Switching Graphs and Digraphs Associated with Total Reconstructed Sets from Two Projection Data

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Abstract

We discuss the structure of the total reconstructed sets from given two projections of a plane figure. In case uniqueness does not hold, we have many solutions which are related among them via modifications of switching components. We let this structure correspond a graph or a digraph. Our primary aim is to overview the origin of complexity of non-uniqueness of the reconstruction, and to obtain way to a good solution. But the results are interesting also as a new source of examples of graphs.

keywords discrete tomography, two projections, switching component, switching graph

1 Introduction

We reconsider the reconstruction problem of plane figures from their two projection data parallel to the axes. In the case of continuous figures, the solvability and the uniqueness were first discussed by G. Lorentz [8]. Its discrete version was first discussed by Ryser [10], in terms of a matrix with only two kinds of entries 0,1. The condition for the uniqueness of the solution, and algorithms to obtain a solution when it is not unique, were essentially solved by them. It was also shown by Ryser that any two solutions are connected to each other by a series of modification of switching components. There are many further works on this problem [5, 6, 7] etc. We too considered a problem of finding better solutions by means of weight function estimating the goodness of the solution [2]. Here we pose a new problem of considering the structure of the set of whole solutions, by means of a graph attached to it. This point of view also enables to see clearly what is essential for the structure of the solution set. Our final aim is to know how the good reconstruction image is located in this graph and how to reach it. In this report we present general properties of switching graphs together with many examples.

After we prepared our former version [3], we found that T. Yung Kong and Gabor T. Herman [4] had already proposed the same notion with the name of *Ryser graph* and used it also to find a better reconstruction. Since Ryser himself did not consider any graph, we keep here our naming of switching graph, and newly add the notion of switching digraph which naturally arises from our type distinction for the switching components.

2 Review of known results to be used

Discrete tomography is usually considered as a point counting problem for the subsets of integral lattice $\mathbb{Z}^2$ in the plane. We shall, however, treat a plane figure F consisting of unit squares with the integral corner points, since it is more related to the continuous case and naturally arises as the discretization of the latter. We shall call a unit square a cell and denote it by the coordinates of its lower-left corner (x, y).

Thus the x-projection $f_y(x)$ of a discrete figure F means the locally constant function on the x-axis counting the cells of F over these points, which just corresponds to the value of line integrals along the y-axis of the characteristic function $f(x, y)$ of F. We identify $f_y(x)$ with its below-graph

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set $P = \{(x, y) \in \mathbf{R}^2;\ 0 \le y \le f_y(x)\}$. We use similar symbols $f_x(y)$ and $Q = \{(x, y) \in \mathbf{R}^2;\ 0 \le x \le f_x(y)\}$ for the other projection to the y-axis.

$f_{xy}(x)$ denotes the re-projection of $f_x(y)$ or Q to the x-axis, which is defined for $x \ge 0$ by

$$f_{xy}(x) = \mathrm{meas}(\{y; f_x(y) \ge x\}). \tag{1}$$

This apparently complex notation will become familiar if it is noticed that the last suffix always indicates the axis along which the last projection was made. $f_{yx}(y)$ is defined in a similar way, replacing the role of x, y. Furthermore $f_{yxy}(x)$ denotes the re-projection of $f_{yx}(y)$, which does not change the associated figure but is merely introduced to adjust the independent variable of the related function from y to x. (See Fig. 18 for the illustration.)

The reconstruction problem is to determine $f(x, y)$ from $f_y(x)$ and $f_x(y)$. The solvability condition is

$$f_{xy}(x) \ge f_{yxy}(x) \quad \text{for } \forall x \in \mathbf{R}, \quad \text{and} \quad f_{xy}(\infty) = f_{yxy}(\infty).$$

The uniqueness condition is

$$f_{xy}(x) = f_{yxy}(x) \quad \text{for } \forall x \in \mathbf{R}.$$

A plane figure F is called non-unique or unique if the reconstruction problem from its projection data has a solution other than F or not. A *switching component* $(x_1, y_1) \in F$, $(x_2, y_2) \in F$ is a pair of cells in F such that $(x_1, y_2) \notin F$, $(x_2, y_1) \notin F$. If we replace the former pair by the latter, the two projections do not change. It is a typical example of non-unique figure. It is known that the existence of a switching component is an equivalent condition of non-uniqueness, and any two solutions can be deformed to each other by a series of appropriate switching of such pairs (Ryser [10]). A switching component is called *type* 1 if $(x_2 - x_1)(y_2 - y_1) > 0$, i.e. if it makes a positive inclination, and *type* 2 otherwise.

Kong-Herman [4] and independently we in [2] tried to find a better figure among the reconstructions by modification of the switching components, employing a goodness function or a weight function estimating the quality of the figure. We especially tried to deform all type 2 pairs to type 1. Intuitively this modification makes the figure more condensed around the principal diagonal $x = y$ of the plane.

3 Examples of switching graphs and digraphs

Given a plane figure F we make a graph G, of which the vertices are the solutions of the reconstruction problem, that is, all the plane figures with the same x- and y-projections. We draw an edge between two vertices if the corresponding figures deform to each other by just one switching of a pair. We call thus obtained graph the *switching graph* for F. It was formerly introduced by [4] and called Ryser graph, as mentioned in the Introduction. We further give the direction to each edge so that it is oriented from type 2 to type 1 modification, and thus define the *switching digraph*. The following are examples of switching digraphs for small size figures. Of course they become examples of switching graphs if we ignore the directions of the edges.

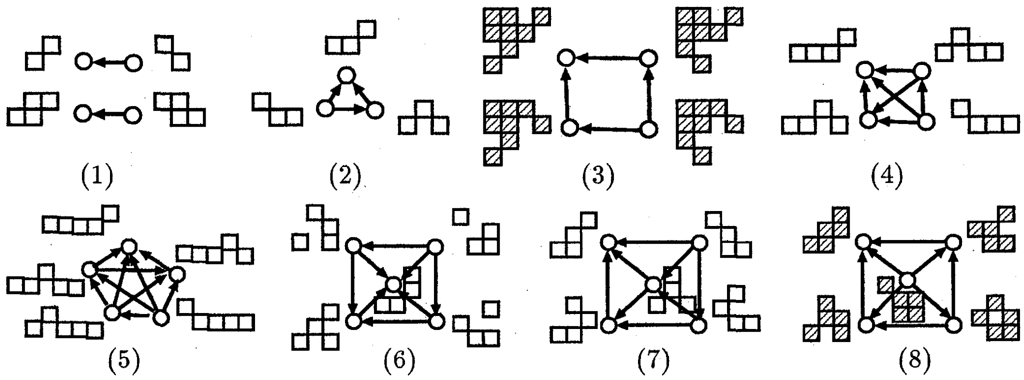
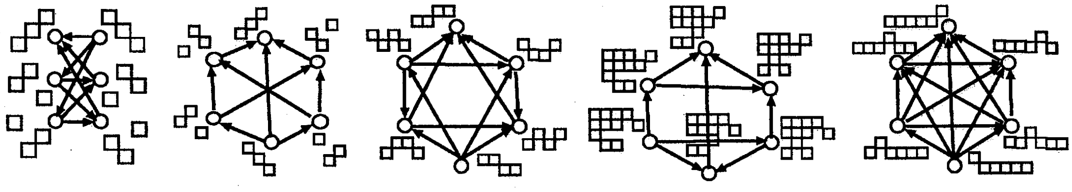


Figure 1: Examples of switching graphs for small patterns

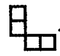
These seemingly enumerate all the switching graphs of order at most 5, although we have not yet checked it rigorously. The following are fundamental examples of order 6. There are apparently different ones, e.g. (9) and (9)′ are the same graph drawn in a different way:

Switching Graphs and Digraphs



(9) (9)′ (10) (11) (12)

Figure 2: hexagons

There are several symmetries which can be easily noticed. The reflection with respect to the principal diagonal, or equivalently the exchange of the x and y coordinates applied to the figure does not change its switching digraph. Similarly, the horizontal and the vertical reflection, or the rotation by 90 degree do not change the switching graph, but reverses the direction of all the edges in the digraph. More generally, the switching graph is invariant under permutation of columns or rows of the original figure, but some of the edges may change the direction in its switching digraph. Thus the switching graphs (6) and (7) in Fig. 1 are essentially equivalent, although they do not contain literally any common pattern. Notice that in the discrete case the rearrangement of Lorentz is achieved by these permutations applied to the x- resp. y projections. Therefore in constructing the switching graph, we can always assume without loss of generality, if we like, that both projections are monotone decreasing. This would reduce these two equivalent figures to another, unique figure like .

On the contrary, (8) is not related with these in the same way, although it has the same switching graph. Whereas two graphs in Fig. 1 (1), the equality of the graph is rather obvious because of the middle wall which serves nothing to the switching, it is not so obvious for (7) and (8) of Fig. 1: Between these figures there is a vertex of which the corresponding patterns are in the inclusion relation, but are not in the other vertices. The reason why their switching graphs agree will be clarified by Proposition 4.10. We will call a figure *minimal* if it has the least number of cells among those with the same switching graph. We do not know yet if after normalization to decreasing order of projections, a minimal figure is unique to the given graph up to the obvious diagonal symmetry.

In general, it is difficult to determine the switching graph for a given figure of general size. In this section we give examples of switching (di-)graphs of regular figures which can be determined for general size.

Lemma 3.1 *The switching graph of a simple diagonal linear figure $F = \{(i,i); i = 1,2,\dots,n\}$ is a 2-partite regular graph of degree $\dfrac{n(n-1)}{2}$ with $n!$ vertices.*

Proof. This is simple because, for such a figure, switching is nothing but a transposition of two columns (or rows). There are $n(n-1)/2$ such possibility from any state (or solution). Moreover, the transposition changes the parity, which proves the bipartiteness. QED

In the sequel, we let Δ_n denote the graph given by Lemma 3.1 to use it as a building block for more complicated graphs. Thus $v(\Delta_n) = n!$ and $e(\Delta_n) = \dfrac{n(n-1)}{4}n!$. Examples are Δ_3 of Fig. 2 (9) and Δ_4 of Fig. 14 left.

In order to describe more complicated examples, we introduce new notation. We hoped to use traditional notation in the graph theory (e.g. [1]). But these constructions do not seem to be popular there.

Definition 3.1 Given two graphs, G, H, we denote by $G \xrightarrow{k} H$ a new graph consisting of vertices and edges of G, H augmented by new edges, which connect each vertex of G with appropriate k vertices of H. (We do not make precise the concrete correspondence of vertices. In the sequel, it is made to maximally preserve the symmetry. Also, the arrow does not intend the directed graph.) In the special case $H = G$, $G \xrightarrow{1} G$ will be simply denoted by $G - G$.

Given three graphs G, H, K such that K is contained as an induced subgraph of G, H (that is, together with the original edges between the vertices), we denote by $G \underset{K}{\cup} H$ the pasting of G, H through the sticking tab K. That is, we overlap G and H through the common subgraph. By $G \underset{K}{\xrightarrow[\cup]{k}} H$ we denote the graph $G \underset{K}{\cup} H$ supplied with further edges which go from each vertex in $G \setminus K$ to k appropriate vertices of $H \setminus K$. In the special case $G = H$, $G \underset{K}{\xrightarrow[\cup]{1}} H$ will be simply denoted by

$G \underset{K}{\overline{\cup}} G$.

By Poly$(n, G, *)$ we denote a complex graph consisting of regular n-gon such that each vertex is a copy of the graph G and that any pair of vertices (G, G) is connected by the rule $*$. Here $*$ stands for

$\emptyset$ if there are no edges added between G's,
$-$ if an edge is added to all pairs of vertices of G's,
$\xrightarrow{k}$ if there go k edges from every vertex to selective G's.
$\underset{K}{\cup}$ if G's are pasted through the common subgraph K,
$\underset{K}{\overline{\cup}}$ if G's are connected with this rule

In the last two cases, it is supposed that G contains mutually disjoint $n-1$ induced subgraphs each isomorphic to K which serve as different sticking tabs to paste each other. Also, the edges are added in a way preserving as much the symmetry as possible.

Poly$(n, G, \underset{K}{\cup})$ is abbreviated to Poly(n, G, K). For example, if G is the regular hexagonal graph and K is the segment, then Poly$(4, G, K)$ becomes a tetrahedron-like graph as in Fig. 3.

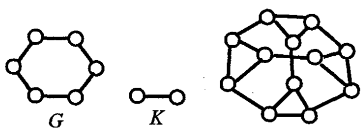


Figure 3: Example of Poly$(4, G, K)$.

Now we have

Lemma 3.2 *Let us denote by L_n the switching graph of a simple diagonal linear figure plus one element $F = \{(i,i); i = 1, 2, \ldots, n\} \cup \{(0,1)\}$. (See Fig. 4.) It is recursively imaged as follows:*

$$\begin{array}{c} \mathrm{Poly}\left(\frac{(n-1)(n-2)}{2}, \Delta_{n-1}, \xrightarrow{2(n-3)}\right) \\ {}^{2}\swarrow \quad \searrow^{2} \\ \Delta_n \underset{\Delta_{n-1}}{\overline{\cup}} \Delta_n \end{array}$$

Hence the number of vertices $v(L_n)$ of this graph is equal to $\frac{(n+1)!}{2}$, and the number of edges $e(L_n) = \frac{(n-1)(n+2)!}{8}$.

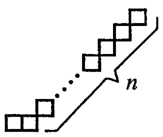


Figure 4: Original pattern of Lemma 3.2.

Proof. It is clear that the required graph contains two induced subgraphs isomorphic to Δ_n, corresponding to those derived patterns in which the cell $(0,1)$ resp. $(1,1)$ is fixed. They are connected to each other by a subgraph Δ_{n-1} corresponding to those patterns in which both of these two cells are fixed. Each vertex of one $\Delta_n \setminus \Delta_{n-1}$ is connected by an edge to the corresponding vertex of the other $\Delta_n \setminus \Delta_{n-1}$ which corresponds to the exchange of the cells in x-level 0, 1. The remaining patterns are classified by the position of two cells sitting in the base row of y-level 0 which came down in place of original ones. There are ${}_{n-1}C_2 = \frac{(n-1)(n-2)}{2}$ possibilities from $(i,1)$, $i = 2, \ldots, n$. Once these two cells are fixed, the remaining cells constitute a figure equivalent to Δ_{n-1}. Every vertex of each Δ_{n-1} has just two edges to each of $\Delta_n \setminus \Delta_{n-1}$ which correspond to the switching operation returning the floating cell in $x = 0$ or 1 to the original base row employing one of these cells now at the base row. Also, there is one edge from every vertex of one Δ_{n-1} to another with one common base cell, corresponding to the exchange of the other base cells. Hence $2(n-3)$ edges from one group

Δ_{n-1} arises. Thus the above graph structure is confirmed. The number of vertices is a mere simple combinatorial calculation:

$$\begin{aligned} v(L_n) &= 2v(\Delta_n) - v(\Delta_{n-1}) + \frac{(n-1)(n-2)}{2}v(\Delta_{n-1}) = 2n! - (n-1)! + \frac{(n-1)(n-2)}{2}(n-1)! \\ &= \frac{1}{2}\{(4n-2+(n-1)(n-2)\}(n-1)! = \frac{(n+1)!}{2}. \end{aligned}$$

Now the number of edges is equal to

$$\begin{aligned} e(L_n) =&2e(\Delta_n) - e(\Delta_{n-1}) + v(\Delta_n) - v(\Delta_{n-1}) + \frac{(n-1)(n-2)}{2}e(\Delta_{n-1}) \\ &+ 4\frac{(n-1)(n-2)}{2}v(\Delta_{n-1}) + 2(n-3)\frac{(n-1)(n-2)}{2}v(\Delta_{n-1})\frac{1}{2} \\ =&\frac{n(n-1)}{2}n! - \frac{(n-1)(n-2)}{4}(n-1)! + n! - (n-1)! \\ &+ \frac{(n-1)(n-2)}{2}\frac{(n-1)(n-2)}{2}(n-1)! \\ &+ 2(n-1)(n-2)(n-1)! + (n-3)\frac{(n-1)(n-2)}{2}(n-1)! \\ =&\frac{(n-1)}{8}\{4n^2 - 2(n-2) + 8 + (n-1)(n-2)^2 + 16(n-2) + 4(n-2)(n-3)\}(n-1)! \\ =&\frac{(n-1)(n+2)!}{8}. \end{aligned}$$

We shall give another interpretation for this graph in Corollary 4.14, which produces a very simple solution. QED

In the sequel, we let L_n denote this graph. An example is L_3 of Fig. 14 right. Notice that the added element may be any of (i,j) with $i \notin [1,\ldots,n]$ and $j \in [1,\ldots,n]$, or vice versa. On the other hand, if both i,j are outside the range $[1,\ldots,n]$, the graph reduces to the case in Lemma 3.1 with n increased by 1.

Lemma 3.3 *The switching graph of a simple diagonal linear figure plus one element* $F = \{(i,i); i = 1,2,\ldots,n\} \cup \{(1,2)\}$ *is a pasting of pairs of* $n-1$ *copies of* L_{n-1} *to each other through the tab* L_{n-2}*, with the remaining vertices tied by an edge, namely,* Poly$(n-1, L_{n-1}, \underset{L_{n-2}}{\sqcup})$.

Hence the number of vertices is equal to $(n+2)(n-1)(n-1)!/4$*, and the number of edges is* $(n^2+9n-4)(n-1)(n-2)(n-1)!/16$.

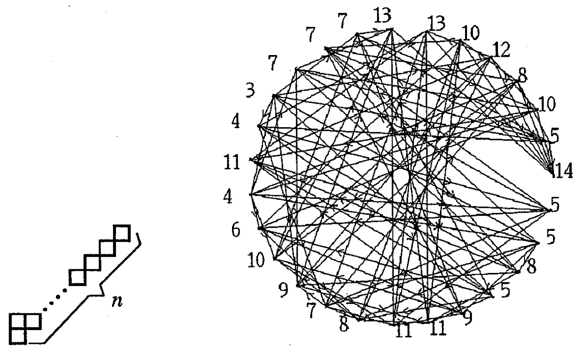


Figure 5: Original pattern of Lemma 3.3 and example of switching graph.

Proof. The patterns are classified by the new place of the cell $(0,1)$ in the first column. There are $n-1$ choices. When this position is fixed, the remaining freedom is just that of L_{n-1}. (Imagine the figure obtained by removing the row containing this floating cell.) Among two such, there are common vertices which correspond to the figures obtained by moving the cell $(1,2)$ to the position of the other's floating cell. These form the same graph as L_{n-2}. (Imagine that after removing these two rows, the remaining cells constitute L_{n-2} where the row with y-level 2 has two cells. They are not in the leftmost positions, but no matter.) Thus the total number of vertices is equal to

$$\begin{aligned} (n-1)v(L_{n-1}) - \frac{(n-1)(n-2)}{2}v(L_{n-2}) &= \frac{(n-1)n!}{2} - \frac{(n-1)(n-2)}{2}\frac{(n-1)!}{2} \\ &= \frac{(n+2)(n-1)(n-1)!}{4}. \end{aligned}$$

Switching Graphs and Digraphs

The number of edges is

$$\begin{aligned}
&(n-1)e(L_{n-1}) - \frac{(n-1)(n-2)}{2}e(L_{n-2}) + \frac{(n-1)(n-2)}{2}(v(L_{n-1}) - v(L_{n-2})) \\
&= (n-1)\frac{(n-2)(n+1)!}{8} - \frac{(n-1)(n-2)}{2}\frac{(n-3)n!}{8} + \frac{(n-1)(n-2)}{2}\frac{n! - (n-1)!}{2} \\
&= \frac{(n^2+9n-4)(n-1)(n-2)}{16}(n-1)!.
\end{aligned}$$

QED

Fig.5 right is a computer-generated example of digraph for $n = 4$. (Figures denote the value of the standard weight function (2) shifted by 15.) Notice that we can choose any (i, j), $1 \le i, j \le n$, $i \ne j$ as the off-diagonal cell to obtain the same switching graph. Thus Fig. 1 (7) serves as an example of Lemma 3.3 for $n = 3$. On the other hand, if either of i, j is outside the range $[1, \dots, n]$, it reduces to the case in Lemma 3.2.

Lemma 3.4 *The switching graph of two slided shells each of size m, n as in Fig. 6 has $\frac{(m+n)!}{m!n!}$ vertices. It is regular of degree mn, hence the total number of edges is $\frac{(m+n)!}{2(m-1)!(n-1)!}$. Especially, in case $n = 1$ it is a complete graph of order $m + 1$.*

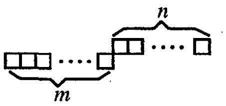


Figure 6: Original pattern of Lemma 3.4.

Proof. Actually, each derived pattern can be characterized by the position of the cells moved upward from the m cells on the ground and those equal number of cells moved downward from upstairs. Therefore the total number is equal to

$$\sum_{k=0}^{\min\{m,n\}} {}_mC_k \cdot {}_nC_k = \sum_{k=0}^{\min\{m,n\}} {}_mC_k \cdot {}_nC_{n-k}.$$

This is the coefficient of the term x^n in the expansion of $(x+1)^{m+n}$, hence $\frac{(m+n)!}{m!n!}$. We shall give a clearer explanation for this value in Corollary 4.14. For any such pattern, the possibility of switching, namely the degree at this vertex, is clearly the number of pairs of a cell from the ground and a cell from upstairs, hence mn. QED

The case $m = n = 2$ gives 6 vertices and 12 edges, as verified in Fig. 2 (11). Cases of complete graph are found in Fig. 1 (4), (5) and Fig. 2 (12).

Lemma 3.5 *The switching graph of m by n rectangle (which is an example of unique figure) plus one element $F = \{(i, j); i = 1, 2, \dots, m, j = 1, 2, \dots, n\} \cup \{(0, 0)\}$ is a kind of suspension:*

$$\begin{array}{c} v_0 \\ {}^{mn}\downarrow \\ G_{m,n} \end{array}$$

where $G_{m,n}$ is a regular graph of mn vertices with degree $(m-1) + (n-1)$.

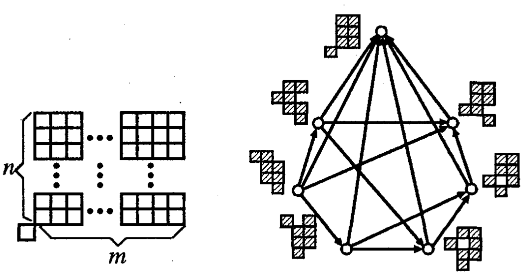


Figure 7: Original pattern of Lemma 3.5 and example for $m = 2, n = 3$.

Proof. The added one cell can switch with any of the othewise immovable cells. Thus from the special vertex v_0 corresponding to the original figure there start mn edges to new figures. Since we cannot make two holes in the rectangle they are all the vertices of our graph. Once a cell is exiled from the rectangle, the produced hole arrows to swap each of the outside cell with another in the same row resp. column. Thus the vertices other than v_0 has $(m-1)+(n-1)$ edges. QED

Fig. 1 (8) is an example with $m = n = 2$. Fig. 7 right shows another example with the structure of suspension made visible.

The switching graph of a general unique figure with n elements plus one isolated element (i.e. with no common element in its column or row) is too complicated to describe, as is easily recognized. This case shows that whereas we have a hope to have simpler graph if we have enough elements, even a block, which in itself could be considered as one element, may blow up by simple addition of one independent element.

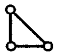
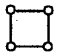
4 Properties of switching graph

We now give several abstract assertions concerning the structure of the switching graph. First we examine characteristic features of the switching graph. The following is fundamental, but a mere paraphrase of Ryser's theorem that any two reconstructions can be deformed to each other by a series of switching operation:

Proposition 4.1 *The switching graph is connected.*

The following lemma is very useful:

Lemma 4.2 *(***triangle-square lemma***) If a switching graph contains a path of the form*  *,*

then it contains either  *or*  *completing the original path.*

Proof. If independent pairs P, Q and R, S switch in the two successive edges, then we can obviously obtain a square, as in Fig. 8 A. If the switching occurs among three cells P, Q, and P′, R, where P′ is the new position of P, then we can also obtain a sqare, as in Fig. 8B, C, according to whether Q, R is a switching component or not. Thus the remaining is the case where the third cell R is in the same column or row as one of the first switching component, say, Q. See Fig. 8 D. This time we can obviously obtain a triangle as shown in the figure. QED

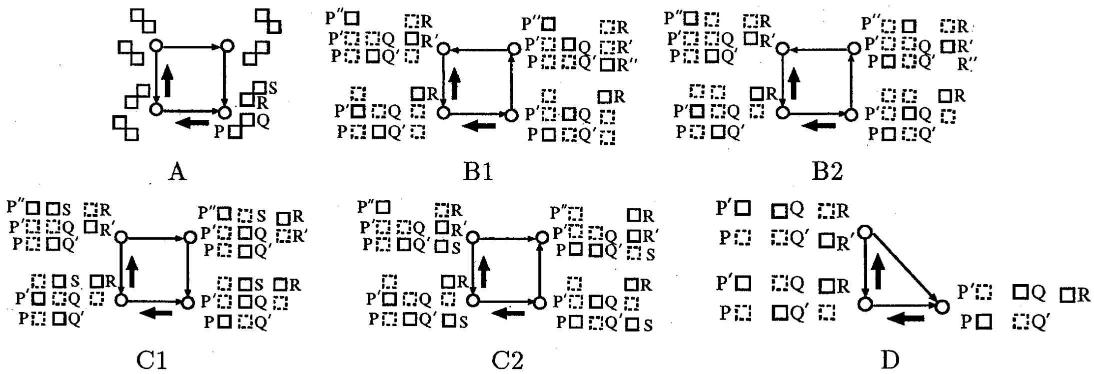


Figure 8: Square-Triangle subgraphs

We have proved the following in the same time:

Corollary 4.3 *Any primitive square subgraph of a switching graph agrees with either of the* 4 *patterns:* A, B1, B2, C1, C2 *in* Fig. 8.

The following assertions are obtained at once from Lemma 4.2:

Proposition 4.4 *A switching graph has no dead end except for the one with only one edge.*

Proposition 4.5 *There is no primitive cycle of length ≥ 5 in a switching graph. That is, every cycle has a shortcut of length ≤ 4 (possibly passing through a new vertex in case of length 4).*

Lemma 4.6 (Path association lemma) *If there is a subgraph as below, we have a new path of length* ≤ 2 *between the vertices* P *and* S.

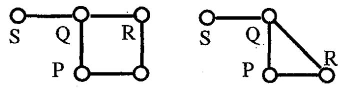


Figure 9: Figure for Lemma4.6

Proof. If QS switches a pair totally different from that for QR, then we have clearly a new path of length two from S to P back of the type Fig. 8 A. Thus assume that PQ and QS switches among three cells common with PQ and QR. Then, If the three cells are on the separate rows and columns, there is a square path containing PQS and not PQR as povided by either one of B1, B2 or of C1, C2 of Fig. 8. If not, there is a new triangular path PQS as in Fig. 8 D. QED

Proposition 4.7 *Except for the three graphs, line segment (Fig.* 1 (1)), *triangle (Fig.* 1 (2)) *and the square (Fig.* 1 (3), *the degree at any vertex of a switching graph is at least* 3.

Proof. Except for these three special graphs there exists at least one vertex of degree ≥ 3 in view of Proposition 4.5. Thus if there is a vertex P of degree 2, then there is one such adjacent to a vertex say, Q, of degree ≥ 3. By Lemma 4.2, PQ is contained in a cycle of length 3 or 4. From Q starts anothe edge QS. Thus by the above lemma, P has another edge. QED

We now examine relations between switching graphs and geometrical operations on the figures.

Proposition 4.8 (Direct product) *The switching (di-)graph of the block figure as in the left of Fig.* 10, *where* G *resp.* H *(by abuse of notation) stands for a figure corresponding to the switching (di-)graph* G *resp.* H, *is the direct product* $G \times H$ *of the two (di-)graphs: Its vertices are the product set* $v(G) \times v(H)$, *and its edges are those between* (a, c) *and* (b, c) *inherited from the one between* a *and* b *in* G, *and those between* (a, c) *and* (a, d) *inherited from the one between* c *and* d *in* H.

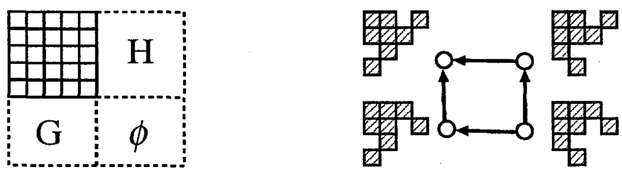


Figure 10: Direct product: left: general image; right: example.

The proof may be obvious. Recall that if we simply make the direct sum of two patterns G, H without the disturbing block as above, we have scarcely any means to control the resulting graph. In particular, it is not completely determined by the two switching graphs corresponding to G, H, as remarked after the proof of Lemma 3.5. By the way, we shall show that for the square graph of the special example in the right of Fig. 10, this figure is minimal and essentially unique as such.

Lemma 4.9 *A figure of which the switching graph is the square contains the pattern of Fig.* 10 *to the right. Any other cell, if exists, does not concern the switching.*

Proof. Denote by 1 to 4 the vertices. Assume that the figure corresponding to vertex 1 has a switching component P, Q (without loss of generality we can assume their position as in Fig. 11 to the left), and that it goes to P′, Q′ in vertex 2. We divide the cases here. First assume that a new cell, say, Q′ switches with another cell R and goes to Q″ in passing to vertex 3. Q″ is in the same column as Q, Q′ as in the left of Fig. 11. Then the three cells P′, Q″, R′ all have to return to the original position by passage from $3 \to 4 \to 1$. For this to be possible, either P′ and Q″ or P′ and R′ should be switched in passage from 2 to 3, and not both should be possible, because otherwise three edges would emanate from vertex 2. If the switching of P′ and R′ is blocked, there should be a cell in position S in the figure. But then the switching P′ and Q″ should be possible, and position P″ should be vacant, which would produce three edges at vertex 1, corresponding to the switchings P-Q, P-R, P-S, a contradiction. The matter is the same for another choice.

Thus in passage from 2 to 3, a different pair, say, R, S should be switched. Then to disturb e.g. excessive switching of Q, R, we should have a cell in either of T, U. If we have T, we should have no cell at U, because otherwise T-U would switch after the switching of P-Q, R-S. By similar argument we conclude that there should be cells at all the positions at upper left region and no cells at the lower right region, to block the excessive switching. We thus obtain essentially a same figure as in Fig. 10 to the right. QED

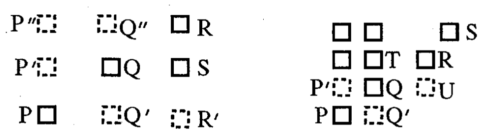


Figure 11: Figure for proof of Lemma 4.9

The following may be obvious.

Proposition 4.10 (Complement) Let F be a figure contained in the rectangle I of the integral lattice. Then the figure $CF := I \setminus F$ has the same switching graph as F.

In fact, a switching component of type 1 (resp. 2) in F corresponds to a switching component of type 2 (resp. 1) in CF. Every pattern derived from F by switching corresponds to just one pattern derived from CF. Notice that surrounding a figure by a rectangular frame of cells does not change the switching graph. Therefore the result does not depend on the choice of I.

Corollary 4.11 *The switching graph for the figure* $F = \{(i,0); 1 \le i \le m\} \cup \{(0,j); 1 \le j \le n\}$ *in Fig.* 12 *agrees with the one for the figure in Lemma* 3.5.

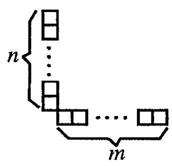


Figure 12: Figure of Corollary 4.11.

The following assertion may sound somewhat like Hadamard-Oka's principle of making the problem easier by raising the dimension.

Proposition 4.12 (Contraction) *Assume that a row of a pattern contains m cells, each of which is unique in their columns. Then the switching graph G for this pattern can be obtained from the graph H of the new pattern which has these m cells in new independent m rows as follows: Search the vertices of H which correspond to the permutations of m cells in these new m rows. Contract these to one vertex and make a reduced graph. If these m cells are not in separate columns, delete the corresponding vertices, together with the edges emanating from these to other vertices. The degree at each vertex is diminished by* $\dfrac{m(m-1)}{2}$ *by this contraction, and if there are deleted vertices, more edges are deleted. Thus a regular graph produces a regular graph again by this process if and only if the remaining cells are all in different columns. The same assertion holds for a column.*

Proof. The relation of these m cells at this special row with other cells are the same in both graphs. The difference lies only in the fact that while they cannot exchange among them within this rows without help of other cells in the original pattern, they can do freely in the modified one. Thus $m!$ cells in the latter are united to one cell in each group of patterns with the same configuration of the remaining cells. Thus the number of vertices reduce by the factor $m!$. Concerning the edges, those corresponding to the switching among these m cells disappear, whereas those to the remaining cells are conserved. QED

The following special case is very useful:

Corollary 4.13 *The switching graph, denoted by* $L_{n,m}$*, of the pattern of cells* $\{(i,1); 0 \le i \le m-1\} \cup \{(j+m-1,j); 1 \le j \le n\}$ *as in Fig.* 13*, is obtained from* Δ_{m+n-1} *by reducing each subgraph* Δ_m *in the latter to one vertex. Hence the number of vertices is equal to* $\dfrac{(n+m-1)!}{m!}$*. The graph is* *regular, with degree* $\frac{(n-1)(n+2m-2)}{2}$*, hence has* $\frac{(n-1)(n+2m-2)(n+m-1)!}{4m!}$ *edges in total. As a particular case, the switching graph* L_n *of the pattern in Lemma* 3.2 *can be obtained from that of* Δ_{n+1} *in Lemma* 3.1 *by identifying a pair of patterns corresponding to the switching of the two cells in the lowest row. Thus the graph is reduced to a half. Namely,* $v(L_n) = \frac{(n+1)!}{2}$*. Also, the graph is regular with degree* $\frac{(n+2)(n-1)}{2}$*, hence* $e(L_n) = \frac{(n+2)(n-1)}{2}\frac{(n+1)!}{4} = \frac{(n-1)(n+2)!}{8}$.

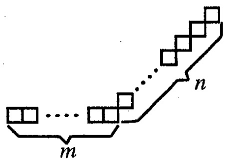


Figure 13: The total number of cells is $n+m-1$.

In fact, the number of edges emanating from a vertex is equal to

$$\frac{(n+m-1)(n+m-2)}{2} - \frac{m(m-1)}{2} = \frac{(n-1)(n+2m-2)}{2}.$$

Notice that L_n in Lemma 3.2 corresponds to $L_{n,2}$ here, and the result agrees with the former one.

Notice also that the case $n=2$ here is the same as the case $n=1$ in Lemma 3.4. We can generalize this as follows:

Corollary 4.14 *The result of Lemma* 3.4 *can be obtained from* Δ_{n+m} *by applying the contraction process of Proposition* 4.12 *twice, separately to the group of* n*- resp.* m*-cells. Hence the number of vertices is equal to* $\frac{(m+n)!}{m!n!}$.

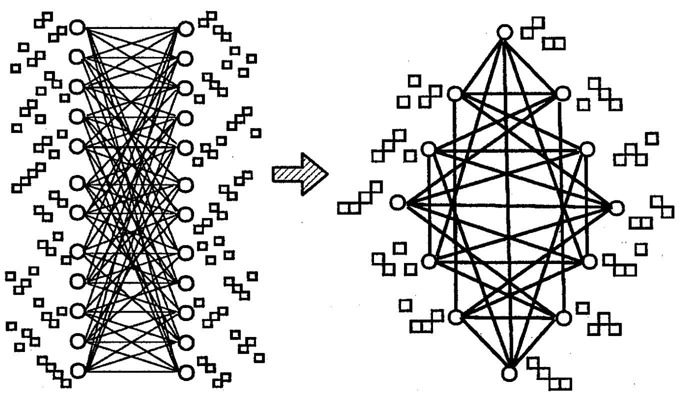


Figure 14: The graph contraction procedure–1.

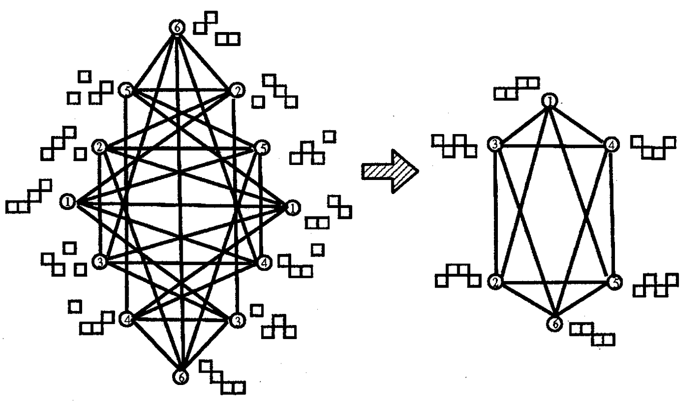


Figure 15: The graph contraction procedure–2.

Fig. 14, Fig. 15 are examples of contraction. As long as the pattern contains only one cell in each column, the contraction procedure continues to work, producing a regular graph. In Fig. 15 the correspondence of the vertices is shown by the numbers in the vertex circles.

Once a column contains more than one cells, however, there appear vanishing cells by contraction, and the resulting graph is no more regular. Henceforth, the contraction becomes more and more complicated. Thus an idea to produce all the switching graphs via contraction from the simple one Δ_n, where n is the total number of cells, and thus deduce their properties, seems hopeless. Here is such an example. (Vanishing vertices are those without numbering).

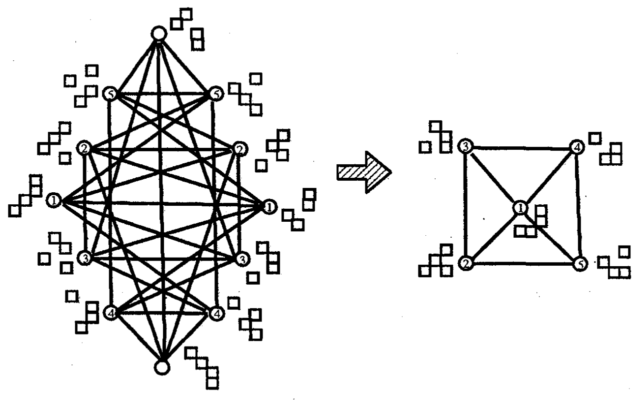


Figure 16: The graph contraction procedure–3.

5 Big examples and modification strategy

As we have seen above, the switching graph of any plane figure has no dead end. Thus we can expect to find a good weight function which makes the switching graph a partially ordered set (or poset for short) in which the target reconstruction pattern is characterized as its global maximal element, and there is no other local maximum. We recall here our standard weight function employed in [2]:

$$f(x,y) = \sum_{(x,y)\in F} xy. \tag{2}$$

It has the following invariance properties for the operations on the original figure: It is invariant by the diagonal reflection; It changes by an additive constant by the shift; It changes sign and additive constant by horizontal and vertical reflection or by the 90 degree rotation. Moreover, this weight function is well coherent with the type modification:

Proposition 5.1 *The switching of type* 2 *to type* 1 *always increases the weight function* (2). *Thus the lattice structure implies the automatic achievement of the maximal element by this type modification. Especially, a switching digraph does not contain a cycle with circularly directed edges.*

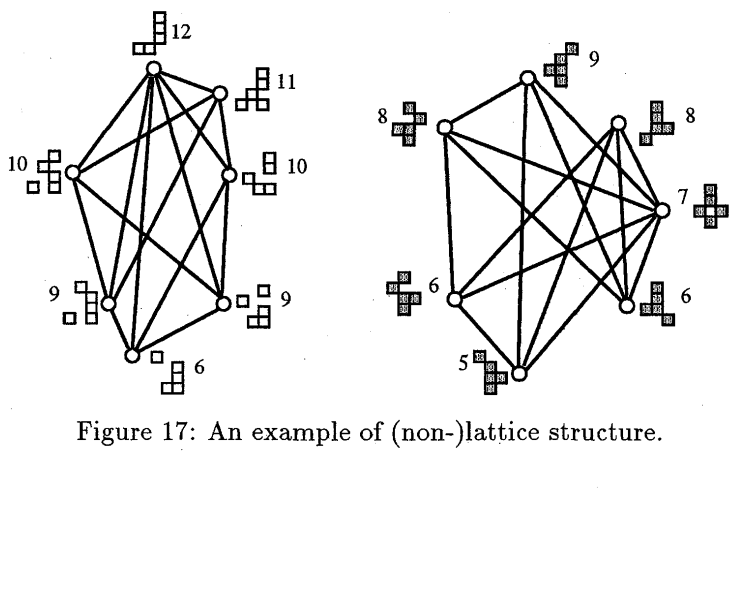


Figure 17: An example of (non-)lattice structure.

Switching Graphs and Digraphs

For simple figures as in Fig. 1-2, the weight function (2) makes them a lattice, and the type modification really attains the maximal element. But the matter is not so simple for large figures. In fact, the main difficulty in type modification process in [2] was the existence of many local maxima for practical complicated figures, and we needed to employ a kind of genetic algorithm with "mutation", that is, intended recession of type 1 to type 2, to get rid of them. It is a very interesting problem to find a criterion to see if a given switching graph becomes a lattice with this weight function (2), and if not, whether there is always a better practical weight function to make the graph a lattice. (In an abstract sense, such one always exists in view of Proposition 4.1 and 4.4.) Here we only show an example of switching graph which has a lattice structure for one weight function and which does not for another one. The left in Fig. 15 has a natural lattice structure with the standard weight function (2), whereas the right, obtained from the former by exchange of rows and columns, hence yielding the same switching graph, has a local maximum. The latter may be considered as the former with the weight function modified by this exchange of columns and rows.

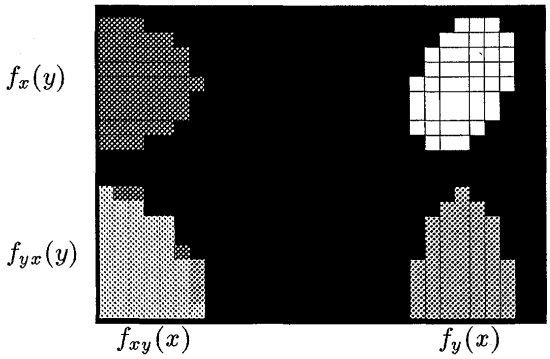


Figure 18: Discrete slant ellipse and its projections

As remarked in [2], whether the global maximum is the desired solution or not is another interesting problem. It is experimentally known that so is the slant ellipse, which is our continuous target of concern. So let us suppose this for the moment. Local maxima arise even for rather coarse discretization of such a natural convex figure. Fig. 18 is an example of descretized slant ellipse. How many vertices and edges does the switching graph of this relatively simple pattern have?

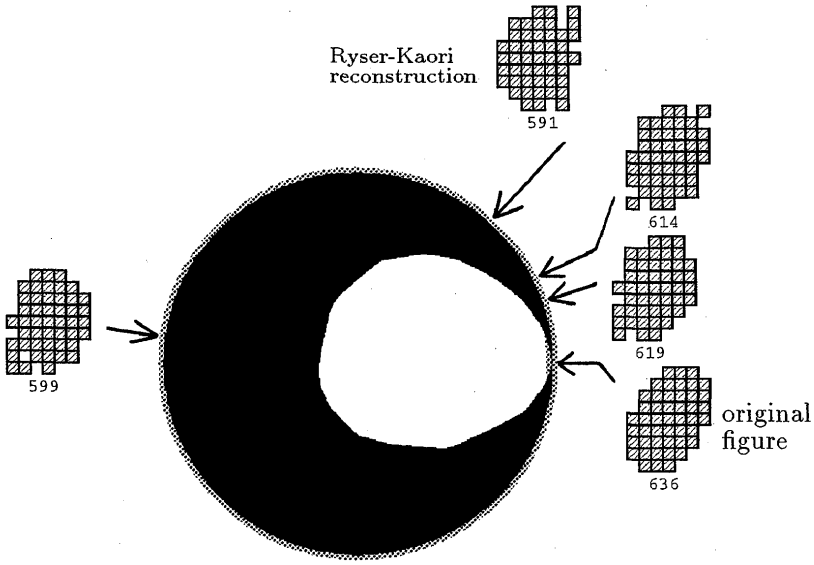


Figure 19: Computer-generated switching graph for the above figure

The answer, produced by our computer program, is rather astonishing. We have 3976 vertices, 54920 edges, the degree ranging $25 \leq d \leq 40$. In the computer-generated figure Fig. 19, vertices are arrainged uniformly on the circle in the discovered order. Edges are drawn by white segments.

The original figure (slant ellipse), Ryser-Kaori's solution of deterministic reconstruction (cf. [2]), and some examples of local maxima encountered in the process of random modifications are indicated by the arrows.

This example clearly shows that we have little chance to attain the true maximum by a continuation of good luck. We need a new strategy. Also, to treat figures of practical size, it is almost impossible to construct the whole switching graph. We are now trying a kind of dynamical algorithm to construct only a necessary neighborhood of the path in the switching graph which may lead from an initial reconstruction to the final solution which is the true maximum.

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