ *and there exists an* $(\alpha, \beta) \neq (0,0)$ *such that* $P_2(\Gamma)(\alpha,\beta) = 0$ *and* $P_3(\Gamma)(\alpha,\beta) \neq 0$,

(B) $Area(\Gamma) = 0$, $G(\Gamma) = (0,0)$, $\det M(\Gamma) \neq 0$ *and for all* $(\alpha, \beta) \neq (0,0)$ *such that* $P_2(\Gamma)(\alpha,\beta) = 0$, $P_3(\Gamma)(\alpha,\beta) = 0$ *holds,*

(C) $Area(\Gamma) = 0$, $G(\Gamma) = (0,0)$, $\det M(\Gamma) = 0$ *and there exists an* $(\alpha, \beta) \neq (0,0)$ *such that* $P_2(\Gamma)(\alpha,\beta) = 0$ *and* $P_3(\Gamma)(\alpha,\beta) \neq 0$,

(D) $Area(\Gamma) = 0$, $G(\Gamma) = (0,0)$, $\det M(\Gamma) = 0$ *and for all* $(\alpha, \beta) \neq (0,0)$ *such that* $P_2(\Gamma)(\alpha,\beta) = 0$, $P_3(\Gamma)(\alpha,\beta) = 0$ *holds,*

we say Γ has the type (A) or (B) or (C) or (D).

The properties (A) $\sim$ (D) are invariant under congruence and reflection of a palette diagram. So we regard palette diagrams up to congruence and reflection to be equivalent.

A *Feynman diagram* is defined by a polygonal path γ in $\mathbb{R}^2$ consisting of a path H in x-positive direction with length 1 and a path V in y-positive direction with length 1. Such a γ has its expression

$$\gamma = H^{n_1} * V^{n_2} * H^{n_3} * V^{n_4} * \cdots * H^{n_{2!!\,1}} * V^{n_{2!}}, \quad n_1, n_2, \dots, n_{2p} \in \mathbb{Z}^{!}, \tag{2}$$

where $*$ denotes the product of paths and H^n stands for the n-fold composite of H. From a palette diagram Γ we can obtain infinitely many closed Feynman diagrams $\gamma \subset \mathbb{R}^2$ which have distinct expressions such that the value of the winding number ρ on the square domain S_i is $p_i, i = 1, 2, \ldots, n$. From a Feynman diagram γ in (2), we obtain a word

$$W_{\gamma^*}(f,g) = f^{(n_1)} \circ g^{(n_2)} \circ f^{(n_3)} \circ g^{(n_4)} \circ \cdots \circ f^{(n_{2p-1})} \circ g^{(n_{2p})}$$

of $f, g, f^{(-1)}, g^{(-1)}$ for two formal diffeomorphisms $f, g \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ by substituting H and V in the expression of γ for f and g respectively, where $f^{(m)}$ stands for the m-fold iteration of f, and $\widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ denotes the group of all formal diffeomorphisms of $\mathbb{C}$ fixing 0

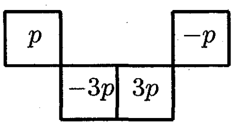
$$\widehat{\mathrm{Diff}}(\mathbb{C}, 0) = \{f(z) = c_1 z + c_2 z^2 + \cdots | c_1 \neq 0, c_i \in \mathbb{C}\}$$

and $\circ$ denotes the composite of mappings. We call $W_{\gamma^*}(f,g)$ the word accompanied with a Feynman diagram γ.

In [2] the author have classified all 22 basis of connected palette diagrams consisting of 4 unit weighted squares into the above 4 types (A) $\sim$ (D) and determined the palette diagrams with the type (A) (Theorem 3.1 [2]). And many relations in $\widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ of the form $W_{\gamma^*}(f,g) = \mathrm{id}$, which admits solutions f, g of non commuting formal diffeomorphisms tangent to the identity with prescribed z^2-term, can be obtained by applying a theorem (Theorem 2.3 [1]), which gives a sufficient condition of closed Feynman diagram γ such that the relation $W_{\gamma^*}(f,g) = \mathrm{id}$ admits solutions, to the above theorem. Some results in [1] on the existence or non existence of relations in $\widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ are obtained by calculating coefficients L_k of z^k-term, $k = 4, 5, \ldots$, of the logarithm of the word $W_{\gamma^*}(f,g)$ in terms of logarithms $a_1\partial_z, a_2\partial_z$ of f, g. For $a_1\partial_z = (a_{12}z^2 + a_{13}z^3 + \cdots)\partial_z, a_2\partial_z = (a_{22}z^2 + a_{23}z^3 + \cdots)\partial_z$, let $A_i = \begin{bmatrix} a_{1i} \\ a_{2i} \end{bmatrix}$ and $K_i = a_{1i}x + a_{2i}y$ for $i = 2, 3, \ldots$. And $|A_i A_j|$ denotes the determinant of the 2×2 matrix $\begin{bmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{bmatrix}$.

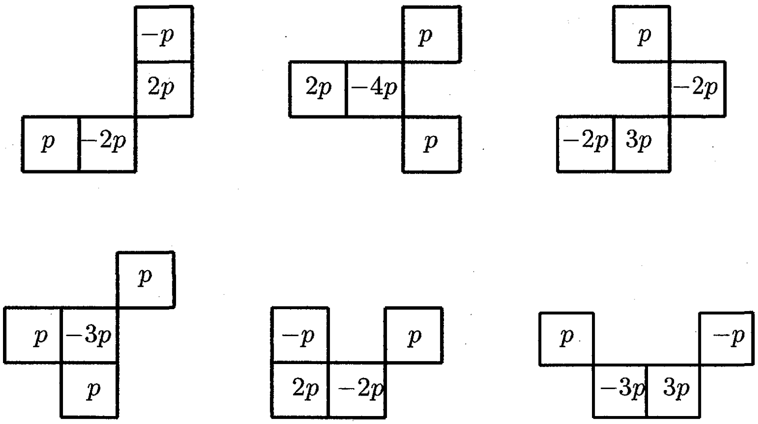
2. A correction in [2] and Theorem 1 on classification of connected palette diagrams consisting of 4 unit weighted squares

The author also would like to correct Theorem 3.1 (and Theorem 3.2) in [2]. I missed the computation of $P_3(\Gamma_{15})(1,0)$ in the page 13, that is, $P_3(\Gamma_{15})(1,0) = 0$ is a mistake. The truth is that $P_3(\Gamma_{15})(1,0) = -6p \neq 0$ (See Remark 1 for detailed computation). Hence the following palette diagram, which is classified into the type (B) by mistake, is classified into the type (A).



Also in this paper we prove the next theorem as a supplement of Theorem 5.4 [2] by making mention of even solutions $f, g \neq \mathrm{id}$ with $(f''(0) : g''(0)) = (\alpha : \beta)$, where (α, β) is an arbitrary complex vector except for the one in Definition 1.

Theorem 1. *A connected palette diagram Γ without area and moment consisting of 4 unit weighted squares except for two palette diagrams with the type (D) such that for any closed Feynman diagram γ obtained from Γ the relation $W_{\gamma^*}(f,g) = id$ admits solutions of formal diffeomorphisms $f, g \neq id$ non commuting and tangent to the identity equals only one of the following 6 palette diagrams. Here p is an arbitrary non zero integer. And then $(f''(0) : g''(0)) = (\alpha : \beta)$, where (α, β) is one satisfying $P_2(\gamma)(\alpha, \beta) = 0$ and $P_3(\gamma)(\alpha, \beta) \neq 0$.*



3. Proof of Theorem 1

To prove Theorem 1, the next Lemma 1 and Lemma 2 are essential. Lemma 2 is the second statement of Proposition 16.1 [1], but a little different form from it.

Lemma 1. *Let Γ be a palette diagram with the type (B) or (D) and let (α, β) a vector such that $P_2(\Gamma)(\alpha,\beta) = P_3(\Gamma)(\alpha,\beta) = 0$. Let ρ denote the winding number of any Feynman diagram γ obtained from Γ and D its support. Then*

$$\iint_D \rho {K_2}^n dx \wedge dy = 0, \quad n = 1, 2, \ldots, \tag{3}$$

holds for $K_2 = \alpha x + \beta y$.

Proof. We have only to compute the left-hand side of (3) for palette diagrams with the type (B) or (D) in Theorem 3.2 and Theorem 3.4 in [2]. For simplicity of the computation, we may assume that one of vertices of a palette diagram is the origin since conditions (B) and (D) are invariant under congruence.

For example for a palette diagram of (B-6) in [2],

$$\begin{aligned}
\iint_D \rho y^n dx \wedge dy &= p\int_0^1 y^n dy \int_0^1 dx - p\int_0^1 y^n dy \int_1^2 dx - p\int_1^2 y^n dy \int_2^3 dx \\
&\quad + p\int_1^2 y^n dy \int_3^4 dx = 0, \\
\iint_D \rho (x-2y)^n dx \wedge dy &= p\int_0^1 dy \int_0^1 (x-2y)^n dx - p\int_0^1 dy \int_1^2 (x-2y)^n dx \\
&\quad - p\int_1^2 dy \int_2^3 (x-2y)^n dx + p\int_1^2 dy \int_3^4 (x-2y)^n dx = 0,
\end{aligned}$$

and for a palette diagram of (D-22) in [2],

$$\begin{aligned}
\iint_D \rho (x-y)^n dx \wedge dy &= p\int_0^1 dy \int_0^1 (x-y)^n dx + q\int_1^2 dy \int_1^2 (x-y)^n dx \\
&\quad -(3p+2q)\int_2^3 dy \int_2^3 (x-y)^n dx + (2p+q)\int_3^4 dy \int_3^4 (x-y)^n dx = 0,
\end{aligned}$$

and so on. □

Lemma 2 ([1], Proposition 16.1). *Assume* $A_1 = 0$, $A_2 \neq 0$ *and*

$$|A_2A_3| = |A_2A_4| = \cdots = |A_2A_i| = 0, \ |A_2A_{i+1}| \neq 0$$

for an $i \geq 2$. *If the moment condition*

$$Moment_n(\gamma) = \iint_D \rho {K_2}^n dx \wedge dy = 0, \quad n = 0, 1, \ldots, i-1,$$

holds, then

$$L_{2i+2} = \iint_D \rho \left\{ \frac{(-1)^{i+1}(2i-1)!}{i!(i-2)!} {K_2}^i + (i-1)K_{i+1} \right\} dK_2 \wedge dK_{i+1}$$

and in addition if $L_{2p+2} = 0$, *then*

$$L_{2i+3} = \iint_D \rho \left\{ \frac{(-1)^{i+2}(2i)!}{(i+1)!(i-2)!} {K_2}^{i+1} + (-1)^{i+1} S {K_2}^{i-1} K_3 - 2i(i-1) K_2 K_{i+1} \right\} dK_2 \wedge dK_{i+1},$$

where $S = 0$ *for* $i = 2$ *and* $S > 0$ *for* $i > 2$.

Proof of Theorem 1.

In the followings, for a palette diagram Γ, γ denotes an arbitrary Feynman diagram obtained from Γ.

1. Let Γ be a palette diagram consisting of 4 unit weighted squares with the type (A).

For an $(\alpha, \beta) \in \mathbb{C}^2$ satisfying the condition that $P_2(\Gamma)(\alpha, \beta) = 0$ and $P_3(\Gamma)(\alpha, \beta) \neq 0$, the relation $W_{\gamma^*}(f, g) = \mathrm{id}$ admits formal solutions $f, g \neq \mathrm{id}$ non commuting and tangent to the identity (Theorem 5.4 [2]).

For an $(\alpha, \beta) \in \mathbb{C}^2$ except for the above one, $P_2(\Gamma)(\alpha, \beta) \neq 0$. Hence if $W_{\gamma^*}(f, g) = \mathrm{id}$ admits formal solutions $f, g \neq \mathrm{id}$ tangent to the identity with (α, β), then f, g commute by Theorem 5.1 [2].

2. Let Γ be a palette diagram consisting of 4 unit weighted squares with the type (B).

Let $A_2 = (\alpha, \beta) \in \mathbb{C}^2$ satisfy the condition that $P_2(\Gamma)(\alpha, \beta) = 0$ and $P_3(\Gamma)(\alpha, \beta) = 0$. Assume $A_3, \ldots, A_{i+1}$ satisfy $|A_2A_3| = |A_2A_4| = \cdots = |A_2A_i| = 0$ and $|A_2A_{i+1}| \neq 0$ for an arbitrary $i \geq 2$, then $L_4 = \cdots = L_{2i+2} = 0$ and

$$L_{2i+3} = -2i(i-1) \iint_D \rho K_2 K_{i+1} dK_2 \wedge dK_{i+1} \tag{4}$$

from Lemma 1 and Lemma 2. And the value of (4) is not 0 since $|A_2A_{i+1}| \neq 0$ and $P_2(\Gamma)(\alpha, \beta) = 0$. Hence if $L_{2i+3} = 0$, then $|A_2A_{i+1}| = 0$. Therefore if $W_{\gamma^*}(f, g) = \mathrm{id}$ admits solutions $f, g \neq \mathrm{id}$ tangent to the identity with (α, β), then f, g commute by Theorem 5.3 [2].

For an $(\alpha, \beta) \in \mathbb{C}^2$ except for the above one, $P_2(\Gamma)(\alpha, \beta) \neq 0$. Hence if $W_{\gamma^*}(f, g) = \mathrm{id}$ admits formal solutions $f, g \neq \mathrm{id}$ tangent to the identity with (α, β), then f, g commute by Theorem 5.1 [2].

3. Let Γ be a palette diagram consisting of 4 unit weighted squares with the type (C).

Let $(\alpha, \beta) \in \mathbb{C}^2$ satisfy the condition that $P_2(\Gamma)(\alpha, \beta) = 0$ and $P_3(\Gamma)(\alpha, \beta) \neq 0$. If $W_{\gamma^*}(f, g) = \mathrm{id}$ admits formal solutions $f, g \neq \mathrm{id}$ tangent to the identity with (α, β), then f, g commute (Theorem 5.5 [2]).

For an $(\alpha, \beta) \in \mathbb{C}^2$ except for the above one, $P_2(\Gamma)(\alpha, \beta) \neq 0$. Hence if $W_{\gamma^*}(f, g) = \mathrm{id}$ admits formal solutions $f, g \neq \mathrm{id}$ tangent to the identity with (α, β), then f, g commute by Theorem 5.1 [2].

4. Let Γ be a palette diagram consisting of 4 unit weighted squares with the type (D).

Let $(\alpha, \beta) \in \mathbb{C}^2$ satisfy the condition that $P_2(\Gamma)(\alpha, \beta) = 0$ and $P_3(\Gamma)(\alpha, \beta) = 0$. Assume $A_3, \ldots, A_{i+1}$ satisfy $|A_2A_3| = |A_2A_4| = \cdots = |A_2A_i| = 0$ and $|A_2A_{i+1}| \neq 0$ for an arbitrary $i \geq 2$, then $L_4 = \cdots = L_{2i+2} = 0$ and

$$L_{2i+3} = -2i(i-1) \iint_D \rho K_2 K_{i+1} dK_2 \wedge dK_{i+1} \tag{5}$$

from Lemma 1 and Lemma 2. Here the value of (5) is 0 for both (D-2) and (D-22) in [2]. Hence whether $W_{\gamma^*}(f,g) = \mathrm{id}$ admits formal solutions $f, g \neq \mathrm{id}$ non commuting and tangent to the identity or not is not known for now by our study.

For an $(\alpha, \beta) \in \mathbb{C}^2$ except for the above one, $P_2(\Gamma)(\alpha,\beta) \neq 0$. Hence if $W_{\gamma^*}(f,g) = \mathrm{id}$ admits formal solutions $f, g \neq \mathrm{id}$ tangent to the identity with (α,β), then f, g commute by Theorem 5.1 [2].

Therefore Theorem 1 is proved. □

Remark 1. The following is the computation of $P_3(\Gamma_{15})(1,0)$.

$$\begin{aligned}
P_3(\Gamma_{15})(1,0) &= p\int_{b+1}^{b+2} dy \int_a^{a+1} x^3 dx - 3p\int_b^{b+1} dy \int_{a+1}^{a+2} x^3 dx + 3p\int_b^{b+1} dy \int_{a+2}^{a+3} x^3 dx \\
&\quad -p\int_{b+1}^{b+2} dy \int_{a+3}^{a+4} x^3 dx \\
&= p\left[\frac{x^4}{4}\right]_a^{a+1} - 3p\left[\frac{x^4}{4}\right]_{a+1}^{a+2} + 3p\left[\frac{x^4}{4}\right]_{a+2}^{a+3} - p\left[\frac{x^4}{4}\right]_{a+3}^{a+4} \\
&= p\frac{(a+1)^4 - a^4}{4} - 3p\frac{(a+2)^4-(a+1)^4}{4} + 3p\frac{(a+3)^4-(a+2)^4}{4} \\
&\quad -p\frac{(a+4)^4-(a+3)^4}{4} \\
&= -6p.
\end{aligned}$$

$P_3(\Gamma_{15})(1,0)$ is also computed by Mathematica by the following commands written in boldface. Their outputs are written under them.

Integrate[x^3, {x, a, a + 1}, {y, b + 1, b + 2}]
Integrate[x^3, {x, a + 1, a + 2}, {y, b, b + 1}]
Integrate[x^3, {x, a + 2, a + 3}, {y, b, b + 1}]
Integrate[x^3, {x, a + 3, a + 4}, {y, b + 1, b + 2}]
$Out[1] = \frac{1}{4} + a + \frac{3a^2}{2} + a^3$
$Out[2] = \frac{15}{4} + 7a + \frac{9a^2}{2} + a^3$
$Out[3] = \frac{65}{4} + 19a + \frac{15a^2}{2} + a^3$
$Out[4] = \frac{175}{4} + 37a + \frac{21a^2}{2} + a^3$
Simplify[pOut[1] − 3pOut[2] + 3pOut[3] − pOut[4]]
$Out[5] = -6p$

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References

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