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Classification of connected palette diagrams without area and moment to find relations of formal diffeomorphisms

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Abstract

For a polygonal path γ consisting of unit vectors in x-direction and y-direction in the plane $\mathbb{R}^2$, a relation $W_{\gamma^*}(f,g) = \mathrm{id}$ of f, g is defined, where id denotes the identity map of $\mathbb{C}$. Some sufficient conditions of γ so that $W_{\gamma^*}(f,g) = \mathrm{id}$ admits solutions of non commuting formal diffeomorphisms tangent to the identity have already been obtained in [4]. In this paper, we define a palette diagram and classify all connected palette diagrams without area and moment consisting of four unit weighted squares into 4 types to find γ so that $W_{\gamma^*}(f,g) = \mathrm{id}$ admits solutions of non commuting formal diffeomorphisms tangent to the identity among those diagrams. We also give some concrete examples of relations of two formal diffeomorphisms.

1 Introduction

Let us consider a unit square $S = (a, a+1) \times (b, b+1)$, $a, b \in \mathbb{Z}$, in the real 2-plane $\mathbb{R}^2$ with vertices on lattice points, where $(*,*)$ denotes an open interval. We attach a non zero integer $p \in \mathbb{Z}^*$ to S and call it a unit square with weight p, and denote by $[S, p]$. In this paper, we define a *palette diagram* Γ consisting of n unit weighted squares to be a collection of n distinct unit weighted squares

$$\Gamma = \Big\{[S_1, p_1], [S_2, p_2], \ldots, [S_n, p_n]\Big\}. \tag{1}$$

For a palette diagram Γ, we call a collection of n unit squares

$$\tilde{\Gamma} = \Big\{S_1, S_2, \ldots, S_n\Big\}$$

the base of Γ. We say a palette diagram (or its base) is *connected* if every square S_i has at least one vertex in common with another square $S_j, j \neq i$. Hence a palette diagram (or its base) is *non connected* if there exists at least one square which has no vertex in common with any other squares.

The Area and Moment of a palette diagram Γ in (1) (or a Feynman diagram γ obtained from Γ, which is defined in the next) are defined by

$$\mathrm{Area}(\Gamma) = \sum_{i=1}^{n} p_i, \quad G(\Gamma) = \left(\sum_{i=1}^{n} p_i \iint_{S_i} x\, dx \wedge dy, \sum_{i=1}^{n} p_i \iint_{S_i} y\, dx \wedge dy\right)$$

respectively. And for Γ we define the moment matrix $M(\Gamma)$ by the symmetric matrix

$$M(\Gamma) = \begin{bmatrix} \sum_{i=1}^{n} p_i \iint_{S_i} x^2\, dx \wedge dy & \sum_{i=1}^{n} p_i \iint_{S_i} xy\, dx \wedge dy \\ \sum_{i=1}^{n} p_i \iint_{S_i} xy\, dx \wedge dy & \sum_{i=1}^{n} p_i \iint_{S_i} y^2\, dx \wedge dy \end{bmatrix},$$

and define the polynomial $P_k(\Gamma)(\alpha,\beta)$ of two complex variables α,β of degree k with real coefficients by

$$P_k(\Gamma)(\alpha,\beta)=\sum_{i=1}^{n}p_i\iint_{S_i}(\alpha x+\beta y)^k\,dx\wedge dy.$$

We see that the equation of α,β of degree 2

$$P_2(\Gamma)(\alpha,\beta)=m_{11}\alpha^2+2m_{12}\alpha\beta+m_{22}\beta^2=0, \tag{2}$$

where $m_{11}=\sum_{i=1}^{n}p_i\iint_{S_i}x^2\,dx\wedge dy, m_{12}=\sum_{i=1}^{n}p_i\iint_{S_i}xy\,dx\wedge dy,\ m_{22}=\sum_{i=1}^{n}p_i\iint_{S_i}y^2\,dx\wedge dy$, determines two distinct lines in $\alpha\beta$-plane if $\det M(\Gamma)\neq 0$. And (2) determines one line if $\det M(\Gamma)=0$ and $M(\Gamma)\neq 0$.

Definition 1.1. *If a palette diagram Γ has the following property (A) or (B) or (C) or (D) :*

(A) *$Area(\Gamma)=0$, $G(\Gamma)=(0,0)$, $\det M(\Gamma)\neq 0$ and there exists an (α,β), $(\alpha,\beta)\neq(0,0)$, such that $P_2(\Gamma)(\alpha,\beta)=0$ and $P_3(\Gamma)(\alpha,\beta)\neq 0$,*

(B) *$Area(\Gamma)=0$, $G(\Gamma)=(0,0)$, $\det M(\Gamma)\neq 0$ and there exists an (α,β), $(\alpha,\beta)\neq(0,0)$, such that $P_2(\Gamma)(\alpha,\beta)=P_3(\Gamma)(\alpha,\beta)=0$,*

(C) *$Area(\Gamma)=0$, $G(\Gamma)=(0,0)$, $\det M(\Gamma)=0$ and there exists an (α,β), $(\alpha,\beta)\neq(0,0)$, such that $P_2(\Gamma)(\alpha,\beta)=0$ and $P_3(\Gamma)(\alpha,\beta)\neq 0$,*

(D) *$Area(\Gamma)=0$, $G(\Gamma)=(0,0)$, $\det M(\Gamma)=0$ and there exists an (α,β), $(\alpha,\beta)\neq(0,0)$, such that $P_2(\Gamma)(\alpha,\beta)=P_3(\Gamma)(\alpha,\beta)=0$,*

we say Γ has the type (A) or (B) or (C) or (D).

For a palette diagram Γ in (1) there exists a closed polygonal path γ in $\mathbb{R}^2$ which have integers $p_1,p_2,\ldots,p_n$ as values of the winding number of γ on the square domains $S_1,S_2,\ldots,S_n$ respectively. Here the winding number $\rho(\gamma)=\rho(\gamma)(x,y)$ at a point $(x,y)\in\mathbb{R}^2$ of a closed path γ is the number that γ winds around (x,y). We call this γ a closed *Feynman diagram* obtained from Γ. We note that for a palette diagram Γ there exist infinitely many (but countable) distinct Feynman diagrams. For example, there exist three distinct Feynman diagrams $\gamma_1,\gamma_2,\gamma_3$ obtained from a palette diagram $\Gamma=\{[S,1]\}$ in Figure 1. The base point of a diagram may be chosen arbitrarily in $\mathbb{R}^2$.

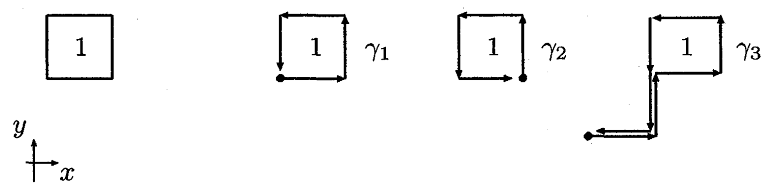


Figure 1: Examples of Feynman diagrams $\gamma_1,\gamma_2,\gamma_3$ obtained from $\Gamma=\{[S,1]\}$

Let H be a unit horizontal vector in x-positive direction and V a unit vertical vector in y-positive direction in $\mathbb{R}^2$. Then in general, a Feynman diagram γ is expressed as

$$\gamma=H^{n_1}*V^{n_2}*H^{n_3}*V^{n_4}*\cdots*H^{n_{2p-1}}*V^{n_{2p}},\quad n_1,n_2,\ldots,n_{2p}\in\mathbb{Z}^*, \tag{3}$$

where $*$ denotes the composite of paths and H^n stands for the n-fold composite of H. For example

$$\gamma_1=H*V*H^{-1}*V^{-1}=\{H^{-1},V^{-1}\},\qquad \gamma_2=V*H^{-1}*V^{-1}*H=\{V^{-1},H\},$$

$$\gamma_3=H*V*H*V*H^{-1}*V^{-1}*V^{-1}*H^{-1}=H*V*\gamma_1*V^{-1}*H^{-1}$$

for the diagrams in Figure 1, where $\{H, V\}$ denotes $H^{-1} * V^{-1} * H * V$. From γ in (3), we obtain a word

$$W_{\gamma^*}(f,g) = f^{(n_1)} \circ g^{(n_2)} \circ f^{(n_3)} \circ g^{(n_4)} \circ \cdots \circ f^{(n_{2p-1})} \circ g^{(n_{2p})}$$

of $f, g, f^{(-1)}, g^{(-1)}$ for two holomorphic diffeomorphisms $f, g \in \mathrm{Diff}(\mathbb{C}, 0)$ by substituting H and V in the expression of γ by f and g respectively, where $f^{(m)}$ stands for the m-fold iteration of f, and $\mathrm{Diff}(\mathbb{C}, 0)$ denotes the group of germs of holomorphic diffeomorphisms of $\mathbb{C}$ fixing 0

$$\mathrm{Diff}(\mathbb{C}, 0) = \{f(z) = a_1 z + a_2 z^2 + \cdots | a_1 \neq 0, a_i \in \mathbb{C}\}$$

and $\circ$ denotes the composite of mappings. We have

$$W_{\gamma_1^*}(f,g) = \{f^{(-1)}, g^{(-1)}\}, \qquad W_{\gamma_2^*}(f,g) = \{g^{(-1)}, f\},$$

$$W_{\gamma_3^*}(f,g) = f \circ g \circ W_{\gamma_1^*}(f,g) \circ g^{(-1)} \circ f^{(-1)}$$

then, where $\{f, g\} = f^{(-1)} \circ g^{(-1)} \circ f \circ g$. The relation of f, g defined for γ in (3) is the equation

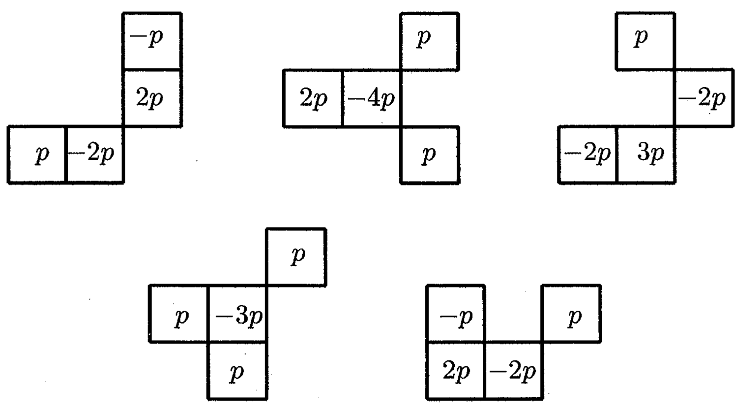
$$W_{\gamma^*}(f,g) = \mathrm{id}. \tag{4}$$

We say a relation (4) have length l if $|n_1| + \cdots + |n_{2p}| = l$.

J. Ecalle and B. Vallet [1] constructed various types of relations of two formal diffeomorphisms tangent to identity in the group $\widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ of formal diffeomorphisms. F. Loray [2] investigated those relations in the study of non solvable subgroups of $\mathrm{Diff}(\mathbb{C}, 0)$ from the view point of real and complex codimension one foliation. While, the structure of non solvable sub groups is not well known [2, 3].

In this paper we classify all connected palette diagrams consisting of four unit weighted squares into the above 4 types (A) $\sim$ (D). The result of the classification is stated in §3 (Theorem 3.1, 3.2, 3.3 and 3.4). The following Theorem 3.1 on the type (A) is important in this paper to obtain relations of formal diffeomorphisms.

Theorem 3.1. *A connected palette diagram Γ consisting of four unit weighted squares with the type (A) equals up to congruence and reflection to one of the diagrams in the below, where p is a non zero integer.*

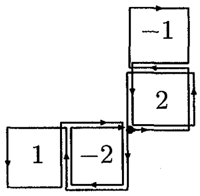


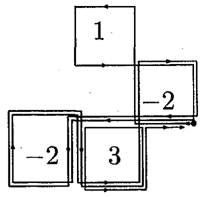
Moreover, we give classification for some non connected palette diagrams (Example 3.1). In §4, we prove the classification theorem. In §5, we apply this classification to obtain relations of two formal diffeomorphisms non commute and tangent to identity. See Theorem 5.4, 5.5 and 5.6. The main theorem in this paper is Theorem 5.4 in the following:

Theorem 5.4. *Assume Γ equals the one of five in the chart in Theorem 3.1, then for all Feynman diagrams γ obtained from Γ, the relation $W_{\gamma^*}(f,g) = id$ admits formal solutions $f, g \neq id$*

non commute and tangent to identity such that $(f''(0) : g''(0)) = (\alpha : \beta)$, *where* (α, β) *satisfies* $P_2(\gamma)(\alpha, \beta) = 0$ *and* $P_3(\gamma)(\alpha, \beta) \neq 0$.

We can obtain many ample relations of formal diffeomorphisms non commute and tangent to identity from a palette diagram in Theorem 3.1 with the type (A) by Theorem 5.4. We give seven examples of relations of formal diffeomorphisms in §5 defined for seven Feynman diagrams concluding the following two diagrams.





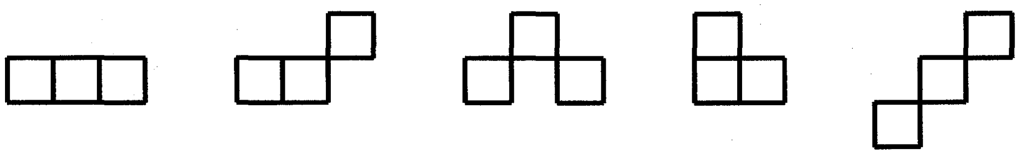
The author will determine the number of relations with minimal length obtained from five palette diagrams in Theorem 3.1. In Appendix, we give the classification of connected palette diagrams without area and moment consisting of two or three unit weighted squares.

2 The number of basis of connected palette diagrams consisting of four unit squares

Here we determine the number of basis of connected palette diagrams consisting of four unit squares by enumerating those basis systematically.

There exist exactly 5 basis of connected palette diagrams consisting of three unit squares in the followings.

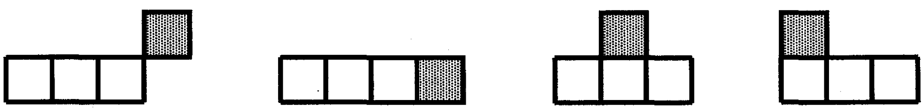
① ② ③ ④ ⑤



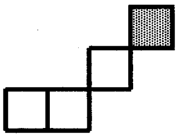
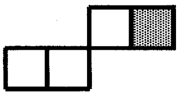
We regard basis of palette diagrams (resp. palette diagrams) up to congruence and reflection to be equivalent. Then we find basis of connected palette diagrams consisting of four unit squares by adding a square to one of ① ~ ⑤.

By adding a square to ①, we find exactly 4 distinct basis (1) ~ (4) of connected palette diagrams consisting of four unit squares in the followings.

(1) (2) (3) (4)



By adding a square to ②, we find exactly 12 distinct basis (5) ~ (16) of connected palette diagrams consisting of four unit squares in addition to (1), (4) in the followings.

(5)

(6)

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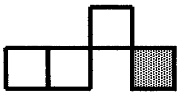
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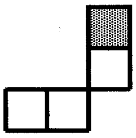
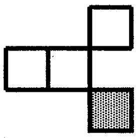
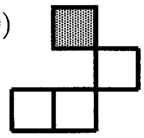
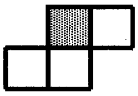


(9)

(10)

(11)

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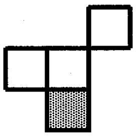
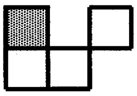
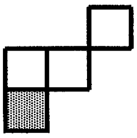
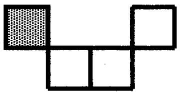
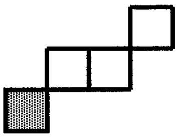
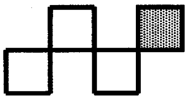
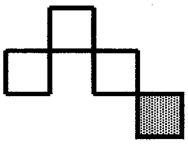
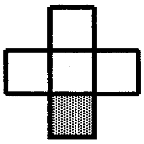
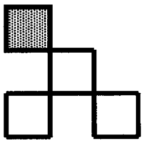
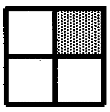
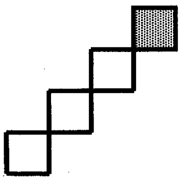


(13)

(14)

(15)

(16)


By adding a square to ③, we find exactly 4 distinct basis (17) $\sim$ (20) of connected palette diagrams consisting of four unit squares in addition to (3), (7), (9), (10), (13) in the followings.

(17)

(18)

(19)

(20)


By adding a square to ④, we find exactly 1 base (21) of connected palette diagrams consisting of four unit squares in addition to (3), (4), (11),(12), (13), (14). And by adding a square to ⑤, we find exactly 1 base (22) of connected palette diagrams consisting of four unit squares in addition to (5), (14).

(21)

(22)


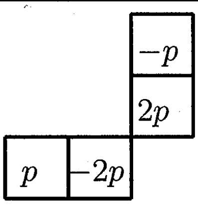
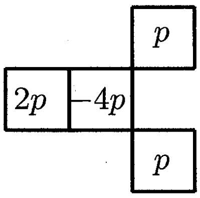
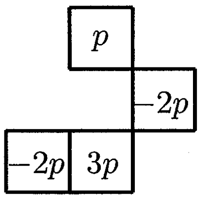
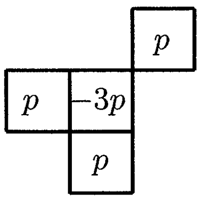
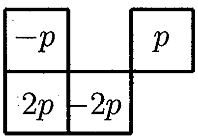
Therefore we obtain the following Lemma 2.1.

Lemma 2.1. *There exist exactly 22 distinct basis of connected palette diagrams consisting of four unit squares.*

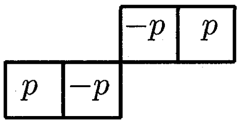
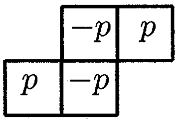
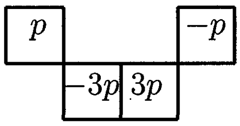
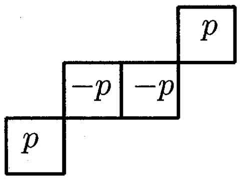
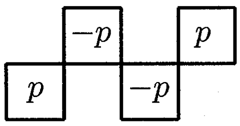
3 Result of the classification of connected palette diagrams without area and moment consisting of four unit weighted squares

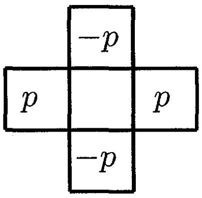
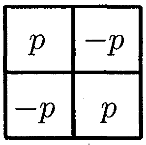
Here we state the classification theorem obtained.

Theorem 3.1. *A connected palette diagram* Γ *consisting of four unit weighted squares with the type (A) equals up to congruence and reflection to one of the diagrams in the chart below, where* p *is a non zero integer.*

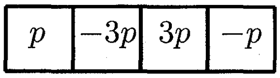
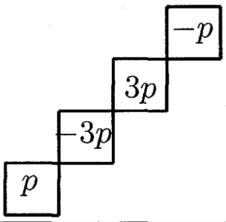
(A)			
N0.	Γ	$M(\Gamma)$	$(\alpha : \beta)$
(8)		$\begin{bmatrix} 2p & 0 \\ 0 & -2p \end{bmatrix}$	$(1 : 1)$
(9)		$\begin{bmatrix} 4p & 0 \\ 0 & 2p \end{bmatrix}$	$(\pm\sqrt{-2} : 2)$
(10)		$\begin{bmatrix} -4p & -2p \\ -2p & 2p \end{bmatrix}$	$(-1 \pm \sqrt{3} : 2)$
(12)		$\begin{bmatrix} 2p & p \\ p & 2p \end{bmatrix}$	$(-1 \pm \sqrt{-3} : 2)$
(13)		$\begin{bmatrix} 2p & 2p \\ 2p & 0 \end{bmatrix}$	$(2 : -1)$

Theorem 3.2. *A connected palette diagram* Γ *consisting of four unit weighted squares with the type (B) equals up to congruence and reflection to one of the diagrams in the chart below, where* p *is a non zero integer.*

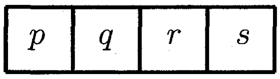
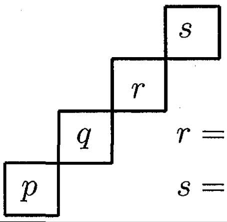
(B)			
N0.	Γ	$M(\Gamma)$	$(\alpha:\beta)$
(6)		$\begin{bmatrix} 4p & p \\ p & 0 \end{bmatrix}$	$(0:1),\ (1:-2)$
(11)		$\begin{bmatrix} 2p & p \\ p & 0 \end{bmatrix}$	$(0:1),\ (1:-1)$
(15)		$\begin{bmatrix} 0 & -3p \\ -3p & 0 \end{bmatrix}$	$(1:0),\ (0:1)$
(16)		$\begin{bmatrix} 4p & 3p \\ 3p & 2p \end{bmatrix}$	$(1:-1),\ (1:-2)$
(17)		$\begin{bmatrix} 4p & 2p \\ 2p & 0 \end{bmatrix}$	$(0:1),\ (1:-1)$

(B)			
No.	Γ	$M(\Gamma)$	$(\alpha:\beta)$
(19)		$\begin{bmatrix} 2p & 0 \\ 0 & -2p \end{bmatrix}$	$(1:1)$, $(1:-1)$
(21)		$\begin{bmatrix} 0 & -p \\ -p & 0 \end{bmatrix}$	$(1:0)$, $(0:1)$

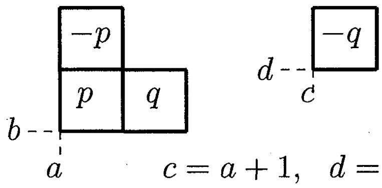
Theorem 3.3. *A connected palette diagram Γ consisting of four unit weighted squares with the type (C) equals up to congruence and reflection to one of the diagrams in the chart below, where p is a non zero integer.*

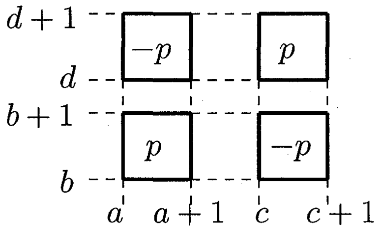
(C)			
No.	Γ	$M(\Gamma)$	$(\alpha:\beta)$
(2)		$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$(\alpha:\beta)$ $\alpha \neq 0$
(22)		$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$(\alpha:\beta)$ $\alpha+\beta \neq 0$

Theorem 3.4. *A connected palette diagram Γ consisting of four unit weighted squares with the type (D) equals up to congruence and reflection to one of the diagrams in the chart below, where p, q, r, s are non zero integers.*

(D)			
No.	Γ	$M(\Gamma)$	$(\alpha:\beta)$
(2)	 $r=-3p-2q, \quad s=2p+q$	$\begin{bmatrix} 2(3p+q) & 0 \\ 0 & 0 \end{bmatrix}$ $3p+q\neq 0$	$(0:1)$
(22)	 $r=-3p-2q$ $s=2p+q$	$\begin{bmatrix} 2(3p+q) & 2(3p+q) \\ 2(3p+q) & 2(3p+q) \end{bmatrix}$ $3p+q\neq 0$	$(1:-1)$

Example 3.1. *There exist examples of non connected palette diagrams consisting of four unit weighted squares with the type (A) or (B) in the followings. (See Example 4.1 and 4.2 in §4).*

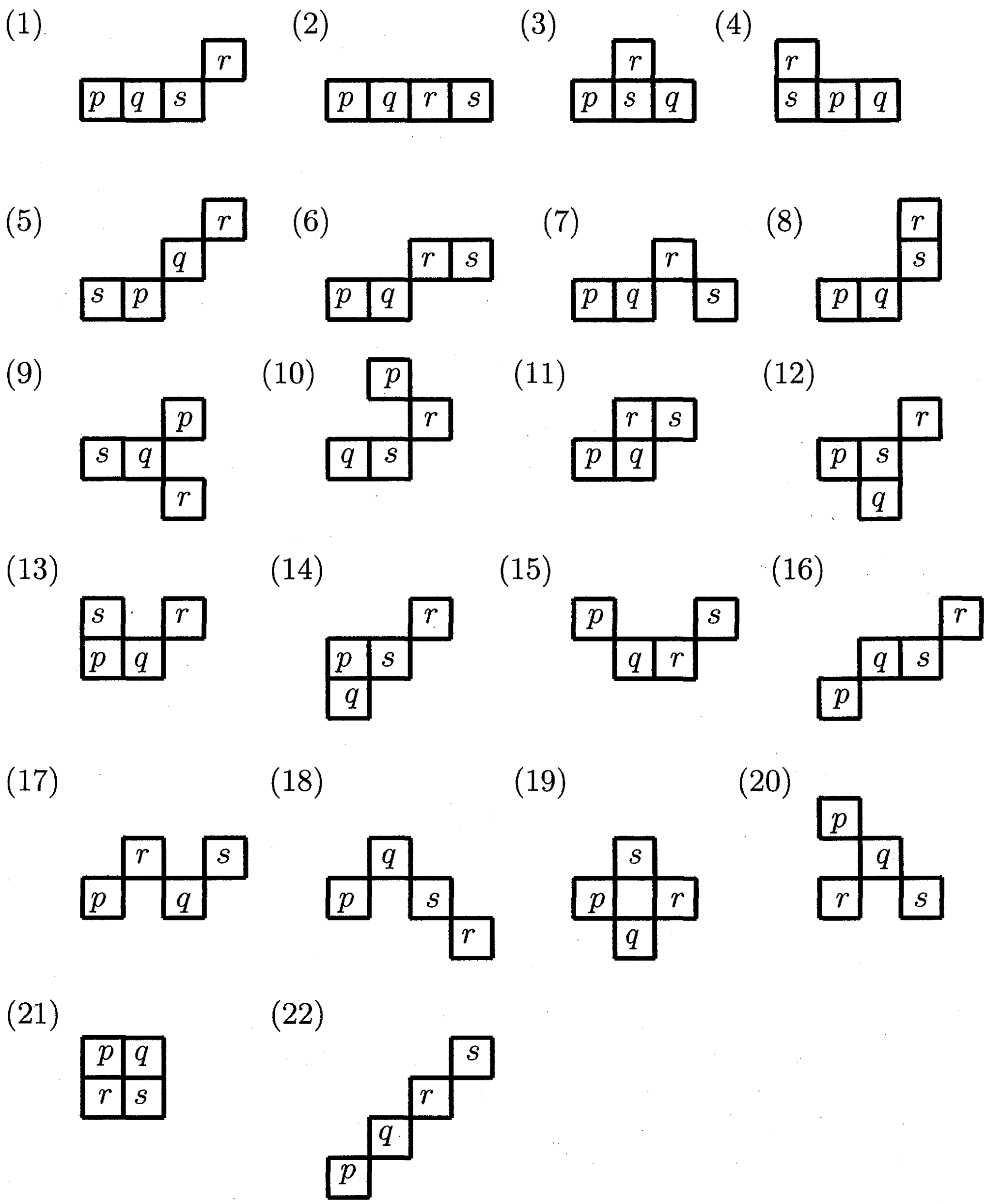
(A)		
Γ	$M(\Gamma)$	$(\alpha:\beta)$
 $c=a+1, \quad d=b-\frac{p}{q}$ $p\neq q, \ p+q\neq 0, \ \frac{p}{q}\in\mathbb{Z}^*$	$\begin{bmatrix} 0 & p \\ p & -\frac{p(p+q)}{q} \end{bmatrix}$	$(p+q:2q)$

(B)		
Γ	$M(\Gamma)$	$(\alpha:\beta)$
	$\begin{bmatrix} 0 & m_{12} \\ m_{12} & 0 \end{bmatrix}$ $m_{12}=(a-c)(b-d)p$	$(1:0), \ (0:1)$

4 Proof of the classification theorem

Here we give the proof of the classification theorem Theorem 3.1 $\sim$ 3.4.

First, we obtain 22 distinct connected palette diagrams consisting of four unit weighted squares in the followings by attaching non zero integers $p, q, r, s \in \mathbb{Z}^*$ to four unit squares of basis obtained in Lemma 2.1 of §2.



The followings are their expressions.

(1) $\Gamma_1 = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+3,a+4)\times(b+1,b+2),r], & [(a+2,a+3)\times(b,b+1),s] \end{matrix}\right\}$

(2) $\Gamma_2 = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+2,a+3)\times(b,b+1),r], & [(a+3,a+4)\times(b,b+1),s] \end{matrix}\right\}$

(3) $\Gamma_3 = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+2,a+3)\times(b,b+1),q] \\ [(a+1,a+2)\times(b+1,b+2),r], & [(a+1,a+2)\times(b,b+1),s] \end{matrix}\right\}$

(4) $\Gamma_4 = \left\{\begin{matrix} [(a+1,a+2)\times(b,b+1),p], & [(a+2,a+3)\times(b,b+1),q] \\ [(a,a+1)\times(b+1,b+2),r], & [(a,a+1)\times(b,b+1),s] \end{matrix}\right\}$

(5) $\Gamma_5 = \left\{\begin{matrix} [(a+1,a+2)\times(b,b+1),p], & [(a+2,a+3)\times(b+1,b+2),q] \\ [(a+3,a+4)\times(b+2,b+3),r], & [(a,a+1)\times(b,b+1),s] \end{matrix}\right\}$

(6) $\Gamma_6 = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+2,a+3)\times(b+1,b+2),r], & [(a+3,a+4)\times(b+1,b+2),s] \end{matrix}\right\}$

(7) $\Gamma_7 = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+2,a+3)\times(b+1,b+2),r], & [(a+3,a+4)\times(b,b+1),s] \end{matrix}\right\}$

(8) $\Gamma_8 = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+2,a+3)\times(b+2,b+3),r], & [(a+2,a+3)\times(b+1,b+2),s] \end{matrix}\right\}$

(9) $\Gamma_9 = \left\{\begin{matrix} [(a+2,a+3)\times(b+2,b+3),p], & [(a+1,a+2)\times(b+1,b+2),q] \\ [(a+2,a+3)\times(b,b+1),r], & [(a,a+1)\times(b+1,b+2),s] \end{matrix}\right\}$

(10) $\Gamma_{10} = \left\{\begin{matrix} [(a+1,a+2)\times(b+2,b+3),p] & [(a,a+1)\times(b,b+1),q] \\ [(a+2,a+3)\times(b+1,b+2),r], & [(a+1,a+2)\times(b,b+1),s] \end{matrix}\right\}$

(11) $\Gamma_{11} = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+1,a+2)\times(b+1,b+2),r], & [(a+2,a+3)\times(b+1,b+2),s] \end{matrix}\right\}$

(12) $\Gamma_{12} = \left\{\begin{matrix} [(a,a+1)\times(b+1,b+2),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+2,a+3)\times(b+2,b+3),r], & [(a+1,a+2)\times(b+1,b+2),s] \end{matrix}\right\}$

(13) $\Gamma_{13} = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+2,a+3)\times(b+1,b+2),r], & [(a,a+1)\times(b+1,b+2),s] \end{matrix}\right\}$

(14) $\Gamma_{14} = \left\{\begin{matrix} [(a,a+1)\times(b+1,b+2),p], & [(a,a+1)\times(b,b+1),q] \\ [(a+2,a+3)\times(b+2,b+3),r], & [(a+1,a+2)\times(b+1,b+2),s] \end{matrix}\right\}$

(15) $\Gamma_{15} = \left\{\begin{matrix} [(a,a+1)\times(b+1,b+2),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+2,a+3)\times(b,b+1),r], & [(a+3,a+4)\times(b+1,b+2),s] \end{matrix}\right\}$

(16) $\Gamma_{16} = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b+1,b+2),q] \\ [(a+3,a+4)\times(b+2,b+3),r], & [(a+2,a+3)\times(b+1,b+2),s] \end{matrix}\right\}$

(17) $\Gamma_{17} = \left\{\begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+2,a+3)\times(b,b+1),q] \\ [(a+1,a+2)\times(b+1,b+2),r], & [(a+3,a+4)\times(b+1,b+2),s] \end{matrix}\right\}$

(18) $\Gamma_{18} = \left\{\begin{matrix} [(a,a+1)\times(b+1,b+2),p], & [(a+1,a+2)\times(b+1,b+2),q] \\ [(a+3,a+4)\times(b,b+1),r], & [(a+2,a+3)\times(b+1,b+2),s] \end{matrix}\right\}$

(19) $\Gamma_{19} = \left\{\begin{matrix} [(a,a+1)\times(b+1,b+2),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(a+2,a+3)\times(b+1,b+2),r], & [(a+1,a+2)\times(b+2,b+3),s] \end{matrix}\right\}$

$$(20)\ \Gamma_{20} = \begin{Bmatrix} [(a,a+1)\times(b+2,b+3),p], & [(a+1,a+2)\times(b+1,b+2),q] \\ [(a,a+1)\times(b,b+1),r], & [(a+2,a+3)\times(b,b+1),s] \end{Bmatrix}$$

$$(21)\ \Gamma_{21} = \begin{Bmatrix} [(a,a+1)\times(b+1,b+2),p], & [(a+1,a+2)\times(b+1,b+2),q] \\ [(a,a+1)\times(b,b+1),r], & [(a+1,a+2)\times(b,b+1),s] \end{Bmatrix}$$

$$(22)\ \Gamma_{22} = \begin{Bmatrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b+1,b+2),q] \\ [(a+2,a+3)\times(b+2,b+3),r], & [(a+3,a+4)\times(b+3,b+4),s] \end{Bmatrix}$$

Here let $s = -p-q-r$ so that diagrams have 0 area.

Secondly we compute the moments of the above 22 diagrams. The results of the computations of the moments and the conditions of non zero integers p, q, r so that $G(\Gamma_i) = (0,0)$ for $i = 1 \sim 22$ are in the followings.

(1) $G(\Gamma_1) = (-2p-q+r, r) \neq (0,0)$.

(2) $G(\Gamma_2) = (-3p-2q-r, 0) = (0,0) \Leftrightarrow r = -3p-2q$.

(3) $G(\Gamma_3) = (-p+q, r) \neq (0,0)$.

(4) $G(\Gamma_4) = (p+2q, r) \neq (0,0)$.

(5) $G(\Gamma_5) = (p+2q+3r, q+2r) \neq (0,0)$.

(6) $G(\Gamma_6) = (-3p-2q-r, -p-q) = (0,0) \Leftrightarrow q = -p, r = -p$.

(7) $G(\Gamma_7) = (-3p-2q-r, r) \neq (0,0)$.

(8) $G(\Gamma_8) = (-2p-q, -p-q+r) = (0,0) \Leftrightarrow q = -2p, r = -p$.

(9) $G(\Gamma_9) = (2p+q+2r, p-r) = (0,0) \Leftrightarrow q = -4p, r = p$.

(10) $G(\Gamma_{10}) = (-q+r, 2p+r) = (0,0) \Leftrightarrow r = q, r = -2p$.

(11) $G(\Gamma_{11}) = (-2p-q-r, -p-q) = (0,0) \Leftrightarrow q = -p, r = -p$.

(12) $G(\Gamma_{12}) = (-p+r, -q+r) = (0,0) \Leftrightarrow p = q = r$.

(13) $G(\Gamma_{13}) = (q+2r, -p-q) = (0,0) \Leftrightarrow p = 2r, q = -2r$.

(14) $G(\Gamma_{14}) = (-p-q+r, -q+r) \neq (0,0)$.

(15) $G(\Gamma_{15}) = (-3p-2q-r, -q-r) = (0,0) \Leftrightarrow q = -3p, r = 3p$.

(16) $G(\Gamma_{16}) = (-2p-q+r, -p+r) = (0,0) \Leftrightarrow r = p, q = -p$.

(17) $G(\Gamma_{17}) = (-3p-q-2r, -p-q) = (0,0) \Leftrightarrow q = -p, r = -p$.

(18) $G(\Gamma_{18}) = (-2p-q+r, q-r) \neq (0,0)$.

(19) $G(\Gamma_{19}) = (-p+r, -p-2q-r) = (0,0) \Leftrightarrow q = -p, r = p$.

(20) $G(\Gamma_{20}) = (-2p-q-2r, 2p+q) \neq (0,0)$.

(21) $G(\Gamma_{21}) = (-r-p, p+q) = (0,0) \Leftrightarrow q = -p, r = -p$.

(22) $G(\Gamma_{22}) = (-3p-2q-r, -3p-2q-r) = (0,0) \Leftrightarrow r = -3p-2q$.

We see that diagrams $\Gamma_i, i = 1,3,4,5,7,14,18,20$ have non zero moment since $p, q, r, -p-q-r \neq 0$. So we need not consider them from now on.

Thirdly we compute the elements of the moment matrix $M(\Gamma_i)$ and find the complex lines $(\alpha : \beta)$ that the equation $P_2(\Gamma_i)(\alpha, \beta) = 0$ determines for the rest of 14 diagrams $i = 2,6,8,9,10,11,12,13,15,16,17,19,21,22$ without area and moment under the above integer conditions. The results of the computations of $M(\Gamma_i)$ are stated in the charts in §3. We see that the determinant of $M(\Gamma_i)$ is not 0 for $i = 6,8,9,10,11,12,13,15,16,17,19,21$ and 0 for $i = 2,22$. And the followings are the equation (2) $P_2(\Gamma_i)(\alpha, \beta) = 0$ and the lines it determines for $i = 2,6,8,9,10,11,12,13,15,16,17,19,21,22$.

(2) $P_2(\Gamma_2)(\alpha,\beta) = 2(3p+q)\alpha^2 = 0 \Leftrightarrow (\alpha:\beta) = (0:1)$ if $3p+q \neq 0$. (α,β) is arbitrary if $3p+q=0$.

(6) $P_2(\Gamma_6)(\alpha,\beta) = 4p\alpha^2 + 2p\alpha\beta = 0 \Leftrightarrow (\alpha:\beta) = (0:1),(1:-2)$.

(8) $P_2(\Gamma_8)(\alpha,\beta) = 2p\alpha^2 - 2p\beta^2 = 0 \Leftrightarrow (\alpha:\beta) = (1:1),(1:-1)$.

(9) $P_2(\Gamma_9)(\alpha,\beta) = 4p\alpha^2 + 2p\beta^2 = 0 \Leftrightarrow (\alpha:\beta) = (\pm\sqrt{-2}:2)$.

(10) $P_2(\Gamma_{10})(\alpha,\beta) = -4p\alpha^2 - 4p\alpha\beta + 2p\beta^2 = 0 \Leftrightarrow (\alpha:\beta) = (-1\pm\sqrt{3}:2)$.

(11) $P_2(\Gamma_{11})(\alpha,\beta) = 2p\alpha^2 + 2p\alpha\beta = 0 \Leftrightarrow (\alpha:\beta) = (0:1),(1:-1)$.

(12) $P_2(\Gamma_{12})(\alpha,\beta) = 2p\alpha^2 + 2p\alpha\beta + 2p = 0 \Leftrightarrow (\alpha:\beta) = (-1\pm\sqrt{-3}:2)$.

(13) $P_2(\Gamma_{13})(\alpha,\beta) = 2p\alpha^2 + 4p\alpha\beta = 0 \Leftrightarrow (\alpha:\beta) = (0:1),(2:-1)$.

(15) $P_2(\Gamma_{15})(\alpha,\beta) = -6p\alpha\beta = 0 \Leftrightarrow (\alpha:\beta) = (1:0),(0:1)$.

(16) $P_2(\Gamma_{16})(\alpha,\beta) = 4p\alpha^2 + 6p\alpha\beta + 2p\beta^2 = 0 \Leftrightarrow (\alpha:\beta) = (1:-1),(1:-2)$.

(17) $P_2(\Gamma_{17})(\alpha,\beta) = 4p\alpha^2 + 4p\alpha\beta = 0 \Leftrightarrow (\alpha:\beta) = (0:1),(1:-1)$.

(19) $P_2(\Gamma_{19})(\alpha,\beta) = 2p\alpha^2 - 2p\beta^2 = 0 \Leftrightarrow (\alpha:\beta) = (1:1),(1:-1)$.

(21) $P_2(\Gamma_{21})(\alpha,\beta) = -2p\alpha\beta = 0 \Leftrightarrow (\alpha,\beta) = (1:0),(0:1)$.

(22) $P_2(\Gamma_{22})(\alpha,\beta) = 2(3p+q)\alpha^2 + 4(3p+q)\alpha\beta + 2(3p+q)\beta^2 = 0 \Leftrightarrow (\alpha:\beta) = (1:-1)$ if $3p+q \neq 0$. (α,β) is arbitrary if $3p+q=0$.

The followings are the values $P_3(\Gamma_i)(\alpha,\beta), i = 2,6,8,9,10,11,12,13,15,16,17,19,21,22$, for (α,β) such that $P_2(\Gamma_i)(\alpha,\beta)=0$.

(2) $P_3(\Gamma_2)(0,1) = 0$ if $3p+q \neq 0$. $P_3(\Gamma_1)(\alpha,\beta) = -6p\alpha^3$ if $3p+q=0$.

(6) $P_3(\Gamma_6)(0,1) = P_3(\Gamma_6)(1,-2) = 0$.

(8) $P_3(\Gamma_8)(1,1) = -12p \neq 0, P_3(\Gamma_8)(1,-1) = 0$.

(9) $P_3(\Gamma_9)(\sqrt{-2},2) = -P_3(\Gamma_9)(-\sqrt{-2},2) = 24\sqrt{-2}p \neq 0$.

(10) $P_3(\Gamma_{10})(-1+\sqrt{3},2) = P_3(\Gamma_{10})(-1-\sqrt{3},2) = 24p \neq 0$.

(11) $P_3(\Gamma_{11})(0,1) = P_3(\Gamma_{11})(1,-1) = 0$.

(12) $P_3(\Gamma_{12})(-1+\sqrt{-3},2) = P_3(\Gamma_{12})(-1-\sqrt{-3},2) = -24p \neq 0$.

(13) $P_3(\Gamma_{13})(0,1) = 0, P_3(\Gamma_{13})(2,-1) = 12p \neq 0$.

(15) $P_3(\Gamma_{15})(1,0) = P_3(\Gamma_{15})(0,1) = 0$.

(16) $P_3(\Gamma_{16})(1,-1) = P_3(\Gamma_{16})(1,-2) = 0$.

(17) $P_3(\Gamma_{17})(0,1) = P_3(\Gamma_{17})(1,-1) = 0$.

(19) $P_3(\Gamma_{19})(1,1) = P_3(\Gamma_{19})(1,-1) = 0$.

(21) $P_3(\Gamma_{21})(1,0) = P_3(\Gamma_{21})(0,1) = 0$.

(22) $P_3(\Gamma_{22})(1,-1) = 0$ if $3p+q \neq 0$. $P_3(\Gamma_{22})(\alpha,\beta) = -6p(\alpha+\beta)^3$ if $3p+q=0$.

Therefore, we see that diagrams $\Gamma_8, \Gamma_9, \Gamma_{10}, \Gamma_{12}, \Gamma_{13}$ are classified into (A), $\Gamma_6, \Gamma_{11}, \Gamma_{15}, \Gamma_{16}, \Gamma_{17}, \Gamma_{19}, \Gamma_{21}$ are classified into (B), and Γ_2, Γ_{22} are classified into (C) or (D). Palette diagrams are lined up in this order in the chart. The author used Mathematica for computations of elements of the moment matrix and values $P_3(\Gamma_i)(\alpha, \beta)$. □

We give examples of non connected palette diagrams consisting of four unit weighted squares with the type (A) or (B) in the followings.

Example 4.1 (Case of non connected palette diagrams consisting of four unit weighted squares with the type (A)). *For a non connected diagram* Γ

$$\Gamma = \left\{ \begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a+1,a+2)\times(b,b+1),q] \\ [(c,c+1)\times(d,d+1),r], & [(a,a+1)\times(b+1,b+2),-p-q-r] \end{matrix} \right\},$$

the moment is

$$G(\Gamma) = (q + r(c-a), -p-q-r+r(d-b))$$

and

$$G(\Gamma) = (0,0) \Leftrightarrow c = a - \frac{q}{r}, \quad d = b + \frac{p+q+r}{r}.$$

And then

$$M(\Gamma) = \begin{bmatrix} \frac{q(q+r)}{r} & -\frac{q(p+q+r)}{r} \\ -\frac{q(p+q+r)}{r} & \frac{(p+q)(p+q+r)}{r} \end{bmatrix}, \quad \det M(\Gamma) = \frac{pq(p+q+r)}{r} \neq 0,$$

and the equation (2) is

$$P_2(\Gamma)(\alpha,\beta) = \frac{q(q+r)}{r}\alpha^2 - \frac{2q(p+q+r)}{r}\alpha\beta + \frac{(p+q)(p+q+r)}{r}\beta^2 = 0. \tag{5}$$

If $q + r = 0$, *(5) becomes*

$$-2pq\alpha\beta + (p+q)p\beta^2 = 0$$

and this determines two lines

$$(\alpha : \beta) = (1:0), \quad (p+q : 2q),$$

and

$$P_3(\Gamma)(1,0) = 0, \quad P_3(\Gamma)(p+q, 2q) = 2pq(p+q)(p-q).$$

Hence Γ *is classified into type (A) if* $p + q \neq 0$ *and* $p - q \neq 0$, *then. If* $q + r \neq 0$, *(5) has solutions*

$$(\alpha : \beta) = (q(p+q+r) \pm \sqrt{-pqr(p+q+r)} : \ q(q+r)).$$

For $(\alpha, \beta) = (q(p+q+r) \pm \sqrt{-pqr(p+q+r)}, q(q+r))$,

$$P_3(\Gamma)(\alpha,\beta) = -\frac{pq^2(q+r)(p+q+r)\{2pqr + q^2 r \pm r\sqrt{-pqr(p+q+r)} + q(r^2 - \sqrt{-pqr(p+q+r)})\}}{r}.$$

Hence Γ *is classified into type (A) if this value is not 0.*

Example 4.2 (Case of non connected palette diagrams consisting of four unit weighted squares with the type (B)). *For a separate diagram* Γ

$$\Gamma = \left\{ \begin{matrix} [(a,a+1)\times(b,b+1),p], & [(a,a+1)\times(d,d+1),-p] \\ [(c,c+1)\times(b,b+1),-p], & [(c,c+1)\times(d,d+1),p] \end{matrix} \right\},$$

$a \neq c, b \neq d$, *we see* $Area(\Gamma) = 0, G(\Gamma) = (0,0)$,

$$M(\Gamma) = \begin{bmatrix} 0 & (a-c)(b-d)p \\ (a-c)(b-d)p & 0 \end{bmatrix}, \quad \det M(\Gamma) \neq 0,$$

and the equation (2) is

$$P_2(\Gamma)(\alpha,\beta) = 2(a-c)(b-d)p\alpha\beta = 0.$$

It determines two lines

$$(\alpha:\beta) = (1:0), \quad (0:1),$$

and

$$P_3(\Gamma)(1,0) = P_3(\Gamma)(0,1) = 0.$$

Hence Γ is classified into type (B).

5 Application of the classification theorem to finding relations of formal diffeomorphisms

Here we explain the application of the classification theorem in §3 to finding relations of two formal diffeomorphisms.

We have obtained the following theorems in [4] for relations of two formal diffeomorphisms in terms of Feynman diagrams.

Theorem 5.1 ([4]). *Let $\gamma \subset \mathbb{R}^2$ be a closed Feynman diagram with Area(γ) $= 0, G(\gamma) = 0$. Assume $W_{\gamma^*}(f,g) = id$ and $f, g \neq id$ are tangent to identity and assume*

$$\iint \rho {K_2}^2 \; dx \wedge dy \neq 0$$

with $K_2 = \frac{1}{2}(f''(0)x + g''(0)y)$. Then f, g commute.

Theorem 5.2 ([4]). *Let $\gamma \subset \mathbb{R}^2$ be a closed Feynman diagram with*

$$\mathit{Area}(\gamma) = 0, G(\gamma) = 0, \iint \rho {K_2}^2 \; dx \wedge dy = 0, \iint \rho {K_2}^3 \; dx \wedge dy \neq 0$$

and assume the moment matrix is non singular, i.e.

$$\det \begin{bmatrix} \iint \rho \; x^2 \; dx \wedge dy & \iint \rho \; xy \; dx \wedge dy \\ \iint \rho \; xy \; dx \wedge dy & \iint \rho \; y^2 \; dx \wedge dy \end{bmatrix} \neq 0.$$

Then $W_{\gamma^*}(f,g) = id$ admits formal solutions $f, g \neq id$ non commute and tangent to identity such that $K_2 = \frac{1}{2}(f''(0)x + g''(0)y)$.

Theorem 5.3 ([4]). *Let $\gamma \subset \mathbb{R}^2$ be a closed Feynman diagram with Area(γ) $= 0, G(\gamma) = 0$. Assume $W_{\gamma^*}(f,g) = id$, $f, g \neq id$ are tangent to identity,*

$$\iint \rho {K_2}^2 \; dx \wedge dy = 0, \iint \rho {K_2}^3 \; dx \wedge dy \neq 0$$

with $K_2 = \frac{1}{2}(f''(0)x + g''(0)y)$, and assume the moment matrix is singular, i.e.

$$\det \begin{bmatrix} \iint \rho \; x^2 \; dx \wedge dy & \iint \rho \; xy \; dx \wedge dy \\ \iint \rho \; xy \; dx \wedge dy & \iint \rho \; y^2 \; dx \wedge dy \end{bmatrix} = 0.$$

Then f, g commute.

In Theorem 5.1, 5.2 and 5.3, ρ denotes the winding number of γ, and the domain of each multiple integral is the domain enclosed by γ.

We obtain the following two lemmas by rephrasing statements of Theorems 5.2 and 5.3 respectively.

Lemma 5.1. *Assume a palette diagram* Γ *has the type (A). Then for all Feynman diagrams* γ *obtained from* Γ, $W_{\gamma^*}(f,g) = id$ *admits formal solutions* $f, g \neq id$ *non commute and tangent to identity such that* $(f''(0) : g''(0)) = (\alpha : \beta)$, *where* (α,β) *satisfies* $P_2(\gamma)(\alpha,\beta) = 0$ *and* $P_3(\gamma)(\alpha,\beta) \neq 0$.

Lemma 5.2. *Assume a palette diagram* Γ *has the type (C). For a Feynman diagram* γ *obtained from* Γ, *assume* $W_{\gamma^*}(f,g) = id$, $f, g \neq id$ *are tangent to identity and* $(f''(0) : g''(0)) = (\alpha : \beta)$, *where* (α,β) *satisfies* $P_2(\gamma)(\alpha,\beta) = 0$ *and* $P_3(\gamma)(\alpha,\beta) \neq 0$. *Then* f, g *commute.*

Then we obtain the following Theorems 5.4 and 5.5 as the application of the classification theorem.

Theorem 5.4. *Assume* Γ *equals the one of five in the chart in Theorem 3.1, then for all Feynman diagrams* γ *obtained from* Γ, *the relation* $W_{\gamma^*}(f,g) = id$ *admits formal solutions* $f, g \neq id$ *non commute and tangent to identity such that* $(f''(0) : g''(0)) = (\alpha : \beta)$, *where* (α,β) *satisfies* $P_2(\gamma)(\alpha,\beta) = 0$ *and* $P_3(\gamma)(\alpha,\beta) \neq 0$.

Theorem 5.5. *Assume* Γ *equals the one of two in the chart in Theorem 3.3. For a Feynman diagram* γ *obtained from* Γ, *assume* $W_{\gamma^*}(f,g) = id$, $f, g \neq id$ *are tangent to identity, and* $(f''(0) : g''(0)) = (\alpha : \beta)$, *where* (α,β) *satisfies* $P_2(\gamma)(\alpha,\beta) = 0$ *and* $P_3(\gamma)(\alpha,\beta) \neq 0$. *Then* f, g *commute.*

If a palette diagram Γ is classified into the type (B) or (D), then there exists an (α,β) such that $P_2(\Gamma)(\alpha,\beta) \neq 0$. Hence then, by rephrasing a statement of Theorem 5.1 we obtain

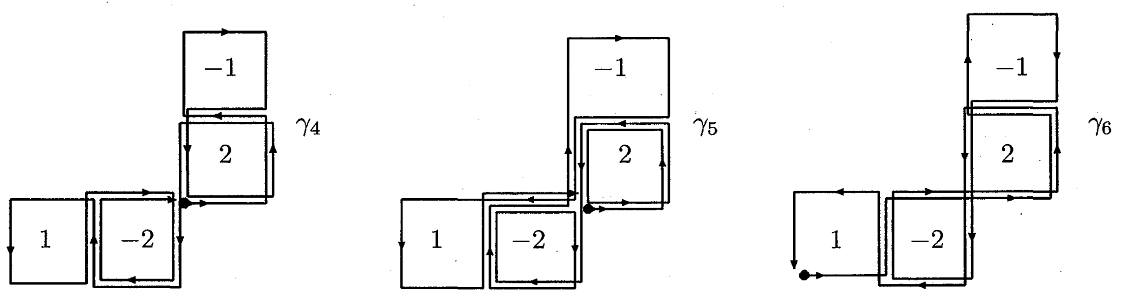
Lemma 5.3. *Assume a palette diagram* Γ *has the type (B) or (D). For a Feynman diagram* γ *obtained from* Γ, *assume* $W_{\gamma^*}(f,g) = id$, $f, g \neq id$ *are tangent to identity and* $(f''(0) : g''(0)) = (\alpha : \beta)$ *for* (α,β) *which is not the solution of* $P_2(\gamma)(\alpha,\beta) = 0$. *Then* f, g *commute.*

And from Lemma 5.3 we obtain

Theorem 5.6. *Assume* Γ *equals the one of nine in the chart in Theorem 3.2 or 3.4. For a Feynman diagram* γ *obtained from* Γ, *assume* $W_{\gamma^*}(f,g) = id$, $f, g \neq id$ *are tangent to identity and* $(f''(0) : g''(0)) = (\alpha : \beta)$, *where* (α,β) *does not satisfy* $P_2(\gamma)(\alpha,\beta) = 0$. *Then* f, g *commute.*

We give some examples of relations of two formal diffeomorphisms non commute and tangent to identity obtained from diagrams classified into (A).

Example 5.1 (Relations of formal diffeomorphisms obtained from Γ_8). *We obtain the following three distinct Feynman diagrams* $\gamma_4, \gamma_5, \gamma_6$ *from a palette diagram* Γ_8 *for* $p = 1$.



For $\gamma_4, \gamma_5, \gamma_6$,

$$
\begin{aligned}
\gamma_4 &= H * V * \{H, V^{-1}\} * H^{-1} * V^{-1} * \{H^{-1}, V^{-1}\} * V^{-1} \\
&\qquad * H^{-1} * \{V^{-1}, H\} * V * H * \{V, H\}, \\
\gamma_5 &= \{H^{-1}, V^{-1}\}^2 * \{V, H\}^2 * V * \{V^{-1}, H^{-1}\} * V^{-1} * H^{-1} * \{H, V\} * H, \\
\gamma_6 &= H * V * H * H * V * H^{-1} * \{V^{-1}, H^{-1}\} * V^{-1} * \{V, H\} \\
&\qquad * \{H^{-1}, V^{-1}\} * V^{-1} * H^{-1} * V * H^{-1} * V^{-1}.
\end{aligned}
$$

And we have

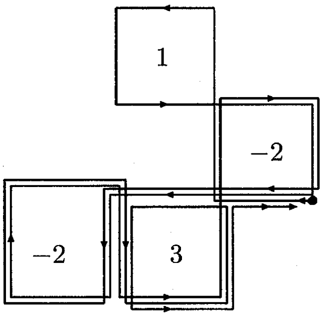
$$\begin{aligned} W_{\gamma_4^*}(f,g) &= f\circ g\circ\{f,g^{(-1)}\}\circ f^{(-1)}\circ g^{(-1)}\circ\{f^{(-1)},g^{(-1)}\}\circ g^{(-1)} \\ &\qquad \circ f^{(-1)}\circ\{g^{(-1)},f\}\circ g\circ f\circ\{g,f\}, \\ W_{\gamma_5^*}(f,g) &= \{f^{(-1)},g^{(-1)}\}^{(2)}\circ\{g,f\}^{(2)}\circ g\circ\{g^{(-1)},f^{(-1)}\} \\ &\qquad \circ g^{(-1)}\circ f^{(-1)}\circ\{f,g\}\circ f, \\ W_{\gamma_6^*}(f,g) &= f\circ g\circ f\circ f\circ g\circ f^{(-1)}\circ\{g^{(-1)},f^{(-1)}\}\circ g^{(-1)}\circ\{g,f\} \\ &\qquad \circ\{f^{(-1)},g^{(-1)}\}\circ g^{(-1)}\circ f^{(-1)}\circ g\circ f^{(-1)}\circ g^{(-1)}. \end{aligned}$$

Since Γ_8 is classified into (A), we see that the relation $W_{\gamma_i^*}(f,g) = id, i = 4,5,6$, admit solutions of formal diffeomorphisms $f, g \neq id$ non commute and tangent to identity with

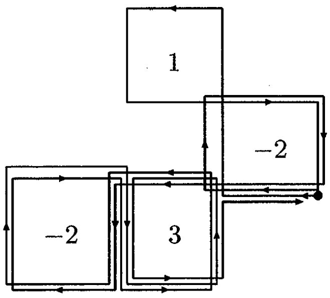
$$(f''(0) : g''(0)) = (1:1)$$

by Theorem 5.1.

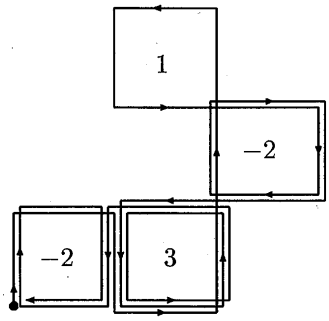
Example 5.2 (Relations of formal diffeomorphisms obtained from Γ_{10}). *We obtain the following three distinct Feynman diagrams $\gamma_7, \gamma_8, \gamma_9$ from Γ_{10} for $p = 1$.*



γ_7



γ_8



γ_9

For $\gamma_7, \gamma_8, \gamma_9$,

$$\begin{aligned} \gamma_7 &= H^{-1}*V*\{V^{-1},H\}*H*V^{-1}*H^{-2}*\{V,H\}*V^{-1}*H*V \\ &\qquad *\{V^{-1},H^{-1}\}*H^{-1}*V^{-1}\{H,V^{-1}\}*\{V,H^{-1}\}*V^{-1}*H*V*H, \\ \gamma_8 &= H^{-1}*V*\{V^{-1},H\}*H*V^{-1}*H^{-1}*V*H*V^{-1}*H^{-2} \\ &\qquad *\{V,H\}*\{V,H^{-1}\}*\{V,H\}*\{V,H^{-1}\}*V^{-1}*H*V*H, \\ \gamma_9 &= V*H*V^{-1}*H*V^2*\{V^{-1},H\}*\{H^{-1},V\}*H*V^{-1}*H^{-1}*\{H,V\}^2 \\ &\qquad *H^{-1}*\{V,H\}*V^{-1}*H^{-1}. \end{aligned}$$

And we have

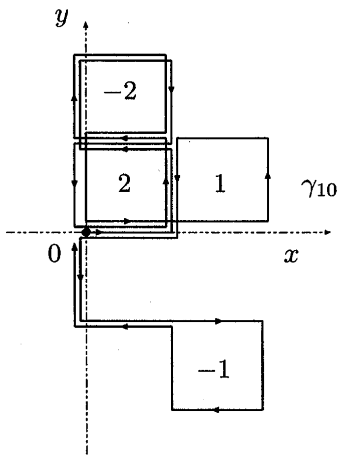
$$
\begin{aligned}
W_{\gamma_7^*}(f,g) &= f^{(-1)} \circ g \circ \{g^{(-1)}, f\} \circ f \circ g^{(-1)} \circ f^{(-2)} \circ \{g, f\} \circ g^{(-1)} \circ f \circ g \\
&\quad \circ\{g^{(-1)}, f^{(-1)}\} \circ f^{(-1)} \circ g^{(-1)}\{f, g^{(-1)}\} \circ \{g, f^{(-1)}\} \circ g^{(-1)} \circ f \circ g \circ f, \\
W_{\gamma_8^*}(f,g) &= f^{(-1)} \circ g \circ \{g^{(-1)}, f\} \circ f \circ g^{(-1)} \circ f^{(-1)} \circ g \circ f \circ g^{(-1)} \circ f^{(-2)} \\
&\quad \circ\{g, f\} \circ \{g, f^{(-1)}\} \circ \{g, f\} \circ \{g, f^{(-1)}\} \circ g^{(-1)} f \circ g \circ f, \\
W_{\gamma_9^*}(f,g) &= g \circ f \circ g^{(-1)} \circ f \circ g^{(2)} \circ \{g^{(-1)}, f\} \circ \{f^{(-1)}, g\} \circ f \circ g^{(-1)} \circ f^{(-1)} \\
&\quad \circ\{f, g\}^{(2)} \circ f^{(-1)} \circ \{g, f\} \circ g^{(-1)} \circ f^{(-1)}.
\end{aligned}
$$

Since Γ_{10} is classified into (A), we see that the relation $W_{\gamma_i^*}(f,g) = id, i = 7,8,9,$ admit solutions of formal diffeomorphisms $f, g \neq id$ non commute and tangent to identity with

$$(f''(0) : g''(0)) = (-1 \pm \sqrt{3} : 2)$$

by Theorem 5.1.

Example 5.3 (Relations of formal diffeomorphisms obtained from a non connected diagram in Example 3.1). *We obtain a Feynman diagram γ_{10} from a non connected palette diagram classified into (A) in Example 3.1 for $p = 2, q = 1, a = b = 0$ (then $c = 1, d = -2$).*



For γ_{10},

$$
\begin{aligned}
\gamma_{10} &= H * V * H^{-1} * \{V^{-1}, H^{-1}\} * \{V, H^{-1}\} * \{V^{-1}, H^{-1}\} * V^{-1} * H \\
&\quad *\{H^{-1}, V^{-1}\} * H^{-1} * V^{-1} * H * \{H^{-1}, V\} * H^{-1} * V,
\end{aligned}
$$

and

$$
\begin{aligned}
W_{\gamma_{10}^*}(f,g) &= f \circ g \circ f^{(-1)} \circ \{g^{(-1)}, f^{(-1)}\} \circ \{g, f^{(-1)}\} \circ \{g^{(-1)}, f^{(-1)}\} \circ g^{(-1)} \circ f \\
&\quad \circ\{f^{(-1)}, g^{(-1)}\} \circ f^{(-1)} \circ g^{(-1)} \circ f \circ \{f^{(-1)}, g\} \circ f^{(-1)} \circ g.
\end{aligned}
$$

Then the relation $W_{\gamma_{10}^*}(f,g) = id$ admits solutions of formal diffeomorphisms $f, g \neq id$ non commute and tangent to identity with

$$(f''(0) : g''(0)) = (3 : 2)$$

by Lemma 5.1.

For a Feynman diagram γ obtained from a palette diagram with the property (A), we see that the relation

$$W(f,g) \circ W_{\gamma^*}(f,g) \circ W(f,g)^{(-1)} = \mathrm{id},$$

always admits solutions of formal diffeomorphisms $f, g \neq \mathrm{id}$ non commute and tangent to identity for an arbitrary word $W(f,g)$ of $f, g, f^{(-1)}, g^{(-1)}$ since the path $W(H,V) \circ \gamma \circ W(H,V)^{(-1)}$ also has the property (A).

Appendix

Here we give the classification of connected palette diagrams without area and moment consisting of two or three unit weighted squares. There is no diagram without area consisting of one unit weighted square, so we need not consider this case.

Appendix 1. Case of two unit squares

There exist exactly two basis of connected palette diagrams consisting of two unit squares. The followings are the two connected palette diagrams without area obtained by attaching non zero integers $p, -p$ to two unit squares of the basis.

Their expressions are

$$\Gamma_1 = \left\{ [(a, a+1) \times (b, b+1), p], [(a+1, a+2) \times (b, b+1), -p] \right\}$$

and

$$\Gamma_2 = \left\{ [(a, a+1) \times (b, b+1), p], [(a+1, a+2) \times (b+1, b+2), -p] \right\}$$

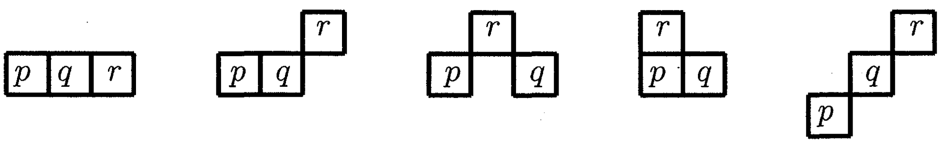
respectively. For them

$$\begin{aligned} G(\Gamma_1) &= (p\frac{2a+1}{2} - p\frac{2a+3}{2}, 0) = (-p, 0) \neq (0,0). \\ G(\Gamma_2) &= (p\frac{2a+1}{2} - p\frac{2a+3}{2}, p\frac{2b+1}{2} - p\frac{2b+3}{2}) = (-p, -p) \neq (0,0). \end{aligned}$$

So we need not consider this case.

Appendix 2. Case of three unit squares

There exist exactly five basis of connected palette diagrams consisting of three unit squares as we see in §2. The followings are the five connected palette diagrams obtained by attaching non zero integers p, q, r to three unit squares of the basis.



The followings are their expressions, where $r = -p - q$ so that diagrams have zero area.

1. $\Gamma_1 = \Big\{ [(a, a+1) \times (b, b+1), p], [(a+1, a+2) \times (b, b+1), q],$
$[(a+2, a+3) \times (b, b+1), -p-q] \Big\},$

2. $\Gamma_2 = \Big\{ [(a, a+1) \times (b, b+1), p], [(a+1, a+2) \times (b, b+1), q],$
$[(a+2, a+3) \times (b+1, b+2), -p-q] \Big\},$

3. $\Gamma_3 = \Big\{[(a, a+1) \times (b, b+1), p], [(a+2, a+3) \times (b, b+1), q],$
$[(a+1, a+2) \times (b+1, b+2), -p-q]\Big\},$

4. $\Gamma_4 = \Big\{[(a, a+1) \times (b, b+1), p], [(a+1, a+2) \times (b, b+1), q],$
$[(a, a+1) \times (b+1, b+2), -p-q]\Big\},$

5. $\Gamma_5 = \Big\{[(a, a+1) \times (b, b+1), p], [(a+1, a+2) \times (b+1, b+2), q],$
$[(a+2, a+3) \times (b+2, b+3), -p-q]\Big\}.$

The followings are the moments and conditions of non zero integers p, q such that $G(\Gamma_i) = (0,0), i = 1,2,3,4,5$, for the above five diagrams.

1. $G(\Gamma_1) = (-2p-q, 0) = (0,0) \Leftrightarrow q = -2p.$
2. $G(\Gamma_2) = (-2p-q, -p-q) \neq (0,0).$
3. $G(\Gamma_3) = (-p+q, -p-q) \neq (0,0).$
4. $G(\Gamma_4) = (q, -p-q) \neq (0,0).$
5. $G(\Gamma_5) = (-2p-q, -2p-q) = (0,0) \Leftrightarrow q = -2p.$

Since $\Gamma_2, \Gamma_3, \Gamma_4$ have non zero moment, we need not consider them. For Γ_1,

$$M(\Gamma_1) = \begin{bmatrix} 2p & 0 \\ 0 & 0 \end{bmatrix}, \qquad \det M(\Gamma_1) = 0.$$

And

$$P_2(\Gamma_1)(\alpha, \beta) = 2p\alpha^2 = 0$$

determines one line $(0:1)$, and $P_3(\Gamma_1)(0,1) = 0$. Hence Γ_1 is classified into (D). And for Γ_5,

$$M(\Gamma_5) = \begin{bmatrix} 2p & 2p \\ 2p & 2p \end{bmatrix}, \qquad \det M(\Gamma_5) = 0.$$

And

$$P_2(\Gamma_5)(\alpha, \beta) = 2p\alpha^2 + 4p\alpha\beta + 2p\beta^2 = 0$$

determines one line $(1:-1)$, and $P_3(\Gamma_5)(1,-1) = 0$. Hence Γ_5 is classified into (D).

Therefore we can not obtain relations of formal diffeomorphisms tangent to identity from connected palette diagrams without area and moment consisting of two or three unit weighted squares.

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