Estimates of the Besov norm on a bounded fractal lateral boundary and the boundedness of operators

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Abstract Consider a cylindrical domain $\Omega_D = D \times (0,T)$, where D is a bounded domain with fractal boundary in $\mathbf{R}^d$. Let u be a λ-Hölder continuous function on $\overline{\Omega_D}$ with respect to the parabolic metric ρ. We estimate the Besov norm of the restriction of u to $S_D = \partial D \times [0,T]$ by the $L^p(\Omega_D)$-norm of the sum of $|\nabla_y u(Y)|\mathrm{dist}(Y,S_D)^{\lambda_1}$ and $|D_{d+1}u(Y)|\mathrm{dist}(Y,S_D)^{\lambda_2}$ for suitable λ_1 and λ_2. We apply it to show the boundedness of an operator on the Besov space on S_D and use the result to prove the boundedness of the operator with respect to the double layer heat potentials.

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1. Introduction

Let D be a bounded domain in $\mathbf{R}^d$ and suppose that the boundary ∂D of D is a β-set ($d-1 \leq \beta < d$), i.e., there is a positive Radon measure μ satisfying

$$b_1 r^\beta \leq \mu(B(z,r) \cap \partial D) \leq b_2 r^\beta \tag{1.1}$$

for all $r \leq r_0$ for some r_0 and all $z \in \partial D$. Here $B(x,r)$ is an open ball with center x and radius r, and b_1, b_2 are constants independent of r, x. Such a measure μ is called a β-measure. We fix a β-measure μ.

We consider a bounded cylinderical domain $\Omega_D = D \times (0,T)$ for the above domain D and denote by S_D the lateral boundary $\partial D \times [0,T]$ of Ω_D.

We use the measure $\mu_T = \mu \times \mathcal{L}^1_{[0,T]}$ on S_D. Here $\mathcal{L}^1_{[0,T]}$ stands for the restriction of the 1-dimensional Lebesgue measure to $[0,T]$. We also use the parabolic metric. Recall that the parabolic metric ρ in $\mathbf{R}^{d+1}$ is defined by

$$\rho(X,Y) = \sqrt{|x-y|^2 + |t-s|}$$

for $X=(x,t)$, $Y=(y,s)$ and x, $y \in \mathbf{R}^d$, t, $s \in \mathbf{R}$.

We may suppose that $\partial D \subset B(0,R/2)$ for some $R \geq 1$ and $r_0 = 3R$ in (1.1).

Let $p \geq 1$ and $\alpha > 0$. We denote by $L^p(\mu_T)$ the set of all L^p-functions defined on S_D with respect to μ_T and by $\Lambda^p_\alpha(S_D)$ the space of all function in $L^p(\mu_T)$ such that

$$\int_{S_D}\int_{S_D} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y) < \infty.$$

For $f \in \Lambda^p_\alpha(S_D)$ the Besov norm of f is defined by

$$\|f\|_{\alpha,p} = \Big(\int_{S_D} |f(X)|^p d\mu_T(X)\Big)^{1/p} + \Big(\int_{S_D}\int_{S_D} \frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X) d\mu_T(Y)\Big)^{1/p}. \tag{1.2}$$

Instead of balls we consider parabolic cylinders. The parabolic cylinder with center $X = (x,t)$ and radius r is defined by

$$C(X,r) = \{Y = (y,s);\ |x-y| < r,\ |t-s| < r^2\}.$$

In this paper we consider the boundedness of an operator A on the space $\Lambda^p_\alpha(S_D)$ on the fractal lateral boundary S_D. It is often difficult to prove directly that A is bounded on $\Lambda^p_\alpha(S_D)$, because D may have a fractal boundary. So we consider a sufficient smooth function u defined on $\overline{\Omega_D}$ and estimate the second part of the Besov norm (1.2) of the restriction of u to S_D by the $L^p(\Omega_D)$-norm of $|\nabla u(Y)| \mathrm{dist}(Y,S_D)^{\eta_1}$ and $|D_{d+1}u(Y)| \mathrm{dist}(Y,S_D)^{\eta_2}$ for suitable $\eta_1,\ \eta_2 > 0$. Here

$$\nabla_y u(Y) = \Big(\frac{\partial u}{\partial y_1}(Y), \cdots, \frac{\partial u}{\partial y_d}(Y)\Big)$$

and

$$D_{d+1}u(Y) = \frac{\partial u}{\partial s}(Y).$$

To do them, we consider a uniform domain by O. Martio and J. Sarvas (cf. [MS]). Recall that D is called a uniform domain if there exist constants a and b such that each pair of points $x_0,\ y_0 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$l(\gamma) \leq a|x_0 - y_0| \tag{1.3}$$

and

$$\min\{l(\gamma(x_0,x)), l(\gamma(x,y_0))\} \leq b\, \mathrm{dist}\,(x,\partial D) \quad \text{for all } x \in \gamma. \tag{1.4}$$

Here $l(\gamma)$ (resp. $\gamma(x_0,x)$) stands for the Euclidean length of γ (resp. the part of γ between x_0 and x). We may assume that $a \geq 1$ and $b > 1$.

Note that an (ϵ,∞) domain, which was introduced by P. W. Jones in [J], is a uniform one (cf. [V]). Therefore a Lipschitz domain is uniform and the snow flake domain is also uniform.

Let $Y = (y,s) \in \mathbf{R}^d \times \mathbf{R}$. We denote by $\delta(Y)$ (resp. $\delta(y)$) the distance of Y from S_D with respect to ρ (resp. the Euclidean distance of y from ∂D). We easily see that $\delta(Y) = \delta(y)$ for $Y = (y,s) \in \mathbf{R}^d \times [0,T]$.

In a uniform domain D we gave the following theorem, which estimates the Besov norm on ∂D of a function u defined on $\overline{D}$ by the $L^p(D)$-norm of $|\nabla u|\delta(y)^\eta$ (cf. [W5]).

Theorem A. *Assume that D is a bounded uniform domain such that ∂D is a β-set ($d-1 \leq \beta < d$). Let $1 < p < \infty$. If $1-(d-\beta) < \alpha < 1-(d-\beta)/p$, $\alpha+(d-\beta)/p < \lambda < 1$ and u is λ-Hölder continuous on $\overline{D}$ and of C^1 in D, then*

$$\int_{S_D}\int_{S_D} \frac{|u(x)-u(z)|^p}{|x-z|^{\beta+p\alpha}} d\mu(x) d\mu(z) \leq c \int_D |\nabla u(y)|^p \delta(y)^{p-p\alpha-d+\beta} dy,$$

where c is a constant independent of u.

Corresponding to this theorem we shall prove the following theorem in §3.

Theorem 1. *Assume that D is a bounded uniform domain such that ∂D is a β-set ($d-1 \leq \beta < d$). Further, let $p > 1$, $p - p\alpha - d + \beta > 0$ and $\alpha + (d-\beta)/p < \lambda < 1$. If u is of class C^1 in Ω_D and λ-Hölder continuous on $\overline{\Omega_D}$ with respect to ρ, then*

$$\begin{aligned} &\int_{S_D}\int_{S_D} \frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{\beta+2+\alpha p}} d\mu_T(X) d\mu_T(Y) \\ &\leq c\left(\int_{\Omega_D} |\nabla_y u(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY + \int_{\Omega_D} |\frac{\partial}{\partial s} u(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dY\right), \end{aligned}$$

where c is a constant independent of u.

In [W4] we proved that there exists a bounded linear extension operator $\mathcal{E}$ from $L^p(\mu_T)$ to $L^p(\mathbf{R}^{d+1})$ such that $\mathcal{E}(f)$ is of C^∞ in $(\mathbf{R}^d \setminus \partial D) \times (0,T)$ and supp $\mathcal{E}(f) \subset B(0,2R) \times (-2, T+2)$. Using this extension operator $\mathcal{E}$ and proved the following theorem in [W4].

Theorem B. *Let $p > 1$, $f \in \Lambda^p_\alpha(S_D)$ and $p - p\alpha - d + \beta > 0$. Then*

$$\begin{aligned} &\int_{(\mathbf{R}^d \setminus \partial D)\times(0,T)} |\nabla \mathcal{E}(f)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY \\ &+ \int_{(\mathbf{R}^d \setminus \partial D)\times(0,T)} |\frac{\partial}{\partial s}\mathcal{E}(f)(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dY \leq c\|f\|^p_{\alpha,p}, \end{aligned}$$

where c is a constant independent of f.

We say that a domain D in $\mathbf{R}^d$ satisfies the condition (b) if there exist positive real numbers c and r_1 such that

$$|B(z,r) \cap D| \geq c r^d$$

for each point $z \in \partial D$ and each positive real number $r \leq r_1$, where $|F|$ stands for the n-dimensional Lebesgue measure of F if $F \subset \mathbf{R}^n$. We note that if D is a uniform domain in $\mathbf{R}^d$, then it satisfies the condition (b) (cf. [W5, Lemma 2.1]).

We define, for $r > 0$ and $0 < c < 1$,

$$F_{r,c} = \{Y \in \Omega_D; cr < \delta(Y) < r\}.$$

In [W3] we showed the following theorem.

Theorem C. *Let D be a domain in $\mathbf{R}^d$ such that ∂D is a compact β-set ($d-1 \leq \beta < d$) and satisfies the condition (b). Further let $1 \leq p < \infty$, $p - p\alpha - d + \beta > 0$ and $\alpha + (d-\beta)/p < \lambda < 1$. If u*

is λ-Hölder continuous on $\overline{D \cap B(0,R)} \times [0,T]$ with respect to the metric ρ, then there is a constant $c_0 < 1$ such that

$$\begin{aligned}
&\int_{S_D}\int_{S_D} \frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha+d-\beta}} d\mu_T(X)d\mu_T(Y) \\
\leq\ & c \liminf_{r\to 0} \int_{F_{r,c_0}}\int_{F_{r,c_0}} \frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dXdY,
\end{aligned}$$

where c is a constant independent of u.

Using this theorem, instead of considering the boundedness of the operator A on $\Lambda^p_\alpha(S_D)$, we shall prove that the operator corresponding to A is bounded from a function space on $(\mathbf{R}^d \backslash \overline{D}) \times (0,T)$ (resp. $D \times (0,T)$) to a function space $D \times (0,T)$ (resp. $(\mathbf{R}^d \backslash \overline{D}) \times (0,T)$). In §4 we have the following theorem.

Theorem 2. *Let D be a bounded uniform domain in $\mathbf{R}^d$ such that ∂D is a β-set $(d-1 \leq \beta < d)$. Further let $p > 1$, $1-(d-\beta) < \alpha < 1-(d-\beta)/p$ and $\alpha + (d-\beta)/p < \lambda < 1$. Assume that A is a bounded linear operator from $\Lambda^p_\alpha(S_D)$ to $L^p(\mu_T)$ and that, if f is a Lipschitz function on S_D with respect to ρ, then Af is λ-Hölder continuous on $\overline{\Omega_D}$ with respect to ρ and Af is of C^1 in Ω_D. Further assume that, for every Lipschitz function on S_D with respect to ρ,*

$$\begin{aligned}
&\int_{D\times(0,T)} |\nabla_y(Af)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY + \int_{D\times(0,T)} |D_{d+1}(Af)(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dY \\
\leq\ & c \int_{(\mathbf{R}^d\backslash\overline{D})\times(0,T)} |\nabla_y \mathcal{E}(f)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY \\
+\ & c \int_{(\mathbf{R}^d\backslash\overline{D})\times(0,T)} |D_{d+1}\mathcal{E}(f)(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dY + c\int_{S_D} |f(Y)|^p d\mu_T(Y),
\end{aligned}$$

where c is a constant independent of f. Then A is a bounded linear operator from $\Lambda^p_\alpha(S_D)$ to $\Lambda^p_\alpha(S_D)$.

We next apply this theorem to potential theory and prove the boundedness of operators with respect to double layer heat potentials.

If D is a bounded domain with smooth boundary, the double layer heat potential Φf of f is defined by

$$\Phi f(X) = -\int_0^T \int_{\partial D} \langle \nabla_y W(X-Y), n_y \rangle f(Y) d\sigma(y) ds \tag{1.5}$$

for $X = (x,t) \in (\mathbf{R}^d \setminus \partial D) \times \mathbf{R}$, where $\langle\ ,\ \rangle$ is the inner product in $\mathbf{R}^d$, n_y is the unit outer normal to ∂D, σ is the surface measure on ∂D and W is the fundamental solution for the heat operator, i.e.,

$$W(X) = W(x,t) = \begin{cases} \frac{\exp\left(-\frac{|x|^2}{4t}\right)}{(4\pi t)^{d/2}} & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The double layer heat potential is important not only physically but also mathematically. For example, R. M. Brown proved that the solution to the initial-Dirichlet problem in a Lipschitz cylinder for the heat operator can be written by a double layer heat potential and the solution to the initial-Neumann problem in a Lipschitz cylinder for the heat operator is given by a single layer heat potential (cf. [B]).

If D is a bounded domain with fractal boundary, then n_y and the surface measure can not be defined. But if f is a C^1-function on $\mathbf{R}^{d+1}$ with compact support, then we see by the Green formula that for $X = (x,t) \in D \times \mathbf{R}$

$$\begin{aligned} \Phi f(X) &= \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y f(Y), \nabla_y W(X-Y) \rangle dy \\ &+ \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} f(Y) \triangle_y W(X-Y) dy \end{aligned}$$

and for $X = (x,t) \in (\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$

$$\begin{aligned} \Phi f(X) &= -\int_0^T ds \int_D \langle \nabla_y f(Y), \nabla_y W(X-Y) \rangle dy \\ &- \int_0^T ds \int_D f(Y) \triangle_y W(X-Y) dy. \end{aligned}$$

Using the extension operator $\mathcal{E}$ in [W4], we define, for $f \in \Lambda_\alpha^p(S_D)$,

$$\begin{aligned} \Phi f(X) &= \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(Y), \nabla_y W(X-Y) \rangle dy \\ &+ \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \mathcal{E}(f)(Y) \triangle_y W(X-Y) dy \end{aligned}$$

for $X \in D \times \mathbf{R}$ and

$$\begin{aligned} \Phi f(X) &= -\int_0^T ds \int_D \langle \nabla_y \mathcal{E}(f)(Y), \nabla_y W(X-Y) \rangle dy \\ &- \int_0^T ds \int_D \mathcal{E}(f)(Y) \triangle_y W(X-Y) dy \end{aligned}$$

for $X \in (\mathbf{R}^d \setminus \overline{D}) \times \mathbf{R}$.

We also define, for $Z \in S_D$,

$$\begin{aligned} K_1 f(Z) &= \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(Y), \nabla_y W(Z-Y) \rangle dy \\ &+ \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - \mathcal{E}(f)(Z)) \triangle_y W(Z-Y) dy \end{aligned}$$

if it is well-defined, and $K_1 f(Z) = 0$ if otherwise.

Similarly we define, for $Z \in S_D$,

$$\begin{aligned} K_2 f(Z) &= -\int_0^T ds \int_D \langle \nabla_y \mathcal{E}(f)(Y), \nabla_y W(Z-Y) \rangle dy \\ &- \int_0^T ds \int_D (\mathcal{E}(f)(Y) - \mathcal{E}(f)(Z)) \triangle_y W(Z-Y) dy \end{aligned}$$

if it is well-defined, and $K_2 f(Z) = 0$ if otherwise.

It is easy to see that the operators K_1 and K_2 are bounded from $\Lambda^p_\alpha(S_D)$ to $L^p(\mu_T)$. But, are they bounded from $\Lambda^p_\alpha(S_D)$ to $\Lambda^p_\alpha(S_D)$?

Using Theorem 2, we shall prove the following theorem in §6.

Theorem 3 . *Let D be a bounded uniform domain in $\mathbf{R}^d$ such that $\mathbf{R}^d \setminus \overline{D}$ satisfies the condition* (b) *and ∂D is a β-set $(d-1 \leq \beta < d)$. Assume that $1 < p < \infty$ and $0 \leq \beta - d + 1 < \alpha < 1 - (d-\beta)/p$. Then the operator K_1 is bounded from $\Lambda^p_\alpha(S_D)$ to $\Lambda^p_\alpha(S_D)$. Further, assume that for every pair of points x_0, $y_0 \in \mathbf{R}^d \setminus \overline{D}$ satisfing $\delta(x_0) \leq r_0$ and $\delta(y_0) \leq r_0$ for some r_0, there is a rectifiable arc $\gamma \subset \mathbf{R}^d \setminus \overline{D}$ satisfying* (1.3) *and* (1.4). *Then K_2 is also bounded from $\Lambda^p_\alpha(S_D)$ to $\Lambda^p_\alpha(S_D)$.*

2. Uniform domains

In this chapter we study a property of Ω_D if D is a uniform domain. At first we note a well-known property of a uniform domain (cf. [V, 2.21 Theorem]).

Lemma D. *Let D be a uniform domain satisfying* (1.3) *and* (1.4). *Then there exists a point $v \in D$ such that*

$$B(v, \frac{d(D)}{8b}) \subset D$$

and for any $x_0 \in D$ we find a curve $\gamma_0 \subset D$, which joins x_0 to v and satisfies

$$l(\gamma_0(x_0, x)) \leq 16ab^2\delta(x) \quad \text{for all } x \in \gamma_0. \tag{2.1}$$

Here $d(D)$ stands for the diameter of D and a, b are the constants in (1.3) and (1.4), respectively.

Using this, we give the following lemma.

Lemma 2.1. *Let D be a unifom domain satisfying* (1.3) *and* (1.4). *Further let $X_0 = (x_0, t_0), Y_0 = (y_0, s_0) \in \Omega_D$ and $c_1\delta(x_0) \leq \delta(y_0) \leq c_2\delta(x_0)$. If j is an integer satisfying $j \geq -1$ and $2^j\delta(x_0) \leq \rho(X_0, Y_0) \leq 2^{j+1}\delta(x_0)$, then there exist cylinders C_0, C_1, $\cdots$, C_m, C'_0, C'_1, $\cdots$, $C'_{m'}$ ($C_k = B(z_k, r_k) \times (a_k, b_k)$, $C'_k = B(z'_k, r'_k) \times (a'_k, b'_k)$) in Ω_D having the following properties:*

(i) *$r_1 \leq \cdots \leq r_m$ and $r'_1 \leq \cdots \leq r'_{m'}$ and $C_m = C'_{m'}$,*

(ii) *$dist(\overline{B(z_k, r_k)}, \partial D) \geq 3r_k$, $dist(\overline{B(z'_k, r'_k)}, \partial D) \geq 3r'_k$,*

(iii) *$x_0 \in B(z_k, cr_k)$, $y_0 \in B(z'_k, cr'_k)$,*

(iv) *If $y \in B(z_k, r_k)$, then $\delta(y) \leq cr_k$,*

(v) *$B(y_k, c'r_k) \subset B(z_k, r_k) \cap B(z_{k+1}, r_{k+1})$ for some $y_k \in D$,*
$B(y'_k, c'r'_k) \subset B(z'_k, r'_k) \cap B(z'_{k+1}, r'_{k+1})$ for some $y'_k \in D$,

(vi) *$|C_k| \leq cr_k^{d+2}$, $|C'_k| \leq c(r'_k)^{d+2}$,*

(vii) *$r_k \leq c\rho(X_0, Y_0)$, $r'_k \leq c\rho(X_0, Y_0)$,*

(viii) *$m \leq c(j+2)$, $m' \leq c(j+2)$.*

Here c, c' are constants depending only on d, D, T, a, b, c_1 and c_2.

Proof. We may assume that $t_0 < s_0$. Since D is a uniform domain, we can find $\gamma_0 \subset D$ which connects x_0 with v in Lemma D and satisfying (2.1). Choose $\eta > 0$ satisfying $l(\gamma_0) = \eta T^{1/2}$.

(1) We first consider the case that $\eta|s_0-t_0|^{1/2}\le|x_0-y_0|$.

Since D is uniform, there exists, for x_0 and y_0, a curve $\gamma\subset D$ satisfying (1.3) and (1.4). We may assume that $a,b\ge 1$. Take the point z' on γ such that $l(\gamma(x_0,z'))=\frac{1}{2}l(\gamma)$. Putting $\rho=\frac{\delta(x_0)}{4b}$, we choose the natural number m such that

$$\rho(1+\frac{1}{4b})^{m-2}<\frac{1}{2}l(\gamma)\le\rho(1+\frac{1}{4b})^{m-1}.$$

(If m is non-positive, then we take m as 1).

We next take z_1, z_2, $\cdots$, z_m on γ as follows:

$$l(\gamma(x_0,z_k))=\rho(1+\frac{1}{4b})^{k-1},\quad(k=1,\cdots,m-1)\quad\text{and}\quad z_m=z'.$$

Set $c=2/\eta^2$ and put

$$t_k=t_0+\min\{c\rho^2(1+\frac{1}{4b})^{2(k-1)},\frac{|s_0-t_0|}{2}\}\quad(k=1,\cdots,m).$$

Since

$$c\rho^2(1+\frac{1}{4b})^{2(m-1)}\ge\frac{c}{4}l(\gamma)^2\ge\frac{c}{4}|x_0-y_0|^2\ge\frac{c}{4}\eta^2|s_0-t_0|=\frac{|s_0-t_0|}{2},$$

it concludes that $t_m=t_0+|s_0-t_0|/2$.

Further, set $r_0=\rho$, $r_k=\frac{\rho}{4b}(1+\frac{1}{4b})^{k-1}$ $(k=1,\cdots,m-1)$, $r_m=\frac{1}{4b}\frac{l(\gamma)}{2}$, and define $(m+1)$-numbers of cylinders as follows:

$$\begin{aligned}C_0&=B(x_0,r_0)\times(t_0,t_1)\\C_k&=B(z_k,r_k)\times(t_0,t_k)\quad(k=1,\cdots,m)\end{aligned}$$

By the same method as in [W5, Lemma 2.2] we see that these cylinders have the properties (i)-(viii).

(2) We next consider the case that $\eta|s_0-t_0|^{1/2}>|x_0-y_0|$.

Let v be the point in D in Lemma D and γ_0 the curve in D joining x_0 to v and satisfying (2.1). Note that $B(v,\frac{d(D)}{8b})\subset D$. Set $b'=12ab^2$. Choose the positive interger n satisfying

$$\rho^2(1+\frac{1}{4b'})^{2(n-2)}<\frac{s_0-t_0}{2}\le\rho^2(1+\frac{1}{4b'})^{2(n-1)}.\tag{2.2}$$

(If n is non-positive, we put $n=1$). Set

$$s_k=t_0+\rho^2(1+\frac{1}{4b'})^{2(k-1)}\quad(k=1,\cdots n-1)$$

and $s_n=t_0+\frac{s_0-t_0}{2}$.

We next take $w_1,\cdots,w_n$ on γ_0 and positive numbers r_1', $\cdots$, r_n' as follows:

$$l(\gamma_0(x_0,w_k))=\eta\rho(1+\frac{1}{4b'})^{k-1}=4b'r_k'\quad(k=1,\cdots,n).$$

Here we note that

$$\eta\rho(1+\frac{1}{4b'})^{n-1}\le\eta(1+\frac{1}{4b'})\frac{|s_0-t_0|^{1/2}}{\sqrt{2}}\le\eta|s_0-t_0|^{1/2}\le\eta T^{1/2}=l(\gamma_0)\tag{2.3}$$

and

$$\operatorname{dist}(w_n,\partial D)\ge\frac{1}{b'}l(\gamma_0(x_0,w_n))=\frac{1}{b'}\eta\rho(1+\frac{1}{4b'})^{n-1}.$$

So we can take $w_1, \cdots, w_n$ on γ_0. Using these points, we define $(n+1)$-numbers of cylinders as follows:

$$
\begin{aligned}
P_0 &= B(x_0,\rho) \times (t_0, t_0+\rho^2) \\
P_k &= B(w_k, r'_k) \times (t_0, s_k) \quad (k=1,\cdots,n).
\end{aligned}
$$

These cylinders have the properties (i)-(viii).

Similarly we join y_0 to v by γ_1 satisfying (2.1) and take $w'_1, \cdots, w'_{n'}$ on γ_1. And we contruct cylinders P'_0, $P'_1, \cdots, P'_{n'}$.

Since D is uniform, there exists a curve γ_2 joining w_n to $w'_{n'}$ satisfying

$$
\begin{aligned}
&l(\gamma_2) \leq a|w_n - w'_n|, \\
&\min\{l(\gamma_2(w_n,x), l(\gamma_2(x,w'_n))\} \leq b\delta(x) \quad \text{for } x \in \gamma_2.
\end{aligned}
$$

Take w'' on γ_2 such that $l(\gamma_2(w_n,w'')) = \frac{1}{2}l(\gamma_2)$ and choosing the natural number l such that

$$
(l-1)\eta\rho\left(1+\frac{1}{4b'}\right)^{n-1} < \frac{l(\gamma_2)}{2} \leq l\eta\rho\left(1+\frac{1}{4b'}\right)^{n-1}.
$$

(If l is non-positive, we put $l=1$.) Further, we define w_{n+k} $(k=1,\cdots,l)$ on γ_2 as follows:

$$
\begin{aligned}
l(\gamma_2(w_n, w_{n+k})) &= (l-1)\eta\rho\left(1+\frac{1}{4b'}\right)^{n-1} \quad (k=1,\cdots,l-1) \\
l(\gamma_2(w_n, w_{n+l})) &= \frac{l(\gamma_2)}{2}.
\end{aligned}
$$

Noting that $l(\gamma_0(x_0,w_n)) \leq \eta|s_0-t_0|^{1/2}$ by (2.3) and

$$
l(\gamma_2) \leq a(|w_n - x_0| + |x_0 - y_0| + |y_0 - w'_n|) \leq 3a\eta|s_0-t_0|^{1/2}, \tag{2.4}
$$

we have

$$
(l-1)\eta\rho\left(1+\frac{1}{4b'}\right)^{n-1} \leq \frac{l(\gamma_2)}{2} \leq \frac{3}{2}a\eta|s_0-t_0|^{1/2},
$$

whence, by (2.2),

$$
l-1 \leq \frac{3}{2}a\eta|s_0-t_0|^{1/2} \times \frac{\sqrt{2}}{\eta|s_0-t_0|^{1/2}} = \frac{3}{2}\sqrt{2}a.
$$

Put

$$
r''_k = \frac{l(\gamma_2(w_n, w_{n+k}))}{4b} \quad (k=1,\cdots,l-1) \qquad r''_l = \frac{l(\gamma_2)}{8b}
$$

and define cylinders by

$$
P_{n+k} = B(w_{n+k}, r''_k) \times (t_0, t_0 + \frac{|s_0-t_0|^{1/2}}{2}) \quad (k=1,\cdots,l).
$$

These cylinders have the properties (i)-(vii).

Similarly we construct cylinders $\{P'_k\}_{k=n'+1}^{n'+l'}$ along $-\gamma_2$. The cylinders $\{P_k\}_{k=1}^{n+l}$ and $\{P'_k\}_{k=1}^{n'+l'}$ are the desired ones. Q.E.D.

3. Estimates by volume integrals of the gradients

In this section we assume that u is a sufficient smooth function on $\overline{\Omega_T}$. Using Theorem C, we shall estimate the Besov norm of the restriction of u to the lateral boundary by the $L^p(\Omega_D)$-norm of the sum of $|\nabla u|^p \times \delta(Y)^{\gamma_1}$ and $|D_{d+1}u|\delta(Y)^{\gamma_2}$ (γ_1 and γ_2 are suitable numbers).

By the same method as in the proof of Lemma 4.1 in [W5] we have the estimate for the Besov norm on the product of two parabolic cylinders as follows:

Lemma 3.1. *If $1 < p < \infty$, $p - p\alpha - d + \beta > 0$, $r > 0$ and $X_0 = (x_0, t_0) \in \Omega_D$. Further let $\delta(x_0) > 2r$ and put $I_1 = (t_0 - r^2, t_0 + r^2) \cap (0, T)$. If u is of C^1-class in Ω_D, then*

$$\int_{I_1} dt \int_{B(x_0,r)} dx \int_{I_1} ds \int_{B(x_0,r)} \frac{|u(X) - u(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dy$$

$$\leq c\left(r^{p-p\alpha-d+\beta} \int_{I_1} ds \int_{B(x_0,r)} |\nabla_y u(Y)|^p dy + r^{2p-p\alpha-d+\beta} \int_{I_1} ds \int_{B(x_0,r)} |D_{d+1}u(Y)|^p dy\right),$$

where c is a constant independent of X_0 and r.

Using this lemma and the covering lemma for parabolic cylinders, we can prove the following lemma (cf. [W5, Lemma 4.2]).

Lemma 3.2. *Let $p - p\alpha - d + \beta > 0$ and u is of C^1-class in Ω_T. For $X = (x, t) \in \Omega_D$ and b in (1.4) put*

$$r_X = \frac{\delta(X)}{4b} \quad \text{and} \quad \eta_{X,r_X} = \frac{\int_{C(X,r_X)} u(Y)dY}{|C(X, r_X)|}.$$

If $0 < r < 1$, $0 < c_1 \leq 1$, $c_2 \geq 1$, then

$$\sum_{j=-1}^{\infty} \int_{F_{r,c_0}} dX \int_{\{c_1 2^j \delta(X) \leq \rho(X,Y) < c_2 2^{j+1}\delta(X)\} \cap F_{r,c_0}} \frac{|u(X) - \eta_{X,r_X}|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dY$$

$$\leq c_3 \int_{\Omega_D} (|\nabla_y u(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} + |D_{d+1}u(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta}) dY,$$

where c_0 is a constant in Theorem C in §1 and c_3 is a constant independent of r, u.

We next prepare a measure which will be used later. Let $0 < \lambda < d - \beta$. The measure ν_λ^+ is defined by

$$\nu_\lambda^+(E) = \int_{\Omega_D \cap E} \delta(y)^{-\lambda} dY \tag{3.1}$$

for a Borel measurable set $E \subset \mathbf{R}^{d+1}$.

We now are ready to prove the main lemma.

Lemma 3.3. *Suppose that D is a bounded uniform domain such that ∂D is a β-set. Let $1 < p < \infty$, $p - p\alpha - d + \beta > 0$ and $\alpha + (d - \beta)/p < \lambda < 1$. If u is of C^1-class on Ω_D and λ-Hölder continuous on $\overline{\Omega_D}$ with respect to ρ, then*

$$\limsup_{r \to 0} \int_{F_{r,c_0}} \int_{F_{r,c_0}} \frac{|u(X) - u(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dX dY$$

$$\leq c\left(\int_{\Omega_D} |\nabla_y u(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY + \int_{\Omega_D} |D_{d+1}u(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dY\right),$$

where c is a constant independent of u.

Proof. Let r be a sufficiently small positive number and c_0 be the number in Theorem C. We write

$$
\begin{aligned}
&\int_{F_{r,c_0}} \frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dY \\
= &\int_{F_{r,c_0}\cap\{\rho(X,Y)<(1/2)\delta(X)\}} \frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dY \\
+ &\int_{F_{r,c_0}\cap\{\rho(X,Y)\geq(1/2)\delta(X)\}} \frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dY \\
\equiv\ & I_1 + I_2.
\end{aligned}
$$

We first estimate I_1. Since $\overline{F_{r,c_0}}$ is compact, there is a covering $\{C(Y_j, \frac{\delta(Y_j)}{12b})\}_{j=1}^m$ of $\overline{F_{r,c_0}}$, where $b > 1$ is the same constant as in (1.4) in §1. Using a covering theorem of Vitali type, we can select a subcovering $\{C(X_k, \frac{\delta(X_k)}{12b})\}_{k=1}^l$ of $\{C(Y_j, \frac{\delta(Y_j)}{12b})\}_{j=1}^m$ such that $\{C(X_k, \frac{\delta(X_k)}{12b})\}_{k=1}^l$ are mutually disjoint and

$$\overline{F_{r,c_0}} \subset \cup_{k=1}^l C(X_k, \frac{\delta(X_k)}{4b}).$$

If $X_k = (x_k, t_k)$, $X = (x,t) \in C(X_k, \frac{\delta(X_k)}{4b})$, $Y = (y,s)$ and $\rho(X,Y) < \frac{1}{2}\delta(X)$, then $Y \in C(X_k, \frac{1}{2}(1+\frac{1}{b})\delta(X_k))$. Putting $b' = \frac{1}{2}(1+\frac{1}{b})$, we see that $b' < 1$. Hence, by Lemma 3.1,

$$
\begin{aligned}
&\int_{F_{r,c_0}\cap C(X_k,\frac{\delta(X_k)}{4b})} I_1 dX \\
\leq &\int_{C(X_k,b'\delta(X_k))\cap\Omega_D} dX \int_{C(X_k,b'\delta(X_k))\cap\Omega_D} \frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dY \\
\leq\ & c_1\delta(X_k)^{p-p\alpha-d+\beta} \int_{C(X_k,b'\delta(X_k))\cap\Omega_D} |\nabla_y u(Y)|^p dY \\
+\ & c_1\delta(X_k)^{2p-p\alpha-d+\beta} \int_{C(X_k,b'\delta(X_k))\cap\Omega_D} |D_{d+1}u(Y)|^p dY.
\end{aligned}
$$

If $Y \in C(X_k, b'\delta(X_k))$, then $\delta(X_k) \leq \delta(Y) + \frac{1}{2}(1+\frac{1}{b})\delta(X_k)$ and hence $\frac{1}{2}(1-\frac{1}{b})\delta(X_k) \leq \delta(Y)$. Consequently we have

$$
\begin{aligned}
&\int_{F_{r,c_0}\cap C(X_k,\frac{\delta(X_k)}{4b})} I_1 dX \\
\leq\ & c_2 \int_{C(X_k,b'\delta(X_k))\cap\Omega_D} \left(|\nabla_y u(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} + |D_{d+1}u(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta}\right) dY.
\end{aligned}
$$

Since $\{C(X_k, \frac{\delta(X_k)}{12b})\}_{k=1}^l$ are mutually disjoint, we conclude that

$$
\begin{aligned}
&\int_{F_{r,c_0}} I_1 dX \\
\leq\ & c_3 \int_{\Omega_D} \left(|\nabla_y u(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} + |D_{d+1}u(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta}\right) dY.
\end{aligned}
$$

We next estimate I_2. Let $X, Y \in F_{r,c_0}$ and $\rho(X,Y) \geq (1/2)\delta(X)$. Using Lemma 2.1 for $X_0 = X$ and $Y_0 = Y$, we can choose cylinders $C_0, \cdots, C_m, C_0', \cdots, C_{m'}'$ such that they have the properties (i)-(viii). Putting

$$
\begin{aligned}
\eta_k &= \frac{1}{|C_k|}\int_{C_k} u(Z)dZ \quad (k=0,\cdots,m) \\
\eta'_k &= \frac{1}{|C'_k|}\int_{C'_k} u(Z)dZ \quad (k=0,\cdots,m')
\end{aligned}
$$

and noting that $\eta_m = \eta'_{m'}$, we write

$$
\begin{aligned}
|u(X)-u(Y)| &\le |u(X)-\eta_0| + \sum_{k=0}^{m-1}|\eta_k-\eta_{k+1}| \\
&+ |u(Y)-\eta'_0| + \sum_{k=0}^{m'-1}|\eta'_k-\eta'_{k+1}| \\
&\equiv J(X) + \sum_{k=0}^{m-1} J_k + J'(Y) + \sum_{k=0}^{m'-1} J'_k.
\end{aligned}
$$

Noting that $\rho(X,Y) \ge \frac{1}{2}\delta(X)$, we have, by Lemma 3.2,

$$
\begin{aligned}
&\int_{F_{r,c_0}} dX \int_{F_{r,c_0}\cap\{\rho(X,Y)\ge(1/2)\delta(X)\}} \frac{J(X)^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}}dY \\
\le\ & c_4\int_{\Omega_D}(|\nabla_z u(Z)|^p\delta(Z)^{p-p\alpha-d+\beta} + |D_{d+1}u(Z)|^p\delta(Z)^{2p-p\alpha-d+\beta})dZ.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\int_{F_{r,c_0}} dX \int_{F_{r,c_0}\cap\{\rho(X,Y)\ge(1/2)\delta(X)\}} \frac{J'(Y)^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}}dY \\
\le\ & c_5\int_{\Omega_D}(|\nabla_z u(Z)|^p\delta(Z)^{p-p\alpha-d+\beta} + |D_{d+1}u(Z)|^p\delta(Z)^{2p-p\alpha-d+\beta})dZ.
\end{aligned}
$$

We next estimate $J_k = |\eta_k - \eta_{k+1}|$. By the similar method as in the proof of Lemma 4.3 in [W5] we have

$$
\begin{aligned}
J_k &\le \frac{c_6}{r_k^{d+2}}\int_{C_k}(|\nabla_z u(Z)|\delta(z) + |D_{d+1}u(Z)|\delta(z)^2)dZ \\
&+ \frac{c_6}{r_{k+1}^{d+2}}\int_{C_{k+1}}(|\nabla_z u(Z)|\delta(z) + |D_{d+1}u(Z)|\delta(z)^2)dZ.
\end{aligned}
$$

Put

$$
I_k = \frac{1}{r_k^{d+2}}\int_{C_k}(|\nabla_z u(Z)|\delta(z) + |D_{d+1}u(Z)|\delta(z)^2)dZ.
$$

We note that $r_k \le c_7\rho(X,Y)$, where c_7 is a constant independent of X, Y and k. Choose $\epsilon > 0$ satisfying $0 < d-\beta-p\epsilon$ and put $\lambda = d-\beta-p\epsilon$. By Lemma 2.1 we see that r_k is comparable $\delta(x_k)$. Noting that $|x-x_k| \le c_8\delta(x_k)$,

$$
\nu_\lambda^+(C_k) \le \int_{C_k}\delta(z)^{-\lambda}dZ \le c_9 r_k^{d+2-\lambda}
$$

and $Z=(z,r)\in C_k$ implies $\delta(z)\leq c_{10}r_k$, we have

$$\begin{aligned} I_k &\leq c_{11}r_k^{\alpha+\epsilon}\frac{\int_{C(X,c_9r_k)\cap\Omega_D}|\nabla_z u(Z)|\delta(z)^{1-\alpha-\epsilon}\delta(z)^{-\lambda}dZ}{r_k^{d+2-\lambda}} \\ &+ c_{11}r_k^{\alpha+\epsilon}\frac{\int_{C(X,c_9r_k)\cap\Omega_D}|D_{d+1}u(Z)|\delta(z)^{2-\alpha-\epsilon}\delta(z)^{-\lambda}dZ}{r_k^{d+2-\lambda}} \\ &\leq c_{12}\rho(X,Y)^{\alpha+\epsilon}(\mathcal{M}(\nu_\lambda^+)(f_1)(X)+\mathcal{M}(\nu_\lambda^+)(f_2)(X)), \end{aligned}$$

where

$$f_1(Z)=|\nabla_z u(Z)|\delta(z)^{1-\alpha-\epsilon},\quad f_2(Z)=|D_{d+1}u(Z)|\delta(z)^{2-\alpha-\epsilon}$$

and $\mathcal{M}(\nu_\lambda^+)(f_i)$ $(i=1,\ 2)$ stands for the Hardy-Littlewood maximal function with respect to the measure ν_λ^+. Hence

$$\sum_{k=0}^{m-1}J_k\leq c_{13}\rho(X,Y)^{\alpha+\epsilon}j(\mathcal{M}(\nu_\lambda^+)(f_1)(X)+\mathcal{M}(\nu_\lambda^+)(f_2)(X)).$$

From this we deduce

$$\begin{aligned} &\int_{\{2^j\delta(X)\leq\rho(X,Y)<2^{j+1}\delta(X)\}\cap F_{r,c_0}}\frac{(\sum_{k=0}^{m-1}J_k)^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}}dY \\ \leq\ & c_{14}(2^j\delta(X))^{-d+\beta+p\epsilon}j^p(\mathcal{M}(\nu_\lambda^+)(f_1)(X)^p+\mathcal{M}(\nu_\lambda^+)(f_2)(X)^p). \end{aligned}$$

Since the maximal function $\mathcal{M}(\nu_\lambda^+)(f_j)$ has the same property as usual one, we have

$$\begin{aligned} &\sum_{j=-1}^{\infty}\int_{F_{r,c_0}}dX\int_{\{2^j\delta(X)\leq\rho(X,Y)<2^{j+1}\delta(X)\}\cap F_{r,c_0}}\frac{(\sum_{k=0}^{m-1}J_k)^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}}dY \\ \leq\ & c_{14}\int_{F_{r,c_0}}\mathcal{M}(\nu_\lambda^+)(f_1)(X)^p+\mathcal{M}(\nu_\lambda^+)(f_2)(X)^p)\delta(x)^{-\lambda}dX \\ \leq\ & c_{15}\int_{\Omega_D}|\nabla_z u(Z)|^p\delta(z)^{p-p\alpha-p\epsilon}\delta(z)^{-d+\beta+p\epsilon}dZ \\ +\ & c_{15}\int_{\Omega_D}|D_{d+1}u(Z)|^p\delta(z)^{2p-p\alpha-p\epsilon}\delta(z)^{-d+\beta+p\epsilon}dZ \\ =\ & c_{16}\int_{\Omega_D}(|\nabla_z u(Z)|^p\delta(Z)^{p-p\alpha-d+\beta}+|D_{d+1}u(Z)|^p\delta(Z)^{2p-p\alpha-d+\beta})dZ. \end{aligned}$$

Similary we can estimate

$$\begin{aligned} &\sum_{j=-1}^{\infty}\int_{F_{r,c_0}}dX\int_{\{2^j\delta(X)\leq\rho(X,Y)<2^{j+1}\delta(X)\}\cap F_{r,c_0}}\frac{(\sum_{k=0}^{m'-1}J_k')^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}}dY \\ \leq\ & c_{17}\int_{\Omega_D}(|\nabla_z u(Z)|^p\delta(Z)^{p-p\alpha-d+\beta}+|D_{d+1}u(Z)|^p\delta(Z)^{2p-p\alpha-d+\beta})dZ. \end{aligned}$$

Thus we have the conclusion. Q.E.D.

Proof of Theorem 1. Let u be of class C^1 in Ω_D and λ-Hölder continuous on $\overline{\Omega_D}$. By Theorem C there exists $c_1>0$ such that

$$\begin{aligned} &\int_{S_D}\int_{S_D}\frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y) \\ \leq\ & c_1\liminf_{r\to 0}\int_{F_{r,c_0}}\int_{F_{r,c_0}}\frac{|u(X)-u(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}}dXdY. \end{aligned}$$

This inequality and Lemma 3.3 lead to the conclusion. Q.E.D.

4. Approximation in $\Lambda_\alpha^p(S_D)$ and proof of Theorem 2

In this section we first approximate a function in $\Lambda_\alpha^p(S_D)$ by Lipschitz functions with respect to ρ and prove Theorem 2.

To approximate a function in $\Lambda_\alpha^p(S_D)$, we choose C^∞-functions ϕ, ψ on $\mathbf{R}^d$, $\mathbf{R}$, respectively such that

$$\phi = 1 \text{ on } B(0,1), \quad 0 \le \phi \le 1, \quad \operatorname{supp} \phi \subset B(0,2)$$

and

$$\psi = 1 \text{ on } [-1,1], \quad 0 \le \psi \le 1, \quad \operatorname{supp} \psi \subset (-2,2).$$

For $n \in \mathbf{N}$ and $X = (x,t) \in S_D$ we define

$$a_n(X) = \int_{S_D} \psi(n^2(t-s))\phi(n(x-y))d\mu_T(Y)$$

and

$$h_n(X,Y) = \frac{1}{a_n(X)}\psi(n^2(t-s))\phi(n(x-y)).$$

Here we note that $a_n(X) \ge cn^{-\beta-2}$ for all $X \in S_D$. Further, define, for $f \in \Lambda_\alpha^p(S_D)$,

$$H_n f(X) = \int_{S_D} h_n(X,Y)f(Y)d\mu_T(Y).$$

We also note that $H_n 1 = 1$. We have the following lemma.

Lemma 4.1. *Let $0 \le \beta - d + 1 < \alpha < 1$ and $f \in \Lambda_\alpha^p(S_D)$. Then $H_n f$ is a Lipschitz function on S_D with respect to ρ and*

$$\|H_n f - f\|_{\alpha,p} \to 0 \quad \text{as } n \to \infty.$$

Proof. It is easy to see that $H_n f$ is a Lipschitz function with respect to ρ on S_D.

We shall first show that $\|H_n f - f\|_p \to 0$ for all $f \in \Lambda_\alpha^p(S_D)$. Put

$$\begin{aligned} J_n f(X) &= H_n f(X) - f(X) \\ &= \int_{S_D} h_n(X,Y)(f(Y) - f(X))d\mu_T(Y). \end{aligned}$$

Then

$$\begin{aligned} |J_n f(X)| &\le c_1 n^{\beta+2} n^{-(\beta+2)/p-\alpha} \int_{C(X,2/n)\cap S_D} \frac{|f(Y)-f(X)|}{\rho(X,Y)^{(\beta+2)/p+\alpha}} d\mu_T(Y) \\ &\le c_2 n^{-\alpha} \left(\int_{S_D} \frac{|f(Y)-f(X)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(Y) \right)^{1/p}. \end{aligned}$$

Hence

$$\int |J_n f(X)|^p d\mu_T(X) \to 0. \tag{4.1}$$

We next put

$$\begin{aligned} A_n &= \{(X,Y)\in S_D\times S_D; \rho(X,Y)\le \frac{8}{n}\}, \\ B_n &= \{(X,Y)\in S_D\times S_D; \rho(X,Y) > \frac{8}{n}\}. \end{aligned}$$

Let $(X,Z)\in A_n$. Noting that $H_n 1 = 1$, we write

$$\begin{aligned} & |H_n f(X) - f(X) - H_n f(Z) + f(Z)| \\ \le\ & |(\int_{S_D} h_n(X,Y) - h_n(Z,Y))(f(Y)-f(X))d\mu_t(Y)| + |f(X)-f(Z)| \\ \equiv\ & I_1(X,Z) + I_2(X,Z). \end{aligned}$$

Noting that, for $(X,Z)\in A_n$,

$$\begin{aligned} & |h_n(X,Y) - h_n(Z,Y)| \\ \le\ & c_3(n^{\beta+3}\rho(X,Z) + n^{\beta+4}\rho(X,Z)^2)\chi_{C(X,10/n)}(Y), \end{aligned}$$

we obtain

$$\begin{aligned} & I_1(X,Z) \\ \le\ & c_3 n^{\beta+3}\rho(X,Z)(1+n\rho(X,Z))n^{(-\beta-2)/p-\alpha}\int_{C(X,10/n)\cap S_D}\frac{|f(Y)-f(X)|}{\rho(X,Y)^{(\beta+2)/p+\alpha}}d\mu_T(Y) \\ \le\ & c_4 n^{1-\alpha}\rho(X,Z)(1+n\rho(X,Z))\left(\int_{C(X,10/n)\cap S_D}\frac{|f(Y)-f(X)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(Y)\right)^{1/p}, \end{aligned}$$

whence, by Lemma 2.3 in [W3],

$$\begin{aligned} & \iint_{A_n}\frac{I_1(X,Z)^p}{\rho(X,Z)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Z) \\ \le\ & c_5\int_{S_D}\int_{S_D}\chi_{C(X,10/n)}(Y)\frac{|f(Y)-f(X)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y) \\ \times\ & \{n^{p-p\alpha}\int_{\rho(X,Z)<10/n}\rho(X,Z)^{-\beta-2-p\alpha+p}d\mu_T(Z) \\ +\ & n^{2p-p\alpha}\int_{\rho(X,Z)<10/n}\rho(X,Z)^{-\beta-2-p\alpha+2p}d\mu_T(Z)\} \\ \le\ & c_6\int_{S_D}\int_{S_D}\chi_{C(X,10/n)}(Y)\frac{|f(Y)-f(X)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y). \end{aligned}$$

From this we see that

$$\iint_{A_n}\frac{I_1(X,Z)^p}{\rho(X,Z)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Z)\to 0 \quad \text{as } n\to\infty.$$

Since

$$\iint_{A_n}\frac{I_2(X,Z)^p}{\rho(X,Z)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Z)\to 0 \quad \text{as } n\to\infty,$$

we have

$$\iint_{A_n}\frac{|H_n f(X) - f(X) - H_n f(Z) + f(Z)|^p}{\rho(X,Z)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Z)\to 0 \quad \text{as } n\to\infty. \tag{4.2}$$

We next consider the integral over B_n. We write, for $(X,Z) \in B_n$,

$$\begin{aligned}
& |H_nf(X) - f(X) - H_nf(Z) + f(Z)| \\
\leq\ & \int_{S_D} h_n(X,Y)|f(Y) - f(X)|d\mu_T(Y) + \int_{S_D} h_n(Z,Y)|f(Y) - f(Z)|d\mu_T(Y) \\
\equiv\ & I_3(X,Z) + I_4(X,Z).
\end{aligned}$$

Then

$$\begin{aligned}
I_3(X,Z) &\leq c_7 n^{\beta+2}\left(\frac{2}{n}\right)^{(\beta+2)/p+\alpha} \int_{S_D} \chi_{C(X,2/n)}(Y)\frac{|f(Y)-f(X)|}{\rho(X,Y)^{(\beta+2)/p+\alpha}}d\mu_T(Y) \\
&\leq c_8 n^{-\alpha}\left(\int_{C(X,2/n)\cap S_D} \frac{|f(Y)-f(X)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(Y)\right)^{1/p},
\end{aligned}$$

whence, by Lemma 2.3 in [W3],

$$\begin{aligned}
& \iint_{B_n} \frac{I_3(X,Z)^p}{\rho(X,Z)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Z) \\
\leq\ & c_9 n^{-p\alpha}\int_{S_D}\int_{S_D} \chi_{C(X,2/n)}(Y)\frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y) \\
\times\ & \int_{\rho(X,Z)\geq 8/n} \rho(X,Z)^{-\beta-2-p\alpha}d\mu_T(Z) \\
\leq\ & c_{10}\int_{S_D}\int_{S_D} \chi_{C(X,2/n)}(Y)\frac{|f(X)-f(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Y).
\end{aligned}$$

From this we see that

$$\iint_{B_n} \frac{I_3(X,Z)^p}{\rho(X,Z)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Z) \to 0 \quad \text{as } n\to\infty.$$

Similarly we also have

$$\iint_{B_n} \frac{I_4(X,Z)^p}{\rho(X,Z)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Z) \to 0 \quad \text{as } n\to\infty.$$

Hence

$$\iint_{B_n} \frac{|H_nf(X)-f(X)-H_nf(Z)+f(Z)|^p}{\rho(X,Z)^{\beta+2+p\alpha}}d\mu_T(X)d\mu_T(Z) \to 0 \quad \text{as } n\to\infty. \tag{4.3}$$

From (4.1), (4.2) and (4.3) we deduce

$$\|H_nf - f\|_{\alpha,p} \to 0.$$

Q.E.D.

We now prove Theorem 2.

Proof of Theorem 2. Let g be a Lipschitz function on S_D with respect to ρ. Then $\mathcal{E}(g)$ is also a Lipschitz one on $\mathbf{R}^d \times [0,T]$ with respect to ρ (cf. §2 in [W4]). Since the function $\mathcal{E}(g)$ is also of C^∞ in Ω_D, we have, by Theorem C and Lemma 3.3,

$$\begin{aligned}
&\int_{S_D}\int_{S_D} \frac{|Ag(X)-Ag(Y)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Y) \\
\leq\ & c_1 \liminf_{r\to 0} \int_{F_{r,c_0}} dX \int_{F_{r,c_0}} \frac{|Ag(X)-Ag(Y)|^p}{\rho(X,Y)^{d+2+p\alpha+d-\beta}} dY \\
\leq\ & c_2 \int_{\Omega_D} |\nabla_y Ag(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY \\
+\ & c_2 \int_{\Omega_D} |D_{d+1} Ag(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dY.
\end{aligned}$$

Let $f \in \Lambda^p_\alpha(S_D)$. By Lemma 4.1 the set $\Lambda_{1,\infty}(S_D)$ of all Lipschitz functions on S_D with respect to ρ is dense in $\Lambda^p_\alpha(S_D)$. We can find $\{f_n\} \subset \Lambda_{1,\infty}(S_D)$ such that $f_n \to f$ in $\Lambda^p_\alpha(S_D)$. By the assumption and Theorem B we have, for $f_n \in \Lambda_{1,\infty}(S_D)$,

$$\begin{aligned}
&\int_{\Omega_D} |\nabla_y Af_n(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY \\
+\ & \int_{\Omega_D} |D_{d+1} Af_n(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dY \\
\leq\ & c_3 \int_{(\mathbf{R}^d\setminus\overline{D})\times(0,T)} |\nabla_y \mathcal{E}(f_n)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY \\
+\ & c_3 \int_{(\mathbf{R}^d\setminus\overline{D})\times(0,T)} |D_{d+1} \mathcal{E}(f_n)(Y)|^p \delta(Y)^{2p-p\alpha-d+\beta} dY + c_3\|f_n\|_p^p \leq c_4\|f_n\|^p_{\alpha,p}.
\end{aligned}$$

Hence

$$\int_{S_D}\int_{S_D} \frac{|Af_n(X)-Af_n(Z)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Z) \leq c_5\|f_n\|^p_{\alpha,p}.$$

Since $\|f_n - f\|_p \to 0$ and $\|Af_n - Af\|_p \to 0$, we can choose a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that

$$f_{n_j}(X) \to f(X)\ \mu_T - \text{a.e. on } S_D \quad Af_{n_j}(X) \to Af(X)\ \mu_T - \text{a.e. on } S_D.$$

From Fatou's lemma

$$\begin{aligned}
&\int_{S_D}\int_{S_D} \frac{|Af(X)-Af(Z)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Z) \\
\leq\ & \liminf_{j\to\infty} \int_{S_D}\int_{S_D} \frac{|Af_{n_j}(X)-Af_{n_j}(Z)|^p}{\rho(X,Y)^{\beta+2+p\alpha}} d\mu_T(X)d\mu_T(Z) \\
\leq\ & c_5 \liminf_{j\to\infty} \|f_{n_j}\|^p_{\alpha,p} = c_5\|f\|^p_{\alpha,p}.
\end{aligned}$$

Thus we have the conclusion. Q.E.D.

5. Boundedness of operators on $\Lambda_{\lambda,\infty}(S_D)$

Let $0 < \lambda \leq 1$. Denote by $\Lambda_{\lambda,\infty}(S_D)$ the family of all λ-Hölder continuous functions on S_D with respect to ρ. We consider $\Lambda_{\lambda,\infty}(S_D)$ for λ satisfying $\beta - (d-1) < \lambda < 1$.

We define, for $f \in \Lambda_{\lambda,\infty}(S_D)$ and $Z \in \mathbf{R}^{d+1}$,

$$\begin{aligned}\phi_1 f(Z) &= \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \langle \nabla_y \mathcal{E}(f)(Y), \nabla_y W(Z-Y)\rangle dy,\\ \phi_2 f(Z) &= \int_0^T ds \int_{B(0,2R)\setminus\overline{D}} (\mathcal{E}(f)(Y) - \mathcal{E}(f)(Z))\triangle_y W(Z-Y)dy.\end{aligned}$$

Let $f \in \Lambda_{\lambda,\infty}(S_D)$ and $X \in S_D$. Choose ϵ satisfying $0<\epsilon<\lambda$. In [W4] we have shown that

$$|\nabla_y \mathcal{E}(f)(Y)| \le c\delta(Y)^{\lambda-1}$$

for $Y \in (\mathbf{R}^d \setminus \overline{D}) \times (0,T)$ and

$$\operatorname{supp} \mathcal{E}(f) \subset B(0,2R) \times (-2, T+2).$$

Consequently

$$\begin{aligned}|\phi_1 f(X)| &\le \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y \mathcal{E}(f)(Y)||\nabla_y W(X,Y)|dy\\ &\le c_1 \int_0^T ds \int_{B(0,2R)\setminus\overline{D}} \delta(Y)^{\lambda-1}\rho(X,Y)^{-d-1}dY\\ &\le c_1 \int_0^T |t-s|^{-1+\epsilon/2} ds \int_{B(x,R)\setminus\overline{D}} \delta(y)^{\lambda-1}|x-y|^{-d+1-\epsilon}dy\\ &+ c_1 \int_0^T ds \int_{B(0,2R)\cap\{|x-y|>R\}} \delta(y)^{\lambda-1}|x-y|^{-d-1}dy,\end{aligned}$$

whence, by Lemma 2.1 in [W1],

$$\int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y \mathcal{E}(f)(Y)||\nabla_y W(X,Y)|dy < \infty.$$

On the other hand we also have, by Lemma 2.1 in [W1],

$$\begin{aligned}|\phi_2 f(X)| &\le \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\mathcal{E}(f)(Y) - f(X)||\triangle_y W(X,Y)|dy\\ &\le c_2 \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\mathcal{E}(f)(Y) - f(X)|\rho(X,Y)^{-d-2}dy\\ &\le c_3 \int_0^T ds \int_{B(0,2R)} \rho(X,Y)^{\lambda-d-2}dy \le c_3 \int_0^T ds \int_{|x-y|>R} |x-y|^{\lambda-d-2}dy < \infty.\end{aligned}$$

Thus we have

$$|\phi_2 f(X)| < \infty.$$

Therefore we can define $K_1 f(X)$ as in §1 for $f \in \Lambda_{\lambda,\infty}(S_D)$ and $X \in S_D$,
Similarly we can also define $K_2 f(X)$ as in §1 for $f \in \Lambda_{\lambda,\infty}(S_D)$ and $X \in S_D$.

Lemma 5.1. *Let $0 \le \beta - (d-1) < \lambda < 1$. If f is λ-Hölder continuous on S_D with respect to ρ, then $\phi_1 f$ and $\phi_2 f$ are also λ-Hölder continuous on $\mathbf{R}^{d+1}$ with respect to ρ.*

Proof. Assume that f is λ-Hölder continuous on S_D and $X,\ Z \in \mathbf{R}^{d+1}$. Set

$$B_1 = \{Y \in (\mathbf{R}^d \setminus \overline{D}) \times (0,T); \rho(X,Y) \le 3\rho(X,Z)\}$$

and

$$B_2 = \{Y \in (\mathbf{R}^d \setminus \overline{D}) \times (0,T); \rho(X,Y) > 3\rho(X,Z)\}.$$

We write

$$\begin{aligned}
&|\phi_1 f(X) - \phi_1 f(Z)| \\
= \ & |\int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \langle \nabla_y \mathcal{E}(f)(Y), (\nabla_y W(X,Y) - \nabla_y W(Z,Y)) \rangle dy| \\
\le \ & \Big(\int_{B_1} |\nabla_y \mathcal{E}(f)(Y)||\nabla_y W(X,Y)| dY + \int_{B_1} |\nabla_y \mathcal{E}(f)(Y)||\nabla_y W(Z,Y)| dY\Big) \\
+ \ & \int_{B_2} |\nabla_y \mathcal{E}(f)(Y)||\nabla_y W(X,Y) - \nabla_y W(Z,Y)| dY \\
\equiv \ & (I_1 + I_2) + I_3.
\end{aligned}$$

Choose ϵ satisfying $0 < \epsilon < 1/2$. Then, by Lemma 2.1 in [W1],

$$\begin{aligned}
I_1 \ \le \ & c_1 \int_{\rho(X,Y)\le 3\rho(X,Z)} \delta(Y)^{\lambda-1} \rho(X,Y)^{-d-1} dY \\
+ \ & c_1 \int_{\rho(Z,Y)\le 4\rho(X,Z)} \delta(Y)^{\lambda-1} \rho(Z,Y)^{-d-1} dY \\
\le \ & c_1 \int_{|t-s|\le 9\rho(X,Z)^2} |t-s|^{-1+\epsilon/2} ds \int_{|x-y|\le 3\rho(X,Z)} \delta(y)^{\lambda-1} |x-y|^{-d+1-\epsilon} dy \\
+ \ & c_1 \int_{|r-s|\le 16\rho(X,Z)^2} |t-s|^{-1+\epsilon/2} ds \int_{|z-y|\le 4\rho(X,Z)} \delta(y)^{\lambda-1} |z-y|^{-d+1-\epsilon} dy \\
\le \ & c_2 \rho(X,Z)^\lambda.
\end{aligned}$$

On the other hand, if $Y \in B_2$, then

$$\begin{aligned}
\rho(Z,Y) \ &\ge \ \rho(X,Y) - \rho(Z,X) \\
&\ge \ \rho(X,Y) - \frac{1}{3}\rho(X,Y) \ge \frac{2}{3}\rho(X,Y).
\end{aligned}$$

Consequently we have

$$\begin{aligned}
I_2 \ &\le \ c_3 \int_{B_2} \delta(Y)^{\lambda-1} \rho(X,Z) \big(\rho(X,Y)^{-d-2} + \rho(Z,Y)^{-d-2}\big) dY \\
&\le \ c_4 \int_{B_2} \delta(Y)^{\lambda-1} \rho(X,Z) \rho(X,Y)^{-d-2} dY.
\end{aligned}$$

If $\rho(X,Y) > 3\rho(X,Z)$, then

$$|x-y| > \frac{3}{2}\rho(X,Z) \quad \text{or} \quad |t-s|^{1/2} > \frac{3}{2}\rho(X,Z).$$

We write

$$\begin{aligned}
I_2 \ \le \ & c_4 \rho(X,Z) \int_{\{|x-y|>(3/2)\rho(X,Z)\}\times(0,T)} \delta(y)^{\lambda-1} \rho(X,Y)^{-d-2} dY \\
+ \ & c_4 \rho(X,Z) \int_{\mathbf{R}^d\setminus\overline{D})\times\{|t-s|>(9/4)\}\rho(X,Z)^2\}} \delta(y)^{\lambda-1} \rho(X,Y)^{-d-2} dY \\
\equiv \ & I_{21} + I_{22}.
\end{aligned}$$

We choose $\epsilon' > 0$ satisfying $\epsilon' < \min\{d-\beta, 1-\lambda\}$. Then, by Lemma 2.1 in [W1],

$$\begin{aligned} I_{21} &\leq c_4\rho(X,Z)\int_{|t-s|\leq\rho(X,Z)^2} ds \int_{|x-y|>(3/2)\rho(X,Z)} \delta(y)^{\lambda-1}|x-y|^{-d-2}dy \\ &+ c_4\rho(X,Z)\int_{|t-s|>\rho(X,Z)^2} |t-s|^{-1+\lambda/2-1/2+\epsilon'/2}ds \\ &\times \int_{|x-y|>(3/2)\rho(X,Z)} \delta(y)^{\lambda-1}|x-y|^{-d-\lambda+1-\epsilon'}dy \\ &\leq c_5\rho(X,Z)^{1+2+\lambda-3} + c_5\rho(X,Z)^{1+\lambda-1+\epsilon'-\epsilon'} = c_6\rho(X,Z)^\lambda \end{aligned}$$

and

$$\begin{aligned} I_{22} &\leq c_7\rho(X,Z)\int_{|t-s|>(9/4)\rho(X,Z)^2} |t-s|^{-1+(\lambda-\epsilon'-1)/2}ds \\ &\times \int_{|x-y|\leq\rho(X,Z)} \delta(y)^{\lambda-1}|x-y|^{-d-\lambda+1+\epsilon'}dy \\ &+ c_7\rho(X,Z)\int_{|t-s|>(9/4)\rho(X,Z)^2} |t-s|^{-1-\epsilon'/2}ds \\ &\times \int_{|x-y|>\rho(X,Z)} \delta(y)^{\lambda-1}|x-y|^{-d+\epsilon'}dy = c_8\rho(X,Z)^\lambda. \end{aligned}$$

Therefore $\phi_1 f$ is λ-Hölder continuous with respect to ρ.

We next consider $\phi_2 f$. We write

$$\begin{aligned} |\phi_2 f(X) - \phi_2 f(Z)| &\leq \int_{B_1} |\mathcal{E}(f)(Y) - \mathcal{E}(f)(X)||\Delta_y W(X-Y)|dY \\ &+ \int_{B_1} |\mathcal{E}(f)(Y) - \mathcal{E}(f)(X)||\Delta_y W(Z-Y)|dY \\ &+ \int_{B_2} |\mathcal{E}(f)(Y) - \mathcal{E}(f)(X)||\Delta_y W(X-Y) - \Delta_y W(Z-Y)|dY \\ &\equiv J_1 + J_2 + J_3. \end{aligned}$$

Then, by Lemma 2.1 in [W3],

$$J_1 \leq c_9\int_{\rho(X,Y)\leq 3\rho(X,Z)} \rho(X,Y)^\lambda\rho(X,Y)^{-d-2}dY \leq c_{10}\rho(X,Z)^\lambda.$$

Similarly $J_2 \leq c_{11}\rho(X,Z)^\lambda$.

Since the inequality $\rho(X,Y) > 3\rho(X,Z)$ implies $\rho(Z,Y) > 2\rho(X,Z)$, we have

$$J_3 \leq c_{12}\rho(X,Z)\int_{\rho(X,Y)>2\rho(X,Z)} \rho(X,Y)^\lambda\rho(X,Y)^{-d-3}dY \leq c_{13}\rho(X,Z)^\lambda.$$

Therefore we see that $\phi_2 f$ is also λ-Hölder continuous on $\mathbf{R}^{d+1}$ with respect to ρ. Q.E.D.

6. Parabolic maximal functions

In this section we introduce a parabolic maximal function, which is useful to investigate the boundedness of an operator from a function space on $(B(0,2R)\setminus\overline{D})\times(0,T)$ to a functon space on

Ω_D. To do it, let $0<\eta<d-\beta$. We define a measure on $(B(0,2R)\setminus\overline{D})\times(0,T)$ by

$$\nu_\eta^-(E):=\int_{E\cap(B(0,2R)\setminus\overline{D})\times(0,T)}\delta(Y)^{-\eta}dY.$$

We recall that, for a Borel measurable set $E\subset\mathbf{R}^{d+1}$,

$$\nu_\eta^+(E):=\int_{E\cap\Omega_D}\delta(Y)^{-\eta}dY.$$

By the same method as in Lemma 3.1 in [W5] we can see that the following assertion holds.

Lemma 6.1. *Assume that $\mathbf{R}^d\setminus\overline{D}$ satisfies the condition* (b). *Let $0<\eta<d-\beta$, $X=(x,t)\in\Omega_D$, $b'>1$ and $3R\geq r\geq b'\delta(x)$. Then*

$$\nu_\eta^+(C(X,r))\leq cr^{d+2-\eta}\leq c'\nu_\eta^-(C(X,r)\cap((B(0,2R)\setminus\overline{D})\times(0,T)))\leq c''r^{d+2-\eta},$$

where c and c' are two constants independent of X and r.

Fix $b'>1$ and let $f\in L^1(\nu_\eta^-)$. The parabolic maximal function $\mathcal{M}(\nu_\eta^+,\nu_\eta^-)(f)$ of f is defined as follows:

$$\mathcal{M}(\nu_\eta^+,\nu_\eta^-)(f)(X)=\sup\{\frac{\int_{C(X,r)}|f(Z)|d\nu_\eta^-(Z)}{\nu_\eta^-(C(X,r))};\, b'\delta(X)\leq r\leq R\}$$

for $X\in\Omega_D$. Note that $\mathcal{M}(\nu_\eta^+,\nu_\eta^-)(f)$ is a function defined on Ω_D.

Using Lemma 6.1 and the covering lemma, we have the following lemma.

Lemma 6.2. (i) *Let $f\in L^1(\nu_\eta^-)$ and $\lambda>0$. Then*

$$\nu_\eta^+(\{X\in\Omega_D;\mathcal{M}(\nu_\eta^+,\nu_\eta^-)(f)(X)>\lambda\})\leq\frac{c}{\lambda}\int_{(B(0,2R)\setminus\overline{D})\times(0,T)}|f(Z)|d\nu_\eta^-(Z).$$

(ii) *$p>1$ and $f\in L^p(\nu_\eta^-)$. Then*

$$\int_{\Omega_D}\mathcal{M}(\nu_\eta^+,\nu_\eta^-)(f)(X)^p d\nu_\eta^+(X)\leq c\int_{(B(0,2R)\setminus\overline{D})\times(0,T)}|f(Z)|^p d\nu_\eta^-(Z).$$

Here c is a constant independent of f and λ.

7. Boundedness of the operators on $\Lambda_\alpha^p(S_D)$

In this section we prove Theorem 3 in §1.

Proof of Theorem 3. First of all, using Theorem 2, we shall prove the boundedness of the operator K_1. We choose $\epsilon>0$ satisfying $\alpha-2\epsilon>0$. Let $f\in\Lambda_\alpha^p(S_D)$ and $Z\in S_D$.

$$\begin{aligned}
&\int_0^T ds\int_{\mathbf{R}^d\setminus\overline{D}}|\nabla_y\mathcal{E}(f)(Y)||\nabla_y W(Z-Y)|dy\\
\leq\;&\Big(\int_0^T ds\int_{\mathbf{R}^d\setminus\overline{D}}|\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p-\alpha-d+\beta}\rho(Z,Y)^{-\beta-2+p\epsilon}dy\Big)^{1/p}\\
\times\;&\Big(\int_0^T ds\int_{\mathbf{R}^d\setminus\overline{D}}\delta(Y)^{q(-1+\alpha+(d-\beta)/p)}\rho(Z,Y)^{q(-d-1+(\beta+2)/p-\epsilon)}dy\Big)^{1/q},
\end{aligned}$$

where $q = p/(p-1)$. Since

$$\begin{aligned}
&\int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \delta(Y)^{q(-1+\alpha+(d-\beta)/p)}\rho(Z,Y)^{q(-d-1+(\beta+2)/p-\epsilon)}dy \\
\leq\ & \int_0^T |r-s|^{-1+q\epsilon/2}ds \\
\times\ & \int_{\mathbf{R}^d\setminus\overline{D}} \delta(y)^{q(-1+\alpha+(d-\beta)/p)}|z-y|^{q(-d-1+(\beta+2)/p-\epsilon)+2-q\epsilon}dy
\end{aligned}$$

and $q(-1+\alpha+(d-\beta)/p)+q(-d-1+(\beta+2)/p-\epsilon)+2-\epsilon q > -d$, we have, by Lemma 2.1 in [W1],

$$\int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} \delta(Y)^{q(-1+\alpha+(d-\beta)/p)}\rho(Z,Y)^{q(-d-1+(\beta+2)/p-\epsilon)}dy < \infty.$$

Hence, by Lemma 2.1 in [W3] and Theorem 1 in [W4],

$$\begin{aligned}
&\int_{S_D} d\mu_T(Z)\Big(\int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y\mathcal{E}(f)(Y)||\nabla_y W(Z-Y)|dy\Big)^p \\
\leq\ & c_1\int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p-p\alpha-d+\beta}dy \\
\times\ & \int_{S_D} \rho(Z,Y)^{-\beta-2+\epsilon p}d\mu_T(Z) \\
\leq\ & c_2\int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p-p\alpha-d+\beta}dy \leq \|f\|^p_{\alpha,p}.
\end{aligned}$$

Therefore ϕ_1 is bound from $\Lambda^p_\alpha(S_D)$ to $L^p(\mu_T)$.

Let $\alpha+(d-\beta)/p<\lambda<1$. We saw by Lemma 5.1 that, if f is a Lipschitz function on S_D with respect to ρ, then $\phi_1 f$ is also a λ-Hölder continuous on $\overline{\Omega_D}$ with respect to ρ.

Let f be a Lipschitz function on S_D with respect to ρ and $Z=(z,r)\in\Omega_D$.

$$\begin{aligned}
|\frac{\partial}{\partial r}\phi_1 f(Z)| \ &\leq\ \int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y\mathcal{E}(f)(Y)||\nabla_y\frac{\partial}{\partial r}W(Z-Y)|dy \\
&\leq\ c_3\int_0^T ds \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_y\mathcal{E}(f)(Y)|\rho(Z,Y)^{-d-3}dy \\
&\leq\ c_3\int_0^T ds \int_{(B(0,2R)\setminus\overline{D})\cap\{\delta(y)\leq R/2\}} |\nabla_y\mathcal{E}(f)(Y)|\rho(Z,Y)^{-d-3}dy \\
&+\ c_3\int_0^T ds \int_{(B(0,2R)\setminus\overline{D})\cap\{\delta(y)> R/2\}} |\nabla_y\mathcal{E}(f)(Y)|\rho(Z,Y)^{-d-3}dy \\
&\equiv\ I_1+I_2.
\end{aligned}$$

Set $q=p/(p-1)$ and $\eta=q(1-\alpha-(d-\beta)/p)$. Then

$$\begin{aligned}
&I_1 \leq c_3\sum_{k=1}^\infty \int_{((\mathbf{R}^d\setminus\overline{D})\times(0,T))\cap\{2^{k-1}\delta(z)<\rho(Z,Y)\leq 2^k\delta(z)\}} |\nabla_y\mathcal{E}(f)(Y)|\rho(Y)^{-d-3}dY \\
\leq\ & c_4\sum_{k=1}^\infty (2^{k-1}\delta(z))^{-d-3}\int_{((B(0,2R)\setminus\overline{D})\times(0,T))\cap\{\rho(Y,Z)\leq 2^k\delta(z)\}} |\nabla_y\mathcal{E}(f)(Y)|\delta(Y)^\eta\delta(Y)^{-\eta}dY.
\end{aligned}$$

Using the parabolic maximal function in §6, we have

$$\begin{aligned} I_1\delta(z)^{\eta+1} &\le c_5\sum_{k=1}^{\infty}\left(2^{k-1}\right)^{-\eta-1}\frac{\int_{((B(0,2R)\setminus\overline{D})\times(0,T))\cap\{\rho(Y,Z)\le 2^k\delta(z)\}}|\nabla_y\mathcal{E}(f)(Y)|dY}{\left(2^k\delta(z)\right)^{d+2-\eta}} \\ &\le c_6\mathcal{M}(\nu_\eta^+,\nu_\eta^-)(|\nabla_y\mathcal{E}(f)|\delta(\cdot)^\eta)(Z). \end{aligned}$$

Here we note that

$$\{Y\in(B(0,2R)\setminus\overline{D})\times(0,T);\rho(Z,Y)\le 2^k\delta(z)\}\subset C(Z,2^{k+1}\delta(z))$$

and, by Lemma 6.3,

$$\begin{aligned} \nu_\eta^-(C(Z,2^{k+1}\delta(z))) &= \int_{C(Z,2^{k+1}\delta(z))\cap((B(0,2R)\setminus\overline{D})\times(0,T))}\delta(y)^{-\eta}dY \\ &\le c_7\left(2^{k+1}\delta(z)\right)^{d+2-\eta}. \end{aligned}$$

Lemma 6.2 yields

$$\begin{aligned} &\int_{\Omega_D}(I_1\delta(Z)^{\eta+1})^p d\nu_\eta^+(Z) \\ \le\ & c_8\int_{\Omega_D}\mathcal{M}(\nu_\eta^+,\nu_\eta^-)(|\nabla_y\mathcal{E}(f)|\delta(\cdot)^\eta)(Z)^p d\nu_\eta^+(Z) \\ \le\ & c_9\int_{(B(0,2R)\setminus\overline{D})\times(0,T)}|\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p\eta}\delta(Y)^{-\eta}dY \\ =\ & c_9\int_{(B(0,2R)\setminus\overline{D})\times(0,T)}|\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p-p\alpha-d+\beta}dY, \end{aligned}$$

whence

$$\int_{\Omega_D}I_1^p\delta(Z)^{2p-p\alpha-d+\beta}dZ\le c_{10}\int_{(B(0,2R)\setminus\overline{D})\times(0,T)}|\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p-p\alpha-d+\beta}dy. \tag{7.1}$$

We next estimate I_2. Since

$$I_2\le c_{11}\int_0^T ds\int_{((B(0,2R)\setminus\overline{D})\times(0,T))\cap\{\delta(y)>R/2\}}|\nabla_y\mathcal{E}(f)(Y)|^p|z-y|^{-d-3}dy$$

and $1-\alpha-(d-\beta)/p>0$, we have

$$\begin{aligned} &I_2\delta(Z)^{2-\alpha-(d-\beta)/p} \\ \le\ & c_{11}\Big(\int_{(B(0,2R)\setminus\overline{D})\times(0,T)}|\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p-p\alpha-d+\beta}dY\Big)^{1/p} \\ \times\ & \Big(\int_{((B(0,2R)\setminus\overline{D})\times(0,T))\cap\{\delta(y)>R/2\}}\delta(Y)^{q(-1+\alpha+(d-\beta)/p)}|z-y|^{-d-1-\alpha-(d-\beta)/p}dY\Big)^{1/q}, \end{aligned}$$

whence

$$\int_{\Omega_D}I_2^p\delta(Z)^{2p-p\alpha-d+\beta}dZ\le c_{12}\int_{(B(0,2R)\setminus\overline{D})\times(0,T)}|\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p-p\alpha-d+\beta}dY. \tag{7.2}$$

Therefore, by (7.1) and (7.2),

$$\begin{aligned} &\int_{\Omega_D}|\frac{\partial}{\partial r}\phi_1 f(Z)|^p\delta(Z)^{2p-p\alpha-d+\beta}dZ \\ \le\ & c_{13}\int_{(B(0,2R)\setminus\overline{D})\times(0,T)}|\nabla_y\mathcal{E}(f)(Y)|^p\delta(Y)^{p-p\alpha-d+\beta}dY. \end{aligned}$$

We next estimate $|\nabla_z \phi_1 f|$. Let $Z \in \Omega_D$. We write

$$
\begin{aligned}
|\nabla_z \phi_1 f(Z)| &\leq \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla_y \mathcal{E}(f)(Y)||\nabla_y \nabla_z W(Z-Y)| dy \\
&\leq c_{14} \int_0^T ds \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla_y \mathcal{E}(f)(Y)| \rho(Z,Y)^{-d-2} dy \\
&\leq c_{14} \int_0^T ds \int_{(B(0,2R)\setminus \overline{D}) \cap \{\delta(y) \leq R/2\}} |\nabla_y \mathcal{E}(f)(Y)| \rho(Z,Y)^{-d-2} dy \\
&+ c_{14} \int_0^T ds \int_{(B(0,2R)\setminus \overline{D}) \cap \{\delta(y) > R/2\}} |\nabla_y \mathcal{E}(f)(Y)| \rho(Z,Y)^{-d-2} dy \\
&\equiv I_3 + I_4.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_3 &\leq c_{15} \sum_{k=1}^{\infty} \int_{(\mathbf{R}^d \setminus \overline{D}) \times (0,T) \cap \{2^{k-1}\delta(z) < \rho(Z,Y) \leq 2^k \delta(z)\}} |\nabla_y \mathcal{E}(f)(Y)| \rho(Z,Y)^{-d-2} dY \\
&\leq c_{16} \sum_{k=1}^{\infty} (2^{k-1}\delta(z))^{-d-2} \int_{(B(0,2R)\setminus \overline{D}) \times (0,T) \cap \{\rho(Y,Z) \leq 2^k \delta(z)\}} |\nabla_y \mathcal{E}(f)(Y)| \delta(Y)^{\eta} \delta(Y)^{-\eta} dY.
\end{aligned}
$$

Using the parabolic maximal function in §6, we have

$$
\begin{aligned}
I_3 \delta(Z)^{\eta} &\leq c_{17} \sum_{k=1}^{\infty} \left(2^{k-1}\right)^{-\eta} \frac{\int_{(B(0,2R)\setminus \overline{D}) \times (0,T) \cap \{\rho(Y,Z) \leq 2^k \delta(z)\}} |\nabla_y \mathcal{E}(f)(Y)| dY}{\left(2^k \delta(z)\right)^{d+2-\eta}} \\
&\leq c_{18} \mathcal{M}(\nu_\eta^+, \nu_\eta^-)(|\nabla_y \mathcal{E}(f)| \delta(\cdot)^{\eta})(Z).
\end{aligned}
$$

From this we deduce

$$
\begin{aligned}
&\int_{\Omega_D} I_3^p \delta(Z)^{p\eta} \delta(Z)^{-\eta} dZ \\
&\leq c_{19} \int_{\Omega_D} \mathcal{M}(\nu_\eta^+, \nu_\eta^-)(|\nabla_y \mathcal{E}(f)| \delta(\cdot)^{\eta})(Z)^p \delta(Z)^{-\eta} dZ \\
&\leq c_{20} \int_{(B(0,2R)\setminus \overline{D}) \times (0,T)} |\nabla_y \mathcal{E}(f)(Y)|^p \delta(Y)^{\eta p} \delta(Y)^{-\eta} dY,
\end{aligned}
$$

whence

$$
\int_{\Omega_D} I_3^p \delta(Z)^{p-p\alpha-d+\beta} dZ \leq c_{21} \int_{(B(0,2R)\setminus \overline{D}) \times (0,T)} |\nabla_y \mathcal{E}(f)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY.
$$

Since we can estimate I_4 in the same way as I_2, we conclude that

$$
\begin{aligned}
&\int_{\Omega_D} |\nabla_z \phi_1 f(Z)|^p \delta(Z)^{p-p\alpha-d+\beta} dZ \\
&\leq c_{22} \int_{(B(0,2R)\setminus \overline{D}) \times (0,T)} |\nabla_y \mathcal{E}(f)(Y)|^p \delta(Y)^{p-p\alpha-d+\beta} dY.
\end{aligned}
$$

Using Lemma 5.1 and Theorem 2, we see that $\phi_1 f$ is bounded on $\Lambda_\alpha^p(S_D)$.

Let us next estimate $\phi_2 f$ for a Lipschitz function f on S_D with respect to ρ. To do it, let $Z \in S_D$. Since

$$
|\phi_2 f(Z)| \leq c_{23} \int_0^T ds \int_{B(0,2R)\setminus \overline{D}} |\mathcal{E}(f)(Y) - \mathcal{E}(f)(Z)| \rho(Z,Y)^{-d-2} dy
$$

$$\begin{aligned}
&\leq c_{23}\Big(\int_0^T ds \int_{B(0,2R)\setminus\overline{D}} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(Z)|^p}{\rho(Z,Y)^{d+2+p\alpha}} dy\Big)^{1/p} \\
&\times \Big(\int_0^T ds \int_{B(0,2R)\setminus\overline{D}} \rho(Z,Y)^{-d-2+q\alpha} dy\Big)^{1/q} \\
&\leq c_{24}\Big(\int_0^T ds \int_{B(0,2R)\setminus\overline{D}} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(Z)|^p}{\rho(Z,Y)^{d+2+p\alpha}} dy\Big)^{1/p},
\end{aligned}$$

we have, by Theorem 2 in [W4],

$$\begin{aligned}
\int |\phi_2 f(Z)|^p d\mu_T(Z) &\leq c_{24} \int d\mu_T(Z) \int_{(B(0,2R)\setminus\overline{D})\times(0,T)} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(Z)|^p}{\rho(Z,Y)^{d+2+p\alpha}} dY \\
&\leq c_{25}\|f\|^p_{\alpha,p}.
\end{aligned}$$

We estimate $|\nabla_z \phi_2 f|$. Let $Z \in \Omega_D$. Then

$$\begin{aligned}
|\nabla_z \phi_2 f(Z)| &\leq c_{26} \int_0^T ds \int_{B(0,2R)\setminus\overline{D}} |\mathcal{E}(f)(Y)-\mathcal{E}(f)(Z)| \rho(Z,Y)^{-d-3} dy \\
&\leq c_{26}\Big(\int_0^T ds \int_{B(0,2R)\setminus\overline{D}} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(Z)|^p}{\rho(Z,Y)^{d+2+p\alpha+d-\beta}} dy\Big)^{1/p} \\
&\times \Big(\int_{\rho(Z,Y)\geq\delta(Z)} \rho(Z,Y)^{-d-2+q\alpha+q(d-\beta)/p-q} dY\Big)^{1/q} \\
&\leq c_{27}\Big(\int_0^T ds \int_{B(0,2R)\setminus\overline{D}} \frac{|\mathcal{E}(f)(Y)-\mathcal{E}(f)(Z)|^p}{\rho(Z,Y)^{d+2+p\alpha+d-\beta}} dy\Big)^{1/p} \rho(Z)^{-(1-\alpha-(d-\beta)/p)}.
\end{aligned}$$

Hence, by Theorem 2 in [W4],

$$\int_{\Omega_D} |\nabla_z \phi_2 f|^p \delta(Z)^{p(1-\alpha-(d-\beta)/p)} dZ \leq c_{28}\|f\|^p_{\alpha,p}. \tag{7.3}$$

On the other hand we have

$$|\frac{\partial}{\partial r}\phi_2 f(Z)| \leq c_{29} \int_0^T ds \int_{B(0,2R)\setminus\overline{D}} |\mathcal{E}(f)(Y)-\mathcal{E}(f)(Z)| \rho(Z,Y)^{-d-4} dy.$$

In the same way as above we have

$$\int_{\Omega_D} |\frac{\partial}{\partial r}\phi_2 f(Z)|^p \delta(Z)\delta(Z)^{p-p\alpha-d+\beta} dZ \leq c_{30}\|f\|^p_{\alpha,p}. \tag{7.4}$$

By (7.3), (7.4), Lemma 5.1 and Theorem 2 we see that $\phi_2 f$ is also bounded on $\Lambda^p_\alpha(S_D)$. Thus we see that K_1 is bounded on $\Lambda^p_\alpha(S_D)$.

Noting that, $\mathbf{R}^d \setminus \overline{D}$ may not be uniform, but each pair of two points near ∂D are joined by a rectifiable arc γ in $\mathbf{R}^d \setminus \overline{D}$, which satisfies (1.3) and (1.4). Theorems 1 and 2 also hold in this case. So we can prove that K_2 is also bounded by the same method as in the proof of that of K_1.

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