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# Estimates of fractional maximal functions in a quasi-metric space

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**Abstract**

Let $M_\alpha$ be the fractional maximal operator in a quasi-metric space $X$. We will prove that $M_\alpha$ is bounded from the Choquet space $L^p(H^\eta_\infty)$ with respect to the $\eta$-Hausdorff capacity $H^\eta_\infty$ to the Choquet space $L^{q,p}(H^\delta_\infty)$ of Lorentz type with respect to the $\delta$-Hausdorff capacity for some $\delta$. To prove it, we use the Choquet integrals with respect to Hausdorff capacities and the dyadic balls introduced by E. Sawyer and R. L. Wheeden.

## 1. Introduction

Fractional maximal functions in $\mathbf{R^n}$ are closely related to the Riesz potentials (cf. [1]). The fractional maximal function $M_\alpha f$ of $f$ with order $\alpha$ is defined by

$$M_\alpha f(x) = \sup \frac{\int_B |f|dx}{|B|^{1-\alpha/n}},$$

where the supremum is taken over all balls $B$ containing $x$ and $|B|$ stands for the $n$-dimensional volume of $B$.

In 1998 D. R. Adams defined a Choquet space $L^{q,p}(H^\delta_\infty)$ of Lorentz type with respect to the Hausdorff capacity $H^\delta_\infty$ and proved that the fractional maximal operator $M_\alpha$ is bounded from $L^p(H^\eta_\infty)$ to $L^{q,p}(H^\delta_\infty)$ for a suitable $\delta$ (cf. Theorem 7 in [2]).

In this paper we estimate the fractional maximal operator $M_\alpha$ by using the Hausdorff capacities in a quasi-metric space $X$.

More precisely, let $X$ be a quasi-metric space with a mapping $\rho$ from $X \times X$ to $[0,\infty)$ having the following properties:

(i) $\rho(x,y) = 0$ if and only if $x = y$,
(ii) $\rho(x,y) = \rho(y,x)$ for all $x,\ y \in X$,
(iii) There is a constant $K \geq 1$ such that

$$\rho(x,y) \leq K(\rho(x,z) + \rho(z,y)) \quad \text{for all } x,\ y,\ z \in X. \tag{1.1}$$

In addition, assume that the diameter of $X$ is finite and set

$$\operatorname{diam} X = R.$$

Furthermore we suppose that there are a nonnegative Radon measure $\mu$ on $X$ and a positive number $d$ such that

$$b_1 r^d \leq \mu(B(x,r)) \leq b_2 r^d \tag{1.2}$$

for all $0 < r \leq R$, where

$$B(x,r) = \{y \in X : \rho(x,y) < r\}.$$

We fix such a measure $\mu$. Let $\alpha > 0$. Using the measure $\mu$ and the positive number $d$, we define the fractional maximal function $M_\alpha f$ of a function $f$ on $X$ with order $\alpha$ by

$$M_\alpha f(x) = \sup \frac{\int_B |f| d\mu}{\mu(B)^{(d-\alpha)/d}}, \tag{1.3}$$

where the supremum is taken over all balls $B$ containing $x$. Here we note that, for a nonnegative function $g$ and a set $E$,

$$\int_E g d\mu := \int_0^\infty \mu(\{x \in E : g(x) > t\}) dt.$$

If $g$ is a $\mu$-measurable function and $E$ is a $\mu$-measurable set, then this integral coincides with the usual one.

Let $0 < \eta \leq d$ and $E$ be a set. Recall that the Hausdorff capacity $H_\infty^\eta$ is defined by

$$H_\infty^\eta(E) := \inf \sum_{j=1}^\infty r(D_j)^\eta, \tag{1.4}$$

where the infimum is taken over all coverings $\{D_j\}$ of $E$ by countable families of balls and $r(D_j)$ stands for the radius of $D_j$.

In a quasi-metric space there is no dyadic cube. Instead of dyadic cubes E. Sawyer and R. L. Wheeden constructed a family of balls in [6] as follows:

**Theorem A.** *Put $\lambda = K + 2K^2$. Then, for each integer $k$, there exists a sequence $\{B_j^k\}_j$ ($B_j^k = B(x_{jk}, \lambda^k)$) of balls of radius $\lambda^k$ having the following properties;*

(i) *Every ball of radius $\lambda^{k-1}$ is contained in at least one of the balls $B_j^k$,*

(ii) $\sum_j \chi_{B_j^k} \leq M$ *for all $k$ in* $\mathbf{Z}$,

(iii) $\hat{B}_i^k \cap \hat{B}_j^k = \emptyset$ *for $i \neq j$, $k \in \mathbf{Z}$, where $\hat{B}_j^k = B(x_{jk}, \lambda^{k-1})$.*

They call these balls $B_j^k$ dyadic balls. Denote by $\mathcal{B}_d$ the family of all dyadic balls.

We denote by $\tilde{M}_\alpha f(x)$ the supremum taken over all dyadic balls containing $x$ in (1.3) and by $\tilde{H}_\infty^\eta(E)$ the infimum taken over all covering of $E$ by countable dyadic balls in (1.4). Then we have

$$\tilde{M}_\alpha f(x) \leq M_\alpha f(x) \leq c\tilde{M}_\alpha f(x) \tag{1.5}$$

and

$$H_\infty^\eta(E) \leq \tilde{H}_\infty^\eta(E) \leq c' H_\infty^\eta(E) \tag{1.6}$$

for some constants $c$, $c'$.

Recall that the Choquet integral of a nonnegative function $g$ over a set $E$ with respect to $H_\infty^\eta$ is defined by

$$\int_E g dH_\infty^\eta = \int_0^\infty H_\infty^\eta(\{x \in E : g(x) > t\}) dt.$$

Under these assumptions and notations we state our theorem, which will be proved in §4.

**Theorem.** *Let $(X, \rho)$ be a quasi-metric space such that diam $X = R$ and there exists a Radon measure $\mu$ on $X$ satisfying* (1.2). *Assume that $0 < \eta \leq d$, $0 \leq \alpha < d$ and $p \leq q$. Put*

$$G_t = \{x : M_\alpha f(x) > t\}. \tag{1.7}$$

(i) *If $\eta/d < p < \eta/\alpha$ and $\delta = q(\eta - \alpha p)/p$, then*

$$\int_0^\infty t^{p-1} H_\infty^\delta(G_t)^{p/q} dt \leq c_1 \int |f|^p dH_\infty^\eta.$$

(ii) *If* $p = \eta/d$ *and* $\delta = \eta(d-\alpha)/d$, *then*

$$\sup_t \left(t^{\delta/(d-\alpha)} H^\delta_\infty(G_t)\right) \le c_2 \int |f|^p dH^\eta_\infty.$$

(iii) *If* $p \ge \eta/\alpha$, *then*

$$\sup_x M_\alpha f(x) \le c_3 \left(\int |f|^p dH^\eta_\infty\right)^{1/p}.$$

## 2. Covering lemmas in a quasi-metric space

Throughout this paper let $(X, \rho)$ be a quasi-metric space with diam $X = R$. The function $\rho$ is called a quasi-metric. Furthermore we assume that there exists a positive Radon measure $\mu$ on $X$ satisfying (1.2) for $0 < r \le R$. For any quasi-metric $\rho$ there exists an equivalent quasi-metric $\rho'$ such that all balls in $X$ are open (cf. [4]). Consequently we may assume that each ball $B(x,r)$ in $X$ is open.

Let $B = B(x,r)$ be a ball and $b$ be a positive real number. The notation $bB$ stands for the ball of radius $br$ centered at $x$ and $r(B)$ stands for the radius of $B$. We often use the following value $\lambda$ defined by

$$\lambda = 2K^2 + K, \tag{2.1}$$

where $K$ is the constant in (1.1).

The following lemma is a covering one of Whitney type by dyadic balls.

**Lemma 2.1.** *Let $G$ be a non-empty open subset of $X$. Then there exists a sequence $\{B(y_j, d_j)\}_j$ of dyadic balls having the properties* (i)-(iii):

(i) $G = \cup_j B(y_j, d_j) = \cup_j B(y_j, \lambda^2 d_j)$, *where $\lambda$ is the number defined by (2.1),*

(ii) *There is a constant $s \ge 1$ such that $B(y_j, sd_j) \cap G^c \ne \emptyset$, where $s$ is independent of $j$ and $G^c$ stands for the complement of $G$.*

(iii) $\sum_j \chi_{B(y_j, \lambda^2 d_j)} \le N$ *for some constant $N$ independent of $j$.*

**Proof.** Let $x \in G$ and put

$$r(x) = \frac{\rho(x, G^c)}{3K^2\lambda^4 h},$$

where $h$ is the constant $h \ge 1$ in the covering lemma of Vitali type (cf. Théorèm (1.2) on p.69 in [3]). Note that $G$ is open and $\rho(x, G^c) = \inf_{y \in G^c} \rho(x,y) > 0$. Since $G \subset \cup_{x\in G} B(x, r(x))$, the covering lemma of Vitali type asserts that there exists a countable subfamily $\{C_j\}$ ($C_j = B(x_j, r_j)$) of $\{B(x, r(x))\}_{x \in G}$ such that $\{C_j\}$ are mutually disjoint and

$$G \subset \cup_j B(x_j, hr_j).$$

For each $j$ we can choose the integer $k(j)$ such that $\lambda^{k(j)-1} < hr_j \le \lambda^{k(j)}$. From Theorem A, (i) we deduce a dyadic ball $B(y_j, \lambda^{k(j)+1})$ such that $B(x_j, hr_j) \subset B(y_j, \lambda^{k(j)+1})$. Put $d_j = \lambda^{k(j)+1}$ and $B_j = B(y_j, d_j)$.

We shall show that the family $\{B_j\}$ of dyadic balls is a desired one.

Noting that $d_j = \lambda^{k(j)+1} < \lambda^2 hr_j$, $\rho(x_j, G^c) = 3K^2\lambda^4 hr_j$ and

$$\rho(x_j, G^c) \le K(\rho(y_j, G^c) + \rho(x_j, y_j)),$$

we have

$$2K\lambda^4 hr_j < \rho(y_j, G^c).$$

Since $\lambda^2 d_j < \lambda^4 h r_j < 2K\lambda^4 h r_j < \rho(y_j, G^c)$, we see that $B(y_j, \lambda^2 d_j) \subset G$.

On the other hand, from $G \subset \cup_j B(x_j, hr_j)$ we deduce $G \subset \cup_j B(y_j, d_j)$.

(ii) Noting that

$$B(y_j, 5K^3\lambda^3 d_j) \supset B(y_j, 5K^3\lambda^4 h r_j) \supset B(x_j, 4K^2\lambda^4 h r_j)$$

and $4K^2\lambda^4 h r_j = (4/3)\rho(x_j, G^c)$, we see that $B(y_j, 5K^3\lambda^3 d_j) \cap G^c \neq \emptyset$. We may put $s = 5K^3\lambda^3$.

(iii) Let $x \in B(y_j, \lambda^2 d_j)$. Then we shall estimate the length of $r_j$ by $c\rho(x, G^c)$ for some $c$ from above and below. Indeed, from

$$\begin{aligned} 3K^2\lambda^4 h r_j &= \rho(x_j, G^c) \\ &\leq K\rho(x, G^c) + K^2\rho(x_j, y_j) + K^2\rho(y_j, x) \\ &< K\rho(x, G^c) + 2K^2\lambda^4 h r_j \end{aligned}$$

we deduce

$$r_j < \frac{\rho(x, G^c)}{K\lambda^4 h}. \tag{2.2}$$

On the other hand, since

$$\rho(x, G^c) \leq K\rho(x_j, G^c) + K^2\rho(x, y_j) + K^2\rho(y_j, x_j) < 5K^3\lambda^4 h r_j,$$

we have

$$r_j > \frac{\rho(x, G^c)}{5K^3\lambda^4 h}. \tag{2.3}$$

Let $z \in \lambda^2 B_j = B(y_j, \lambda^2 d_j)$ for some $j$. We claim that

$$B(y_j, \lambda^2 d_j) \subset B(z, 2\rho(z, G^c)). \tag{2.4}$$

In fact, using (2.2), we have, for any $w \in B(y_j, \lambda^2 d_j)$,

$$\rho(z, w) \leq K(\rho(z, y_j) + \rho(y_j, w)) < 2K\lambda^4 h r_j < 2\rho(z, G^c),$$

which leads to the claim (2.4).

On the other hand, we have, by (2.3),

$$\mu(B(z, 2\rho(z, G^c))) \leq \mu(B(z, 10K^3\lambda^4 h r_j) \leq \mu(B(x_j, 12K^4\lambda^4 h r_j)).$$

Noting (1.2), we have

$$\mu(B(x_j, 12K^4\lambda^4 h r_j)) \leq N\mu(B(x_j, r_j))$$

for some constant $N$, which is independent of $j$. Hence

$$\mu(B(z, 2\rho(z, G^c))) \leq N\mu(B(x_j, r_j)).$$

Let $z$ be in $\lambda^2 B_j$ $(j = 1, \cdots, m)$. Then

$$\begin{aligned} m\mu(B(z, 2\rho(z, G^c))) &\leq N\sum_j \mu(B(x_j, r_j)) \\ &= N\mu(\cup_j B(x_j, r_j)) \leq N\mu(B(z, 2\rho(z, G^c))). \end{aligned}$$

Here we used that balls $\{B(x_j, r_j)\}$ are mutually disjoint and (2.4) holds. Therefore $m \leq N$. Thus we also have (iii).

We next show that, if the multiplicity of a countable family of balls is at most $N$, then it has the following property.

**Lemma 2.2.** *Let $D$ be a ball and $\{D_j\}$ be a countable family of balls such that*

$$\sum_j \chi_{\lambda D_j} \leq N \text{ for some } N. \tag{2.5}$$

*Put*

$$T_D = \{j : D \cap D_j \neq \emptyset,\ r(D) \leq r(D_j)\}.$$

*Then $\#T_D \leq N$.*

**Proof.** If $D \cap D_j \neq \emptyset$ and $r(D) \leq r(D_j)$, then $D \subset \lambda D_j$. The inequality (2.5) yields that $\#\{j : D \subset \lambda D_j\} \leq N$. Hence we have the conclusion.

A countable family of balls has following subfamily which is useful to study the Hausdorff capacity of the union of these balls.

**Lemma 2.3.** *Let $\tau > 0$ and $\{D_j\}$ be a sequence of balls. Then there exists a (finite or countable) subfamily $\{D_{j_k}\}$ of $\{D_j\}$ having the following properties:*
(i) $\sum_{j_k \in S_D} r(D_{j_k})^\tau \leq 2r(D)^\tau$ for each $D \in \mathcal{B}_d$, *where*

$$S_D = \{j_k : D_{j_k} \cap D \neq \emptyset,\ r(D_{j_k}) \leq r(D)\}.$$

(ii) *Let $b > 0$. Then*

$$H^\tau_\infty(\cup_j bD_j) \leq c \sum_k r(D_{j_k})^\tau,$$

*where $c$ is a constant independent of $\{D_j\}$.*

**Proof.** This lemma has been proved in Lemma 2.5 in [7] in case $\{D_j\}$ are dyadic balls. We can prove Lemma 2.3 by the same method as in the proof of it, even if $\{D_j\}$ are not dyadic balls and if $D \in \mathcal{B}_d$.

The integral with respect to the Hausdorff capacity $H^\eta_\infty$ does not always have the property such that, for a nonnegative function $f$,

$$\sum_j \int_{D_j} f dH^\eta_\infty \leq c \int f dH^\eta_\infty \tag{2.6}$$

even if $\{D_j\}$ are mutually disjoint. But, we shall show that the inequality (2.6) holds for a suitable subsequence of $\{D_j\}$ as follows:

**Lemma 2.4.** *Let $\alpha > 0$, $0 < \tau \leq \eta < d$ and $f$ be a nonnegative function. Assume that $\{D_j\}$ be a sequence of balls such that*

$$\sum_j \chi_{\lambda D_j} \leq N$$

*for some constant $N$. If $\{D_{j_k}\}$ is a subsequence of $\{D_j\}$ satisfying* (i) *and* (ii) *in Lemma 2.3 for $\tau$, then*

$$\sum_k \int_{D_{j_k}} f dH^\eta_\infty \leq c \int_{\cup_j D_j} f dH^\eta_\infty, \tag{2.7}$$

*where $c$ is a constant independent of $f$ and $\{D_j\}$.*

**Proof.** Put $F = \cup_j D_j$. We may assume that $\int_F f dH^\eta_\infty < \infty$. This means

$$\int_0^\infty H^\eta_\infty(\{x \in F : f(x) > t\})dt < \infty$$

and hence

$$H^\eta_\infty(\{x \in F : f(x) > t\}) < \infty \quad \text{for } \mu - \text{a.e. } t.$$

By (1.6) we have

$$\tilde{H}^\eta_\infty(\{x \in F : f(x) > t\}) < \infty \quad \text{for } \mu - \text{a.e. } t. \tag{2.8}$$

Fix $t$ satisfying (2.8). For $\epsilon > 0$, there exist balls $\{Q_i\} \subset \mathcal{B}_d$ such that

$$\{x \in F : f(x) > t\} \subset \cup_i Q_i$$

and

$$\sum_i r(Q_i)^\eta < \tilde{H}^\eta_\infty(\{x \in F : f(x) > t\}) + \epsilon. \tag{2.9}$$

Since $\sum_k \chi_{\lambda D_{j_k}} \le N$, the number

$$\#\{j_k : Q_i \cap D_{j_k} \neq \emptyset,\ r(Q_i) \le r(D_{j_k})\}$$

is at most $N$ by Lemma 2.2. Lemma 2.3 yields

$$2r(Q_i)^\tau \ge \sum_{j_k \in S_{Q_i}} r(D_{j_k})^\tau.$$

Noting that $\eta/\tau \ge 1$, we have

$$2^{\eta/\tau} r(Q_i)^\eta \ge \sum_{j_k \in S_{Q_i}} r(D_{j_k})^\eta.$$

Hence

$$\begin{aligned}
&(2^{\eta/\tau}+1)\sum_i r(Q_i)^\eta \\
=\ & 2^{\eta/\tau}\sum_i r(Q_i)^\eta + \frac{1}{N}\sum_i r(Q_i)^\eta N \\
\ge\ & \sum_i \sum_{D_{j_k}\cap Q_i \neq \emptyset, r(D_{j_k}) < r(Q_i)} r(D_{j_k})^\eta + \frac{1}{N}\sum_i \sum_{D_{j_k}\cap Q_i \neq \emptyset, r(D_{j_k}) \ge r(Q_i)} r(Q_i)^\eta \\
=\ & \sum_k \Big( \sum_{D_{j_k}\cap Q_i \neq \emptyset, r(D_{j_k}) < r(Q_i)} r(D_{j_k})^\eta + \frac{1}{N} \sum_{D_{j_k}\cap Q_i \neq \emptyset, r(D_{j_k}) \ge r(Q_i)} r(Q_i)^\eta \Big) \\
\ge\ & \frac{1}{N}\sum_k H^\eta_\infty(D_{j_k} \cap \cup_i Q_i).
\end{aligned}$$

Using (2.9), we have

$$(2^{\eta/\tau}+1)N\tilde{H}^\eta_\infty(\{x \in F : f(x) > t\}) \ge \sum_k H^\eta_\infty(\{x \in D_{j_k} : f(x) > t\}),$$

whence, by (1.6),

$$cH^\eta_\infty(\{x \in F : f(x) > t\}) \ge \sum_k H^\eta_\infty(\{x \in D_{j_k} : f(x) > t\}).$$

Since this holds for $\mu$-a.e. $t$, we have

$$\begin{aligned}\sum_k \int_{D_{j_k}} f dH^\eta_\infty &= \sum_k \int_0^\infty H^\eta_\infty(\{x \in D_{j_k} : f(x) > t\})dt \\ &\leq c\int_0^\infty H^\eta_\infty(\{x \in F : f(x) > t\})dt = c\int_F f dH^\eta_\infty.\end{aligned}$$

Thus we have the conclusion.

The relation between the integral with respect to $\mu$ and the integral with respect to the Hausdorff capacity is as follows:

**Lemma 2.5.** *Let $g$ be a nonnegative function and $0 < \eta \leq d$. Then*

$$\int g d\mu \leq c\left(\int g^{\eta/d} dH^\eta_\infty\right)^{d/\eta},$$

*where $c$ is a constant independent of $g$.*

**Proof.** Using the property (1.2) of $\mu$, we can prove this lemma by the same method as in the proof of Lemma 3 in [5].

## 3. Estimates of fractional functions

In this section we prepare several lemmas with respect to fractional maximal functions.
Let $0 \leq \alpha < d$ and $0 < \eta \leq d$. Here $d$ is the number satisfying (1.2). Recall that

$$G_t = \{x : M_\alpha f(x) > t\}.$$

Noting that $G_t$ is open, we see that there exists a sequence $\{B_j\}$ ($B_j = B(y_j, d_j)$) of dyadic balls satisfying (i)-(iii) in Lemma 2.1. Fix such a covering $\{B_j\}$ of $G_t$.

**Lemma 3.1.** *Assume that $a > 2(3K^3 s)^{d-\alpha} b_1^{(\alpha-d)/d} b_2^{(d-\alpha)/d}$ and $f$ is a nonnegative function. Then, for $x \in G_{at} \cap B_j$,*

$$\frac{at}{2} < \sup\{\frac{\int_B f d\mu}{\mu(B)^{(d-\alpha)/d}} : x \in B,\ r(B) < \frac{r(B_j)}{K}\}.$$

*Here $s$ is the number in Lemma 2.1,* (ii) *and $b_1$, $b_2$ are the constants in (1.2).*

**Proof.** Let $x \in G_{at} \cap B_j$. Then $at \leq I_{1x} + I_{2x}$, where

$$I_{1x} = \sup\{\frac{\int_B f d\mu}{\mu(B)^{(d-\alpha)/d}} : x \in B,\ r(B) < \frac{r(B_j)}{K}\}$$

and

$$I_{2x} = \sup\{\frac{\int_B f d\mu}{\mu(B)^{(d-\alpha)/d}} : x \in B,\ r(B) \geq \frac{r(B_j)}{K}\}$$

Therefore $at/2 \leq I_{1x}$ or $at/2 \leq I_{2x}$. First, assume that $at/2 \leq I_{2x}$. By Lemma 2.1, (ii), we can find $z_j \in G_t^c \cap B(y_j, sr(B_j))$. Let $x \in B$ and $r(B) \geq r(B_j)/K$. Denote by $y$ the center of $B$. Then

$$\begin{aligned}\rho(y, z_j) &\leq K\rho(z_j, y_j) + K^2\rho(y_j, x) + K^2\rho(x, y) \\ &< Ksr(B_j) + K^2 r(B_j) + K^2 r(B) \leq 3K^3 sr(B),\end{aligned}$$

whence $z_j \in B(y, 3K^3 sr(B))$.

Using that $M_\alpha f(z_j) \le t$, we have

$$\begin{aligned} I_{2x} &\le (3K^3 s)^{d-\alpha} b_1^{(\alpha-d)/d} b_2^{(d-\alpha)/d} \sup\{\frac{\int_{3K^3 sB} f d\mu}{\mu(3K^3 sr(B))^{(d-\alpha)/d}} : x \in B,\ r(B) \ge \frac{r(B_j)}{K}\} \\ &\le (3K^3 s)^{d-\alpha} b_1^{(\alpha-d)/d} b_2^{(d-\alpha)/d} t. \end{aligned}$$

Hence $a \le 2(3K^3 s)^{d-\alpha} b_1^{(\alpha-d)/d} b_2^{(d-\alpha)/d}$. This is a contradiction to the assumption. Therefore we see that $at/2 \le I_{1x}$.

**Lemma 3.2.** *Let $t > 0$ and $f$ be a nonnegative function. If*

$$t < \sup\{\frac{\int_B f d\mu}{\mu(B)^{(d-\alpha)/d}} : x \in B,\ r(B) < \frac{r(B_j)}{K}\},$$

*then*

$$t\mu(B_j)^{(d-\alpha)/d} \le c \int_{3KB_j} f d\mu,$$

*where $c$ is a constant independent of $\{B_j\}$.*

**Proof.** Let $x \in B_j$. We choose $\epsilon > 0$ satisfying

$$t < t + 2\epsilon \le \sup\{\frac{\int_B f d\mu}{\mu(B)^{(d-\alpha)/d}} : x \in B,\ r(B) < \frac{r(B_j)}{K}\}.$$

Then there is a ball $B_x$ such that $x \in B_x$, $r(B_x) < r(B_j)/K$ and

$$t + \epsilon \le \frac{\int_{B_x} f d\mu}{\mu(B_x)^{(d-\alpha)/d}}$$

Assume that $B_x \cap G_t^c \ne \emptyset$ and pick $z \in B_x \cap G_t^c$. Then

$$t + \epsilon \le M_\alpha f(z) \le t.$$

This is a contradiction. Therefore $B_x \cap G_t^c = \emptyset$ and hence $B_x \subset G_t$. Denote by $x_0$ the center of $B_x$. Let $y \in B_x$. Then

$$\begin{aligned} \rho(y_j, y) &\le K\rho(y_j, x) + K^2\rho(x, x_0) + K^2\rho(x_0, y) \\ &< Kr(B_j) + 2K^2K^{-1}r(B_j) = 3Kr(B_j). \end{aligned}$$

Hence $B_x \subset B(y_j, 3Kr(B_j))$.

Since $B_j \subset \cup_{x\in B_j} B_x$, we can find, by the covering lemma of Vitali type, a countable subfamily $\{D_k\} \subset \{B_x\}_{x\in B_j}$ such that $\{D_k\}$ are mutually disjoint and $B_j \subset \cup_k hD_k$ for some $h$. Using (1.2), we have

$$\begin{aligned} t\mu(B_j)^{(d-\alpha)/d} &\le t(\sum_k \mu(hD_k))^{(d-\alpha)/d} \le c_1 t \sum_k r(D_k)^{d-\alpha} \\ &\le c_2 t \sum_k \mu(D_k)^{(d-\alpha)/d} \le c_2 \sum_k \int_{D_k} f d\mu \\ &\le c_2 \int_{\cup_k D_k} f d\mu \le c_2 \int_{3KB_j} f d\mu. \end{aligned}$$

Thus we have the conclusion.

We next show that for a sufficiently large number $a$ the integral of $f$ over $3KB_j$ is comparable to the integral of $f$ over $3KB_j \setminus G_{at}$.

**Lemma 3.3.** *Let $f$ be a nonnegative function. Then, for sufficiently large number $a$,*

$$\int_{3KB_j \setminus G_{at}} f d\mu \geq \frac{1}{2} \int_{3KB_j} f d\mu.$$

**Proof.** Let $x \in B_j \cap G_{at}$ and $a > 2(3K^3 s)^{d-\alpha} b_1^{(\alpha-d)/d} b_2^{(d-\alpha)/d}$. From Lemma 3.1 and Lemma 3.2 we deduce

$$\frac{at}{2} \leq c_1 \frac{\int_{3KB_j} f d\mu}{\mu(B_j)^{(d-\alpha)/d}}.$$

Hence, by (1.2),

$$\frac{at}{2} \leq c_1 \Big( \frac{\int_{3KsB_j} f d\mu}{\mu(3KsB_j)^{(d-\alpha)/d}} \Big) \leq c_2 M_\alpha f(z_j) \leq c_2 t,$$

where $z_j \in G_t^c \cap B(y_j, sr(B_j))$ in Lemma 2.1, (ii). We note that the constant $c_2$ is independent of $f$, $j$ and $t$. Therefore we have

$$\int_{3KB_j \cap G_{at}} f d\mu \leq \frac{2c_2}{a} \int_{3KB_j} f d\mu.$$

If we take $a > \max\{4c_2, 2(3K^3 s)^{d-\alpha} b_1^{(\alpha-d)/d} b_2^{(d-\alpha)/d}\}$, then

$$\int_{3KB_j \cap G_{at}} f d\mu \leq \frac{1}{2} \int_{3KB_j} f d\mu.$$

Consequently

$$\int_{3KB_j \setminus G_{at}} f d\mu = \int_{3KB_j} f d\mu - \int_{3KB_j \cap G_{at}} f d\mu \geq \frac{1}{2} \int_{3KB_j} f d\mu.$$

## 4. Proof of Theorem

In this section we prove our theorem.

*Proof of Theorem.* Let $t > 0$, $f$ be a function and $G_t$ be the set defined in (1.7). Recall that $\{B_j\}$ is the covering of $G = G_t$ of Whitney type by dyadic balls in Lemma 2.1. Fix a sufficiently large $a$ satisfying Lemma 3.3. Using Lemmas 3.1, 3.2 and 3.3, we have

$$t\mu(B_j)^{(d-\alpha)/d} \leq c_1 \int_{3KB_j} |f| d\mu \leq c_2 \int_{3KB_j \setminus G_{at}} |f| d\mu. \tag{4.1}$$

Noting that $dp/\eta > 1$, we see, by Hölder's inequality, that

$$t\mu(B_j)^{(\eta-\alpha p)/dp} \leq c_3 \Big( \int_{3KB_j \setminus G_{at}} |f|^{dp/\eta} d\mu \Big)^{\eta/dp}.$$

(i) By (1.2) and Lemma 2.5 we have

$$r(B_j)^{\eta-\alpha p} \leq \frac{c_4}{t^p} \int_{3KB_j \setminus G_{at}} |f|^p dH_\infty^\eta. \tag{4.2}$$

For the sequence $\{D_j\} = \{\lambda B_j\}$ of balls and $\tau = \eta - \alpha p$ Lemma 2.3 asserts that one can choose a subsequence $\{\lambda B_{j_k}\}$ of $\{\lambda B_j\}$ satisfying (i) and (ii) in Lemma 2.3. Note that $\lambda \geq 3K$ and

$\sum_j \chi_{\lambda^2 B_j} \leq N$ by Lemma 2.1, (iii). Therefore we apply Lemma 2.4 to the function $|f|^p \chi_{G_t \setminus G_{at}}$ and the sequence $\{\lambda B_{j_k}\}$ of balls. Then, by Lemma 2.3, (4.2) and Lemma 2.4,

$$\begin{aligned} H_\infty^{\eta-\alpha p}(\cup_j B_j) &\leq c_5 \sum_k r(B_{j_k})^{\eta-\alpha p} \leq \frac{c_6}{t^p} \sum_k \int_{3KB_{j_k}\setminus G_{at}} |f|^p dH_\infty^\eta \\ &\leq \frac{c_7}{t^p} \int_{G_t \setminus G_{at}} |f|^p dH_\infty^\eta. \end{aligned}$$

Hence

$$H_\infty^{\eta-\alpha p}(G_t) \leq \frac{c_7}{t^p} \int_{G_t \setminus G_{at}} |f|^p dH_\infty^\eta. \tag{4.3}$$

On the other hand, let $\{Q_j\}$ be an arbitrary covering of $G_t$ by balls. Since

$$\begin{aligned} \sum_j r(Q_j)^{\eta-\alpha p} &\geq \big(\sum_j r(Q_j)^{q(\eta-\alpha p)/p}\big)^{p/q} \\ &= \big(\sum_j r(Q_j)^\delta\big)^{p/q} \geq H_\infty^\delta(G_t)^{p/q}, \end{aligned}$$

we have

$$H_\infty^\delta(G_t)^{p/q} \leq H_\infty^{\eta-\alpha p}(G_t).$$

Using (4.3), we have

$$\begin{aligned} I &\equiv \int_0^\infty H_\infty^\delta(G_t)^{p/q} t^{p-1} dt \\ &\leq \int_0^\infty H_\infty^{\eta-\alpha p}(G_t) t^{p-1} dt \leq c_8 \int_0^\infty t^{-1}\big(\int_{G_t\setminus G_{at}} |f|^p dH_\infty^\eta\big) dt \\ &= c_8 \int_0^\infty t^{-1}\big(\int_0^\infty H_\infty^\eta(\{x \in G_t \setminus G_{at} : |f(x)|^p > \tau\}) d\tau. \end{aligned}$$

Fubini's theorem yields

$$\begin{aligned} I &\leq c_8 \int_0^\infty H_\infty^\eta(\{|f|^p > \tau\}) d\tau \int_{M_\alpha f(x)/a}^{M_\alpha f(x)} \frac{1}{t} dt \\ &\leq c_8 \log a \int |f|^p dH_\infty^\eta. \end{aligned}$$

Therefore

$$\int_0^\infty H_\infty^\delta(G_t)^{p/q} t^{p-1} dt \leq c_9 \int |f|^p dH_\infty^\eta.$$

Thus we have the conclusion.

(ii) Assume that $p = \eta/d$ and $\delta = \eta(d-\alpha)/d$. From (4.1) and Lemma 2.5 we deduce

$$t\mu(B_j)^{(d-\alpha)/d} \leq c_1 \int_{3KB_j} |f| d\mu \leq c_{10}\big(\int_{3KB_j} |f|^{\eta/d} dH_\infty^\eta\big)^{d/\eta},$$

whence, by (1.2),

$$t^{\delta/(d-\alpha)} r(B_j)^\delta = t^{\eta/d} r(B_j)^{\eta(d-\alpha)/d} \leq c_{11} \int_{3KB_j} |f|^{\eta/d} dH_\infty^\eta. \tag{4.4}$$

Using Lemma 2.3 for $\tau = \delta$ and $\{D_j\} = \{\lambda B_j\}$, there exists a subfamily $\{\lambda B_{j_k}\}$ of $\{\lambda B_j\}$ satisfying (i) and (ii) in Lemma 2.3. By (4.4) and Lemma 2.4 we have

$$\begin{aligned} t^{\delta/(d-\alpha)} H_\infty^\delta(G_t) &\leq c_{12} t^{\delta/(d-\alpha)} \sum_k r(B_{j_k})^\delta \\ &\leq c_{13} \sum_k \int_{3KB_{j_k}} |f|^p dH_\infty^\eta \leq c_{14} \int |f|^p dH_\infty^\eta. \end{aligned}$$

This leads to the conclusion (ii).

(iii) Assume that $p \geq \eta/\alpha$. Noting that $\alpha p - \eta \geq 0$ and

$$\int_B |f| d\mu \leq \Big( \int_B |f|^{dp/\eta} d\mu \Big)^{\eta/dp} \mu(B)^{1-\eta/dp},$$

we have, by Lemma 2.5,

$$\begin{aligned} M_\alpha f(x) &\leq \sup_B \Big( \int_B |f|^{dp/\eta} d\mu \Big)^{\eta/dp} \mu(B)^{(\alpha p-\eta)dp}, \\ &\leq c_{15} \mu(X)^{(\alpha p-\eta)dp} \Big( \int |f|^p dH_\infty^\eta \Big)^{1/p}. \end{aligned}$$

Hence

$$\|M_\alpha f\|_\infty \leq c_{16} \Big( \int |f|^p dH_\infty^\eta \Big)^{1/p}.$$

## References


[1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Grundlehren **314**, Springer, 1996.

[2] D. R. Adams, Choquet integrals in potential theory, *Publ. Mat.* **42** (1998), 3-66.

[3] R. R. Coifman and G. Weiss, Analyse harmonique non-commutative sur certains espaces homogenes, *Lecture Notes in Math.* **242**, Springer, 1971.

[4] R. A. Macías and C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv. in Math.* **33** (1979), 257-270.

[5] J. Orovitg and J. Verdera, Choquet integrals, Hausdorff content and the Hardy-Littlewood maximal operator, *Bull. London Math. Soc.* **30** (1998), 145-150.

[6] E. Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, *Amer. J. Math.* **114** (1992), 813-874.

[7] H. Watanabe, Estimates of maximal functions by Hausdorff contents in a metric space, *Proceedings of IWPT2004, Advanced Studies in Pure Mathematics* (to appear).



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