A functional limit theorem for trimmed sums of multi-indexed i.i.d. random variables with slowly varying tail probability

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Abstract

In this paper, trimmed sums of nonnegative multi-indexed i.i.d. random variables with slowly varying tail probabilities are studied. By using point process method, the author proves a functional limit theorem; i.e., the convergence in law with respect to multi-parameter processes over the space of all functions which are "continuous from above with limits from below" endowed with Skorohod's topology.

1 Introduction

Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed (i.i.d.) nonnegative random variables with common distribution function $F(x)$, and let $S_n = X_1 + X_2 + \cdots + X_n$. Now in this paper, we shall assume that

$$1 - F(x) = l(x) \quad x \geq 0, \tag{1}$$

where $l(x)$ is slowly varying at ∞. We shall assume also $l(x) > 0$ for every $x \geq 0$ to avoid exceptional cases, and denote $\frac{1}{l(x)}$ by $L(x)$; it is sometimes more convenient to consider its reciprocal since $l(x)$ is non-increasing. Notice that $L(x)$ is nondecreasing, slowly varying and right-continuous. Since every linear normalization of S_n leads to a degenerate limiting distribution in this case, we consider a non-linear normalization of Darling type: $\frac{1}{n}L(S_n)$ (see [3] for the results). For the process $\frac{1}{n}L(S_{[nt]})$, the convergence of the semi-group was obtained by [11], and its weak convergence in càdlàg function space was obtained by [6]: Let $X_n \xrightarrow{\mathscr{D}} X$ denote convergence in law of random elements of the function space $D([0,\infty) \to \mathbb{R})$ endowed with J_1-topology. This notation shall also be used to express weak convergence of the laws of random elements of any other topological spaces. When we need to emphasize the space, we write, for example, $X_n \xrightarrow{\mathscr{D}} X \quad in\ D([0,\infty) \to \mathbb{R})$.

Theorem A([6])
Under the assumption (1),

$$\frac{1}{n}L(S_{[nt]}) \xrightarrow{\mathscr{D}} m(t), \quad n \to \infty \quad in\ D([0,\infty) \to \mathbb{R}),$$

where $m(t) = \max_{0<s\leq t} p(s), (m(0) = 0)$*, and p is Poisson point process with intensity measure* $x^{-2}dx$.

The present paper is an extension of Theorem A. We shall treat the case of the trimmed sum and multi-dimensional time parameter. Let $\{X_{i_1,i_2,\cdots,i_d}, i_k \in \mathbb{N}, d = 1,2,\cdots\}$ be a system of multi-indexed i.i.d. nonnegative random variables with common distribution function $F(x)$, and let

$$S_{N_1,\cdots,N_d} = \sum_{i_1 \leq N_1} \cdots \sum_{i_d \leq N_d} X_{i_1,\cdots,i_d} \quad (N_k \in \mathbb{N}).$$

For every $r = 0, 1, 2, \cdots$, we define the *trimmed sum* by

$$S^{(r)}_{N_1,\cdots,N_d} = \begin{cases} S_{N_1,\cdots,N_d} & ,\text{for } r = 0 \\ S_{N_1,\cdots,N_d} - (M^{(1)}_{N_1,\cdots,N_d} + \cdots + M^{(r)}_{N_1,\cdots,N_d}), & \text{for } 1 \le r \le N_1 \cdots N_d \\ 0 & ,\text{for } r > N_1 \cdots N_d, \end{cases}$$

where

$$M^{(k)}_{N_1,\cdots,N_d} = \text{k-th} \max_{\substack{(i_1,\cdots,i_d)\in \\ (0,N_1]\times\cdots\times(0,N_d]}} X_{i_1,\cdots,i_d}.$$

In this paper we shall prove the convergence in law of a process $\frac{1}{n^d}L(S^{(r-1)}_{[nt_1],\cdots,[nt_d]})$ over the function space $D([0,\infty)^d \to \mathbb{R})$ which denotes the totality of real-valued functions which are "continuous from above with limits from below" endowed with Skorohod's topology (see [9] for definition). We usually prove the weak convergence in function space by proving weak convergence of the finite-dimensional distributions and then proving tightness. However, it dose not seem easy to check the well-known conditions for tightness such as Bickel and Wichura's moment condition (see [2] and [1]), thus we shall adopt a direct method: On a suitable probability space we shall construct processes which are distributed like $\frac{1}{n^d}L(S^{(r-1)}_{[nt_1],\cdots,[nt_d]})$, $n = 1, 2, \cdots$ and which converge to a limiting process almost surely in Skorohod's topology. In order to state the limiting process, we define

$$(m^{(r)}(t_1,\cdots,t_d))_{t_k\ge 0}, \quad (m^{(r)}(0,\cdots,0) = (0,\cdots,0)),$$

where

$$m^{(r)}(t_1,\cdots,t_d) = \text{r-th} \max_{\substack{(s_1,\cdots,s_d)\in \\ (0,t_1]\times\cdots\times(0,t_d]}} p(s_1,\cdots,s_d),$$

and p is Poisson point process of d-dimensional time parameter with its mean measure $x^{-2}dx$.

2 Main theorem

Our main result is the following theorem.

Theorem 2.1 *Under the assumption (1), for every* $r = 1, 2, \cdots$

$$\frac{1}{n^d}L(S^{(r-1)}_{[nt_1],\cdots,[nt_d]}) \xrightarrow{\mathscr{D}} m^{(r)}(t_1,\cdots,t_d), \quad n \to \infty \quad \textit{in} \quad D([0,\infty)^d \to \mathbb{R}).$$

We divide the proof into two parts. It is clear that Theorem 2.1 follows immediately from the following two propositions:

Proposition 2.2 For every $t_1, \cdots, t_d > 0$

$$\frac{1}{n^d}L(M^{(r)}_{[nt_1],\cdots,[nt_d]}) \xrightarrow{\mathscr{D}} m^{(r)}(t_1,\cdots,t_d) \quad n \to \infty \;\; \textit{in}\, D([0,\infty)^d \to \mathbb{R}).$$

Proposition 2.3 For every $T_1, \cdots, T_d > 0$

$$\lim_{n\to\infty} \sup_{\substack{(t_1,\cdots,t_d)\in \\ (0,T_1]\times\cdots\times(0,T_d]}} \left| \frac{1}{n^d}L(S^{(r-1)}_{[nt_1],\cdots,[nt_d]}) - \frac{1}{n^d}L(M^{(r)}_{[nt_1],\cdots,[nt_d]}) \right| = 0 \quad \textit{a.s.}$$

We shall prove this theorem only for $r = 1$ and $d = 2$, because the case of $d \ge 3$ and $r \ge 2$ can be treared by similar means as $r = 1$ and $d = 2$ replacing *max* with *r-th max*.

Proof of Proposition 2.2.

Let U be a random variable uniformly distributed on $(0,1)$, and let $V_{i,j}$ $(i, j = 1, 2, \cdots)$ be independent copies of $V = \frac{1}{U}$. Now we shall define a real-valued point function p_n as follows. The domain D_{p_n} of

p_n is $\{(\frac{i_1}{n},\frac{i_2}{n}) : i_1, i_2 = 1,2,\dots\}$ and we put $p_n(t_1,t_2) = \frac{V_{i_1,i_2}}{n^2}$ for $(t_1,t_2) = (\frac{i_1}{n},\frac{i_2}{n})$. Thus the counting measure of p_n is

$$N_{p_n}(dt_1dt_2dx) = \sum_{i_1\le[nt_1]}\sum_{i_2\le[nt_2]} \delta_{(\frac{i_1}{n},\frac{i_2}{n},\frac{V_{i_1,i_2}}{n^2})}(dt_1dt_2dx),$$

where $\delta_{(t_1,t_2,x)}$ denotes Dirac measure located at (t_1,t_2,x). Now notice that

$$\begin{aligned} &E[N_{p_n}([0,t_1]\times[0,t_2]\times(x,\infty))] \\ =\;& \sum_{i_1\le[nt_1]}\sum_{i_2\le[nt_2]} P[\frac{V_{i_1,i_2}}{n^2}\ge x] = \sum_{i_1\le[nt_1]}\sum_{i_2\le[nt_2]} P[\frac{1}{V_{i_1,i_2}}\le\frac{1}{xn^2}] = [nt_1][nt_2]\frac{1}{xn^2} \\ \longrightarrow\;& \frac{t_1t_2}{x} = E[N_p([0,t_1]\times[0,t_2]\times(x,\infty))], \quad n\to\infty. \end{aligned}$$

This implies that the point process $\{p_n\}_n$ converge in law to the process p which was defined at the previous section. (For details see [8]. See also [7].) Then we can construct the process $\hat{p}_n$ which converges to p almost surely on a suitable probability space. No confusion of notation should occur, and thus we shall not distinguish between $\hat{p}_n$ and p_n for simplicity. Consequently, it holds that

$$m_n^{(1)}(t_1,t_2) \xrightarrow{\mathscr{D}} m^{(1)}(t_1,t_2), \quad n\to\infty \; in\, D([0,\infty)^2\to R),$$

where $m_n^{(1)}(t_1,t_2) = \max_{\substack{(s_1,s_2)\in\\ (0,t_1]\times(0,t_2]}} p_n(s_1,s_2)$. Since $X_{i_1,i_2} \overset{\mathscr{L}}{\approx} L^{-1}(V_{i_1,i_2})$, where $\overset{\mathscr{L}}{\approx}$ means equivalence in distribution, we have that

$$\frac{1}{n^2}L(M^{(1)}_{[nt_1],[nt_2]}) \overset{\mathscr{L}}{\approx} \frac{1}{n^2}L(L^{-1}(n^2\cdot m_n^{(1)}(t_1,t_2))).$$

We also have that

$$\frac{1}{n^2}L(L^{-1}(n^2\cdot m_n^{(1)}(t_1,t_2))) \sim m_n^{(1)}(t_1,t_2), \quad n\to\infty,$$

where $f(x)\sim g(x)$ means $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$, because we can show that $L(L^{-1}(x))\sim x\ as\ x\to\infty$ from the fact $L(x)$ is slowly varying. Consequently, we have that

$$\frac{1}{n^2}L(M^{(1)}_{[nt_1],[nt_2]}) \xrightarrow{\mathscr{D}} m^{(1)}(t_1,t_2), \quad n\to\infty \quad in\, D([0,\infty)^2\to R),$$

which completes the proof of Proposition 2.2.

Proof of Proposition 2.3.

Since we have that

$$S_{[nt_1],[nt_2]} \overset{\mathscr{L}}{\approx} \sum_{i_1\le[nt_1]}\sum_{i_2\le[nt_2]} L^{-1}(n^2\cdot\frac{V_{i_1,i_2}}{n^2}) = \int_0^{t_1}\int_0^{t_2}\int_{x>0} L^{-1}(n^2x)N_{p_n}(ds_1ds_2dx),$$

it is sufficient for the proof of this proposition to show that for every $T_1,T_2>0$

$$\lim_{n\to\infty}\sup_{\substack{(t_1,t_2)\in\\(0,T_1]\times(0,T_2]}} \left| \frac{1}{n^2}L(\int_0^{t_1}\int_0^{t_2}\int_{x>0}L^{-1}(n^2x)N_{p_n}(ds_1ds_2dx)) - \frac{1}{n^2}L(L^{-1}(n^2m_n^{(1)}(t_1,t_2)))\right| = 0 \quad a.s. \qquad (2)$$

To this end, we shall divide $\int_0^{t_1}\int_0^{t_2}\int_{x>0}L^{-1}(n^2x)N_{p_n}(ds_1ds_2dx)$ into two parts as follows:

$$\begin{aligned} &\int_0^{t_1}\int_0^{t_2}\int_{0<x<\varepsilon}L^{-1}(n^2x)N_{p_n}(ds_1ds_2dx) + \int_0^{t_1}\int_0^{t_2}\int_{x>0}L^{-1}(n^2x)N_{p_n^\varepsilon}(ds_1ds_2dx) \\ =\;& \qquad X_n^\varepsilon(t_1,t_2) \qquad + \qquad Y_n^\varepsilon(t_1,t_2), \end{aligned}$$

where $p_n^\varepsilon = p_n\cdot 1_{\{p_n\ge\varepsilon\}}$. Notice that

$$\begin{aligned} &\left|\frac{1}{n^2}L(X_n^\varepsilon(t_1,t_2)+Y_n^\varepsilon(t_1,t_2)) - \frac{1}{n^2}L(L^{-1}(n^2m_n^{(1)}(t_1,t_2)))\right| \\ &\le \left|\frac{1}{n^2}L(X_n^\varepsilon(t_1,t_2)+Y_n^\varepsilon(t_1,t_2)) - \frac{1}{n^2}L(Y_n^\varepsilon(t_1,t_2))\right| + \left|\frac{1}{n^2}L(Y_n^\varepsilon(t_1,t_2)) - \frac{1}{n^2}L(L^{-1}(n^2m_n^{(1)}(t_1,t_2))\right|, \end{aligned}$$

and hence it is sufficient to show the following two claims; for every $T_1,T_2>0$,

(i) $\lim_{n\to\infty} \sup_{\substack{(t_1,t_2)\in\\ (0,T_1]\times(0,T_2]}} \mid \frac{1}{n^2}L(Y_n^\varepsilon(t_1,t_2)) - \frac{1}{n^2}L(L^{-1}(n^2 m_n^{(1)}(t_1,t_2)) \mid = 0 \;\; a.s.$

(ii) $\lim_{n\to\infty} \sup_{\substack{(t_1,t_2)\in\\ (0,T_1]\times(0,T_2]}} \mid \frac{1}{n^2}L(X_n^\varepsilon(t_1,t_2) + Y_n^\varepsilon(t_1,t_2)) - \frac{1}{n^2}L(Y_n^\varepsilon(t_1,t_2)) \mid = 0 \;\; a.s.$

Let us begin with (i). We can replace $Y_n^\varepsilon(t_1,t_2)$ by the form of finite sum since $N_{p_n^\varepsilon}$ is finite almost surely. Let $n_0(\varepsilon) = N_{p_n^\varepsilon}((0,\infty)^2, \mathbb{R}_+)$, which is a random variable depending on ε, and let ${}^\varepsilon m_n^{(1)}(t_1,t_2) = \max_{\substack{(s_1,s_2)\in\\ (0,t_1]\times(0,t_2]}} p_n^\varepsilon(s_1,s_2)$. Now it follows that

$$\frac{1}{n^2}L(L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) \le \frac{1}{n^2}L(Y_n^\varepsilon(t_1,t_2)) \le \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) \tag{3}$$

and we shall show that it holds almost surely,

$$\lim_{n\to\infty} \sup_{\substack{(t_1,t_2)\in\\ (0,T_1]\times(0,T_2]}} \mid \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) - \frac{1}{n^2}L(L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) \mid = 0, \tag{4}$$

for every $T_1, T_2 > 0$. To this end, first we observe that

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\substack{(t_1,t_2)\in\\ [\varepsilon,T_1]\cup[\varepsilon,T_2]}} \mid \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) - \frac{1}{n^2}L(L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) \mid \\
&\le \lim_{n\to\infty} \sup_{\substack{(t_1,t_2)\in\\ (0,T_1]\cup(0,T_2]}} \mid \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) - \frac{1}{n^2}L(L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) \mid \\
&\le \lim_{n\to\infty} \sup_{\substack{0<{}^\varepsilon m_n^{(1)}(t_1,t_2)\\ \le {}^\varepsilon m_n^{(1)}(T_1,T_2)}} \{ \mid \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) - {}^\varepsilon m_n^{(1)}(t_1,t_2) \mid \\
&\qquad + \mid {}^\varepsilon m_n^{(1)}(t_1,t_2) - \frac{1}{n^2}L(L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) \mid \}.
\end{aligned}$$

Now we have that for every $\lambda > 0$ and $x > 0$,

$$\lim_{n\to\infty} \frac{L(\lambda L^{-1}(n^2 x))}{n^2} = x, \tag{5}$$

since $L^{-1}(x)$ is rapidly varying and increasing. This convergence is uniform for x on every finite interval, since $\frac{L(\lambda L^{-1}(n^2x))}{n^2}$ is monotone in x and its limiting function is continuous. Therefore for every $T_1, T_2 > 0$, it follows almost surely that

$$\lim_{n\to\infty} \sup_{\substack{(t_1,t_2)\in\\ [\varepsilon,T_1]\times[\varepsilon,T_2]}} \mid \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) - \frac{1}{n^2}L(L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) \mid = 0. \tag{6}$$

Next let us consider the part of $(0,\varepsilon]\times(0,\varepsilon]$. It follows that

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{(t_1,t_2)\in(0,\varepsilon]^2} \mid \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) - \frac{1}{n^2}L(L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(t_1,t_2))) \mid \\
&\le \lim_{n\to\infty} \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(\varepsilon,\varepsilon))) \\
&\le \lim_{n\to\infty} \{\mid \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^\varepsilon m_n^{(1)}(\varepsilon,\varepsilon))) - {}^\varepsilon m_n^{(1)}(\varepsilon,\varepsilon) \mid + {}^\varepsilon m_n^{(1)}(\varepsilon,\varepsilon)\}.
\end{aligned} \tag{7}$$

If we choose $\varepsilon > 0$ so that $\varepsilon \le {}^\varepsilon m_n^{(1)}(t_1,t_2)$, then ${}^\varepsilon m_n^{(1)}(t_1,t_2) = m_n^{(1)}(t_1,t_2)$. Therefore from (5) and the fact that $m_n^{(1)}(t_1,t_2)$ converges to $m^{(1)}(t_1,t_2)$ almost surely as $n\to\infty$, it holds that

$$[\text{RHS of (7)}] = m^{(1)}(\varepsilon,\varepsilon).$$

Since $m^{(1)}(\varepsilon,\varepsilon)$ converges to 0 as $\varepsilon \to 0$, we conclude that almost surely

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup_{\substack{(t_1,t_2)\in\\(0,\varepsilon]^2}} \left| \frac{1}{n^2}L(n_0(\varepsilon)\cdot L^{-1}(n^2\cdot {}^{\varepsilon}m_n^{(1)}(t_1,t_2))) - \frac{1}{n^2}L(L^{-1}(n^2\cdot {}^{\varepsilon}m_n^{(1)}(t_1,t_2)))\right| = 0. \tag{8}$$

We have (4) from (6) and (8), and hence (i) is proved.

Next we shall show (ii). Since N_{p_n} is infinite, we can not adopt the same argument for $X_n^{\varepsilon}(t_1,t_2)$ as that for $Y_n^{\varepsilon}(t_1,t_2)$. Thus we shall use the following proposition.

Proposition 2.4 *For every $\varepsilon > 0$, it holds that*

$$\limsup_{n\to\infty}\frac{1}{n^2}L(X_n^{\varepsilon}(t_1,t_2)) \le \varepsilon, \quad for\ any\ t_1,t_2 \ge 0.$$

We postpone the proof of it until Section 3 and return to the proof of (ii). If we choose ε so that $\varepsilon \le {}^{\varepsilon}m_n^{(1)}$, then from (i) and Proposition 2.4 it holds that for all sufficiently large n

$$\frac{1}{n^2}L(Y_n^{\varepsilon}(t_1,t_2)) \le \frac{1}{n^2}L(X_n^{\varepsilon}(t_1,t_2)+Y_n^{\varepsilon}(t_1,t_2)) \le \frac{1}{n^2}L(2\cdot Y_n^{\varepsilon}(t_1,t_2)). \tag{9}$$

Since $L(x)$ is slowly varying and it has been shown that $\frac{1}{n^2}L(Y_n^{\varepsilon}(t_1,t_2))$ converges uniformly, we have for any $T_1,T_2>0$,

$$\begin{aligned}&\lim_{n\to\infty}\sup_{(t_1,t_2)\in(0,T_1]\times(0,T_2]}\left|\frac{1}{n^2}L(2\cdot Y_n^{\varepsilon}(t_1,t_2)) - \frac{1}{n^2}L(Y_n^{\varepsilon}(t_1,t_2))\right| \\ &= \lim_{n\to\infty}\sup_{\substack{(t_1,t_2)\in\\(0,T_1]\times(0,T_2]}}\frac{1}{n^2}L(Y_n^{\varepsilon}(t_1,t_2))\cdot\left|\frac{L(2\cdot Y_n^{\varepsilon}(t_1,t_2))}{L(Y_n^{\varepsilon}(t_1,t_2))}-1\right| = 0 \quad a.s.\end{aligned} \tag{10}$$

From (9) and (10), (ii) is shown immediately. This completes the proof of Proposition 2.3.

3 Proof of Proposition 2.4

For the proof of Proposition 2.4, we prepare

Lemma 3.1 *Let $L(x)\ge 0, x\ge 0$, be nondecreasing and slowly varying at ∞. Then for every $\delta>0$, there exist an $f_0(x)$ and A_δ such that*

(i) $L^{-1}(x) \le f_0(x) \le L^{-1}((1+\delta)x)$, *for any* $x > A_\delta$,

(ii) $\frac{f_0(x)}{x^2}$ *is nondecreasing,* *for any* $x \ge A_\delta$.

Proof

It is known that $L(x)$ is slowly varying if and if $L(x) = a(x)\exp[\int_1^x \frac{\epsilon(t)}{t}dt]$ for some $a(x)$ and $\epsilon(x)$, where $\epsilon(x)\to 0, a(x)\to a<\infty$ as $x\to\infty$ (see [4]). We can assume that $\epsilon(x)$ is continuously differentiable, and that $a(x)>0$ and $\epsilon(x)>0$ here (see the proof of corollary of [4] p.282). Furthermore from easy computation we have $L^{-1}(x) = \exp[\int_1^{c(x)x}\frac{b(t)}{t}dt]$, where we assume that $c(x)\to c>0, b(x)\to\infty$ as $x\to\infty$.

We shall begin with the proof of (i). Notice that it holds for every $\delta>0$, as x is sufficiently large, $c(x)\le(1+\frac{\delta}{2})c \le (1+\delta)c((1+\delta)x)$, therefore we have that

$$\int_1^{c(x)x}\frac{b(t)}{t}dt \le \int_1^{(1+\frac{\delta}{2})cx}\frac{b(t)}{t}dt \le \int_1^{c((1+\delta)x)(1+\delta)x}\frac{b(t)}{t}dt.$$

Put $f_0(x) = \exp[\int_1^{(1+\frac{\delta}{2})cx} \frac{b(t)}{t} dt]$, and then the proof of (i) is complete. We shall next prove (ii). Since

$$\exp[\int_1^{(1+\frac{\delta}{2})cx} \frac{2}{t} dt] = \{(1+\frac{\delta}{2})\}^2 \cdot x^2,$$

we have immediately that

$$\frac{f_0(x)}{x^2} = \{(1+\frac{\delta}{2})\}^2 \cdot \exp[\int_1^{(1+\frac{\delta}{2})cx} \frac{b(t)-2}{t} dt].$$

Now, since $b(x) \to \infty$ as $x \to \infty$, it follows that $\frac{f_0(x)}{x^2}$ is uniformly increasing as x is sufficiently large, and this completes the proof of Lemma 3.1.

Proof of Proposition 2.4.

We devide $X_n^\varepsilon(t_1, t_2)$ into two parts as follows

$$\int_0^{t_1}\int_0^{t_2}\int_{\substack{0<x<\varepsilon \\ n^2x>A_\delta}} L^{-1}(n^2x) N_{p_n}(ds_1 ds_2 dx) + \int_0^{t_1}\int_0^{t_2}\int_{\substack{0<x<\varepsilon \\ n^2x\le A_\delta}} L^{-1}(n^2x) N_{p_n}(ds_1 ds_2 dx)$$

$$\overset{\text{put}}{=} \quad I \quad + \quad II.$$

Let us begin with the part I. From Lemma 3.1 we have that

$$I \le \int_0^{t_1}\int_0^{t_2}\int_{\substack{0<x<\varepsilon \\ n^2x>A_\delta}} \frac{f_0(n^2x)}{(n^2x)^2}(n^2x)^2 N_{p_n}(ds_1 ds_2 dx) \le \frac{1}{\varepsilon^2} L^{-1}(n^2(1+\delta)\varepsilon) \int_0^{t_1}\int_0^{t_2}\int_{x>0} x^2 N_{p_n}(ds_1 ds_2 dx). \quad (11)$$

Since the convergence of p_n has been shown, it follows that

$$\int_0^{t_1}\int_0^{t_2}\int_{x>0} x^2 N_{p_n}(ds_1 ds_2 dx) \longrightarrow \int_0^{t_1}\int_0^{t_2}\int_{x>0} x^2 N_{p}(ds_1 ds_2 dx) \quad a.s. \quad n \to \infty.$$

Therefore we have that for every $\delta > 0$, it holds that almost surely

$$\limsup_{n\to\infty} \frac{1}{n^2} L\,(\text{R.H.S. of (11)}) = \limsup_{n\to\infty} \frac{1}{n^2} L(L^{-1}(n^2(1+\delta)\varepsilon)) = (1+\delta)\varepsilon.$$

Hence it follows that

$$\limsup_{n\to\infty} \frac{1}{n^2} L(I) \le \varepsilon \quad a.s. \quad (12)$$

Next we shall consider the part II. Note that $[nt_1][nt_2]$ is the total number of points of p_n, and hence we see that for every T_1, T_2 $(0 < t_1 \le T_1, 0 < t_2 \le T_2)$,

$$II \le \int_0^{t_1}\int_0^{t_2}\int_{\substack{0<x<\varepsilon \\ n^2x\le A_\delta}} L^{-1}(A_\delta) N_{p_n}(ds_1 ds_2 dx) \le L^{-1}(A_\delta)[nt_1][nt_2] \le L^{-1}(A_\delta) n^2 T_1 T_2.$$

Since we see that $\lim_{n\to\infty} \frac{1}{n^2} L(n^2) = 0$, we have that

$$\lim_{n\to\infty} \frac{1}{n^2} L(L^{-1}(A_\delta) n^2 T_1 T_2) = \lim_{n\to\infty} \frac{1}{n^2} L(n^2) = 0.$$

Therefore it follows that

$$\lim_{n\to\infty} \frac{1}{n^2} L(II) = 0, \quad (13)$$

and hence from (12) and (13) we have

$$\limsup_{n\to\infty} \frac{1}{n^2} L(X_n^\varepsilon(t_1, t_2)) = \limsup_{n\to\infty} \frac{1}{n^2} L(I + II) \le \varepsilon,$$

which completes the proof of Proposition 2.4.

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