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Asymptotic Analysis of the Confluent Hypergeometric Partial Differential Equations Satisfied by Φ_2, I

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Abstract. The purpose of this paper is to analyze the system of confluent hypergeometric partial differential equations satisfied by Φ_2 in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ from the view point of Gevrey asymptotic analysis and k-summability. This system possesses the singular loci $x = 0, y = 0, x = y$ of regular type and $x = \infty, y = \infty$ of irregular type. We treat it, near the irregular singularity $x = \infty$ or $y = \infty$ except $(x, y) = (\infty, \infty)$.

1 Introduction

The purpose of this paper is to analyze the system of confluent hypergeometric differential equations

$$\begin{aligned} x\frac{\partial^2 w}{\partial x^2} + y\frac{\partial^2 w}{\partial y \partial x} + (\gamma - x)\frac{\partial w}{\partial x} - \beta w = 0 \\ y\frac{\partial^2 w}{\partial y^2} + x\frac{\partial^2 w}{\partial x \partial y} + (\gamma - y)\frac{\partial w}{\partial y} - \beta' w = 0 \\ (x - y)\frac{\partial^2 w}{\partial x \partial y} - \beta'\frac{\partial w}{\partial x} + \beta\frac{\partial w}{\partial y} = 0 \end{aligned} \tag{1.1}$$

in two variables in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ from the view point of Gevrey asymptotic analysis and k-summability. In this paper we assume that $\beta, \beta', \gamma - \beta - \beta', \beta - \gamma, \beta' - \gamma, \beta + \beta' \notin \mathbb{Z}$. This system possesses the singular loci $x = 0, y = 0, x = y$ of regular type and $x = \infty, y = \infty$ of irregular type. We treat it, near the irregular singularity $x = \infty$ or $y = \infty$ except $(x, y) = (\infty, \infty)$. Erdélyi [2, 3] showed that the system admits solutions expressed by

$$\int_C t^{\beta+\beta'-\gamma}(t - x)^{-\beta}(t - y)^{-\beta'} e^t dt, \tag{1.2}$$

where C is a suitably chosen path, and gave various series expansions of solutions and monodromic relations. Furthermore Shimomura [10] gave asymptotic expansions of the solutions expressed by (1.2) and Stokes multipliers using monodoromy matrices.

Our methods are different from them. We explicitly construct formal solutions and actual solutions having Gevrey asymptotic expansions through the Borel-Laplace transform and compute Stokes multipliers by using relations of Borel transforms. In Sections 2 and 3 we give definitions and notations, respectively used in the following sections. We give main results with their proofs, namely asymptotic solutions and Stokes multipliers near $y = \infty$ (x : bounded) in Section 4. In Section 5, without proofs, we give similar results near $x = \infty$ (y : bounded). In Section 6 we give the relation between asymptotic solutions given in Sections 4 and 5 and the solutions given by Shimomura [10].

This is the first of a sequence of papers. In the forthcoming paper we will study the behavior near a singular point $(x, y) = (\infty, \infty)$.

2 Preparations

In general, formal solutions are divergent series near an irregular singularity, but we construct actual solutions which admit asymptotic expansions of Gevrey order. This fact is described by many authors (see, e.g., Balser [1], and Heish and Sibuya [5]). We briefly explain the definition of the asymptotic series of Gevrey order and that of Gevrey asymptotic expansion. Let D be a domain in the complex plane $\mathbb{C}$. Denote by $\mathcal{O}(D)$ the set of holomorphic functions in D. We further denote by $\mathcal{O}(D)[[z]]$ the $\mathbb{C}$-algebra of all formal power series with coefficients in $\mathcal{O}(D)$. We denote by $S(d, \alpha, \rho)$ the sector

$$\{z : |z| > \rho, \ |\arg z - d| < \frac{\alpha}{2}\},$$

where d is an arbitrary real number, α is a positive number and ρ is a non-negative number. When $\rho = 0$, we use the notation $S(d, \alpha)$ instead of $S(d, \alpha, 0)$. Also, a closed sector is a set of the form

$$\bar{S} = \bar{S}(d, \alpha, \rho) = \{z : |z| \geq \rho, |d - \arg z| \leq \frac{\alpha}{2}\}$$

with d, α and ρ as before.

Firstly, we define the asymptotic series of Gevrey order s.

Definition 2.1 *Let s be a positive integer. The formal series*

$$p(u, z) = \sum_{m=0}^{\infty} c_m(u) z^{-m}$$

in $\mathcal{O}(D)[[z]]$ is said to be of Gevrey order s as $z \to \infty$ uniformly on a domain D if there exist non-negative numbers $C_0 > 0$ and $C_1 > 0$ such that

$$|c_m(u)| \leq C_0(\Gamma(m+1))^s C_1^m$$

for $u \in D$ and $m = 0, 1, 2, \ldots$.

We denote by $\mathcal{O}(D)[[z]]_s$ the $\mathbb{C}$-algebra of all formal power series of Gevrey order s in z with coefficients in $\mathcal{O}(D)$ as $z \to \infty$.

Secondly, we define uniformly Gevrey asymptotic expansions.

Definition 2.2 *Set $V = S(d, \alpha, \rho)$ and $W = \bar{S}(d, a, r)$. Let s be a positive number. A function $f(u, z)$ is said to admit an asymptotic expansion $p(u, z) = \sum_{m=0}^{\infty} c_m(u) z^{-m}$ of Gevrey order s as $z \to \infty$ in V uniformly on a domain D in the u-space, if*
(1) $p(u, z) = \sum_{m=0}^{\infty} c_m(u) z^{-m}$ is of Gevrey order s as $z \to \infty$ uniformly on D,
(2) for each W such that $0 < \rho < r$ and $0 < a < \alpha$, there exist non-negative numbers K_W and L_W such that

$$|f(u, z) - \sum_{m=0}^{N-1} c_m(u) z^{-m}| \le K_W (\Gamma(N+1))^s (L_W)^N |z|^{-N}$$

for $u \in D$, $z \in W$, and for $N = 1, 2, \ldots$.

If the conditions given in Definition 2.2 are satisfied, we write for short

$$f(u, z) \simeq_s p(u, z) \quad \text{in } V \text{ uniformly on } D.$$

We denote by $\mathcal{A}_s(V, \mathcal{O}(D))$ the set of functions holomorphic in $(u, z) \in D \times V$ and having an asymptotic expansion of Gevrey order s as $z \to \infty$ uniformly on the domain D.

Furthermore, if the opening of V is greater than $\frac{\pi}{k}$ (where $s = \frac{1}{k}$), namely $2\alpha > \frac{\pi}{k}$, in the above definition, $p(u, z)$ is said to be k-summable in direction d uniformly on the domain D, and $f(u, z)$ is called k-sum of the power series $p(u, z)$ in direction d uniformly on D. We say that $p(u, z)$ is k-summable uniformly on D whenever $p(u, z)$ is k-summable in all directions but finitely many directions (see, e.g., Balser [1]; note that we work near the point at infinity while they do near the origin). We treat only the case $k = 1$ in the following argument. Then $p(u, z)$ is frequently said to be Borel summable uniformly on D and its 1-sum is said to be the Borel sum. Next, we define the Borel transform and the Laplace transform as follows.

Definition 2.3 *Suppose that $\alpha \in \mathbb{C}$ and that $\beta \in \mathbb{C} \setminus \{0, 1, 2, \ldots\}$. Let D be a domain and s be a positive number. For the product of Gevrey formal power series $\sum_{m=0}^{\infty} c_m(u) z^{-m} \in \mathcal{O}(D)[[z]]_s$, $e^{\alpha z}$ and $z^{-\beta}$, namely*

$$p(u, z) = e^{\alpha z} \sum_{m=0}^{\infty} c_m(u) z^{-\beta - m},$$

we define the formal Borel transform of it as below,

$$\hat{\mathfrak{B}} p(u, \eta) = \sum_{m=0}^{\infty} \frac{c_m(u)}{\Gamma(\beta + m)} (\alpha + \eta)^{\beta + m - 1}.$$

If the integral

$$\int_{-\alpha}^{-\alpha+\infty(\theta)} \hat{\mathfrak{B}}p(u,\eta)e^{-z\eta}d\eta$$

converges, then we call it the Laplace transform of $\hat{\mathfrak{B}}p(u,\eta)$ *in direction* θ, *and denote by* $\mathfrak{L}_{(\alpha,\theta)}\hat{\mathfrak{B}}p(u,z)$. *If* $\mathfrak{L}_{(\alpha,\theta)}\hat{\mathfrak{B}}p(u,z)$ *is analytically continuable in* $D \times S(\theta,\gamma)$ *with* $\gamma > \pi$, *we can verify that this integral is the Borel sum of* $p(u,z)$ *in the direction* θ.

3 Notation

Here we employ the notation

$$e^{(\lambda)} := e^{2\pi i\lambda} \quad (\lambda \in \mathbb{C}).$$

By $F(a,b,c;z)$, we denote the hypergeometric function expressed as

$$F(a,b,c;z) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m(1)_m} z^m,$$

and by ${}_1F_1(a,c;z)$ the confluent hypergeomtric function expressed as

$${}_1F_1(a,c;z) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m(1)_m} z^m.$$

Denote by $L_m^{(a)}(s)$ the Laguerre polynomial expressed as

$$L_m^{(a)}(s) = \sum_{j=0}^{m} \binom{m+a}{m-j} \frac{(-s)^j}{(1)_j}.$$

Let $T(\beta,\beta',\gamma,s;u)$ be a power series of the form

$$T(\beta,\beta',\gamma,s;u) = \sum_{\nu=0}^{\infty} \frac{(\beta)_\nu(\beta+\beta'-\gamma+1)_\nu}{(1)_\nu(\beta-\gamma+2)_\nu} s^\nu {}_1F_1(\beta+\beta'-\gamma+\nu+1, \beta-\gamma+\nu+2, s)u^\nu,$$

which converges for $|s| < +\infty$, $|su| < 1$. Let $U(\beta,\beta',\gamma,s;u)$ and $V(\beta,\beta',\gamma,s;u)$ be a formal power series given by

$$U(\beta,\beta',\gamma,s;u) = \sum_{\nu=0}^{\infty} \frac{(\beta)_\nu(\beta-\gamma+1)_\nu}{(1)_\nu} s^\nu {}_1F_1(\beta',\gamma-\beta-\nu,s)u^\nu$$

and

$$V(\beta,\beta',\gamma,s;u) = \sum_{\nu=0}^{\infty} (1-\beta)_\nu P_\nu(\beta,\beta',\gamma,s)u^\nu,$$

where

$$P_\nu(\beta,\beta',\gamma,s) = \sum_{m=0}^{\nu} \frac{(\beta')_m(\gamma-\beta-\beta')_{\nu-m}}{(\beta-\nu)_m(1)_{\nu-m}} L_m^{(\beta-\nu-1)}(s).$$

4 Solutions near $y = \infty$ (x is bounded)

We begin with the case where (x, y) satisfies

$$0 < \arg x < \pi < \arg y < 2\pi, \ \ \pi < \arg(y - x) < 2\pi. \tag{4.1}$$

4.1 Formal solutions

Theorem 4.1 *Near $y = \infty$ (x is bounded) we have formal solutions as follows:*

$$\hat{Y}_- = -\Gamma(1-\beta')e^{-\beta'\pi i}y^{\beta'-\gamma}e^{y}V(\beta', \beta, \gamma, x, 1/y),$$
$$Y_0 = e^{(\beta'-\beta)\pi i}\frac{\Gamma(\beta+\beta'-\gamma+1)\Gamma(1-\beta)}{\Gamma(\beta'-\gamma+2)} \times x^{\beta'-\gamma+1}y^{-\beta'}T(\beta', \beta, \gamma, x, 1/y),$$
$$\hat{Y}_+ = \frac{2\pi i e^{\beta'\pi i}}{\Gamma(\gamma-\beta')}y^{-\beta'}U(\beta', \beta, \gamma, x, -1/y).$$

Proof of Theorem 4.1 We try to find a formal solution of the system (1.1) expressed as

$$w = e^{ay}x^{b}y^{-c}\sum_{j,l\geq 0} w_{j,l}x^{j}y^{-l} \tag{4.2}$$

By substituting (4.2) into

$$x\frac{\partial^2}{\partial x^2}w + y\frac{\partial}{\partial y}\frac{\partial}{\partial x}w + (\gamma - x)\frac{\partial}{\partial x}w - \beta w = 0,$$

we obtain the recurrence formulae,

$$\begin{cases} (b+j+1)(b+j-\gamma-c)w_{j+1,l} - (b+j+\beta)w_{j,l} + a(b+j+1)w_{j+1,l+1} = 0, \\ ab = 0, \\ b(b-1+\gamma-c)w_{0,0} + aw_{0,1} = 0. \end{cases}$$

By substituting (4.2) into

$$y\frac{\partial^2}{\partial y^2}w + x\frac{\partial}{\partial y}\frac{\partial}{\partial x}w + (\gamma - y)\frac{\partial}{\partial y}w - \beta' w = 0,$$

we also obtain,

$$\begin{cases} a(a-1)w_{j,l+2} - (2a(-c-l-1) + a(b+j) + b\gamma - (-c-l-1) - \beta')w_{j,l+1} = 0, \\ a(a-1) = 0, \\ a(a-1)w_{0,1} + (-2ac + ab + c + \gamma a - \beta')w_{0,0} = 0. \end{cases}$$

From the recurrence formulae above, we derive the following:
Case 1) $a = 0, \ b = \beta' - \gamma + 1, \ c = \beta'$;
Case 2) $a = 0, \ b = 0, \ c = \beta'$;

Case 3) $a = 1,\ b = 0,\ c = \gamma - \beta'$.

Case 1) $a = 0,\ b = \beta' - \gamma + 1,\ c = \beta'$:
If we set $w_{0,0} = e^{(\beta'-\beta)\pi i}\frac{\Gamma(\beta+\beta'-\gamma+1)\Gamma(1-\beta)}{\Gamma(\beta'-\gamma+2)}$, then

$$w_1 = e^{(\beta'-\beta)\pi i}\frac{\Gamma(\beta+\beta'-\gamma+1)\Gamma(1-\beta)}{\Gamma(\beta'-\gamma+2)}x^{\beta'-\gamma+1}y^{-\beta'}T(\beta',\beta,\gamma,x,1/y).$$

Case 2) $a = 0,\ b = 0,\ c = \beta'$:
If we set $w_{0,0} = \frac{2\pi i e^{\beta'\pi i}}{\Gamma(\gamma-\beta')}$, then

$$w_2 = \frac{2\pi i e^{\beta'\pi i}}{\Gamma(\gamma-\beta')}y^{-\beta'}U(\beta',\beta,\gamma,x,-1/y).$$

Case 3) $a = 1,\ b = 0,\ c = \gamma - \beta'$:
If we set $w_{0,0} = -\Gamma(1-\beta')e^{-\beta'\pi i}$, then

$$w_3 = -\Gamma(1-\beta')e^{-\beta'\pi i}y^{\beta'-\gamma}e^{y}V(\beta',\beta,\gamma,x,1/y).$$

4.1.1 Gevrey asymptotic estimate

Theorem 4.2 *Assume that R is an arbitrary large positive constant. Then,*
(1) $\hat{Y}_-$ is a power series of Gevrey order 1 of $1/y$ uniformly for $|x| < R$.
(2) Y_0 converges uniformly for $|x| < R$ and $|x| < |y|$.
(3) $\hat{Y}_+$ is a power series of Gevrey order 1 of $1/y$ uniformly for $|x| < R$.

Proof of Theorem 4.2 (1) For all positive integer n,

$$\begin{aligned}
&\left|\sum_{m=0}^{n}\sum_{j=0}^{m}\frac{(\beta)_m(\gamma-\beta-\beta')_{n-m}(1-\beta')_n}{(1)_{n-m}(1)_j(\beta'-n)_m}(-x)^j\binom{m+\beta'-n-1}{m-j}\right| \\
\leq&\left|\sum_{m=0}^{n}\sum_{j=0}^{m}\frac{(\beta)_m}{(1)_m}\frac{(\gamma-\beta-\beta')_{n-m}}{(1)_{n-m}}\frac{\Gamma(1-\beta'+n)\Gamma(\beta'-n)}{\Gamma(\beta'-n+j)}\frac{(1)_m}{(1)_j(1)_{m-j}}\frac{1}{\Gamma(1-\beta')}R^j\right| \\
\leq&\sum_{m=0}^{n}\left|\frac{(\beta)_m}{(1)_m}\right|\left|\frac{(\gamma-\beta-\beta')_{n-m}}{(1)_{n-m}}\right|(1+R)^m \\
&\times\sum_{j=0}^{m}|\Gamma(n-j-\beta'+1)|\left|\frac{1}{\Gamma(1-\beta')}\right|\left|\frac{\sin\pi(\beta'-n+j)}{\sin\pi(\beta'-n)}\right| \\
\leq&\frac{1}{|\Gamma(1-\beta')|}\sum_{m=0}^{n}(1+|\beta-1|)^m(1+|\gamma-\beta-\beta'-1|)^{n-m}(1+R)^m\sum_{j=0}^{n}|\Gamma(n-j-\beta'+1)| \\
\leq&\frac{1}{|\Gamma(1-\beta')|}(1+|\gamma-\beta-\beta'-1|)^n\sum_{m=0}^{n}\left\{\frac{(1+|\beta-1|)(1+R)}{1+|\gamma-\beta-\beta'-1|}\right\}^m\sum_{j=0}^{n}|\Gamma(n-j-\beta'+1)|
\end{aligned}$$

$$\leq \frac{1}{|\Gamma(1-\beta')|}(1+|\gamma-\beta-\beta'-1|)^n \left| \frac{1-\left(\frac{(1+|\beta-1|)(1+R)}{1+|\gamma-\beta-\beta'-1|}\right)^{n+1}}{1-\frac{(1+|\beta-1|)(1+R)}{1+|\gamma-\beta-\beta'-1|}} \right| \sum_{j=0}^{n} |\Gamma(n-j-\beta'+1)|$$

$$\leq \frac{1}{|\Gamma(1-\beta')||1+|\gamma-\beta-\beta'-1|-(1+|\beta-1|)(1+R)|}$$

$$\times \left|(1+|\gamma-\beta-\beta'-1|)^{n+1}-(1+|\beta-1|)^{n+1}(1+R)^{n+1}\right| \sum_{j=0}^{n} |\Gamma(n-j-\beta'+1)|.$$

(2) Put

$$\alpha_{(n,m)} = \min_{1\leq j\leq m} |\beta'-\gamma+n+j+1|.$$

We may suppose that

$$1+\frac{|\beta-1|}{\alpha_{(n,m)}} < a_0$$

is valid for all n. Then we have, for $|x|<R$,

$$\left| \frac{(\beta')_n(\beta+\beta'-\gamma+1)_n}{(1)_n(\beta'-\gamma+2)_n} {}_1F_1(\beta+\beta'-\gamma+n+1, \beta'-\gamma+n+2, x) \right|$$

$$\leq \left| \frac{(\beta')_n(\beta+\beta'-\gamma+1)_n}{(1)_n(\beta'-\gamma+2)_n} \right|$$

$$\times \left| \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(m+1)} \left(1+\frac{\beta-1}{\beta'-\gamma+n+m+1}\right) \cdots \left(1+\frac{\beta-1}{\beta'-\gamma+n+2}\right) \right|$$

$$\leq \left| \frac{(\beta')_n(\beta+\beta'-\gamma+1)_n}{(1)_n(\beta'-\gamma+2)_n} \right| \sum_{m=0}^{\infty} \frac{|x|^m \left(1+\frac{|\beta-1|}{\alpha_{(n,m)}}\right)^m}{\Gamma(m+1)}$$

$$\leq \left| \frac{(\beta')_n(\beta+\beta'-\gamma+1)_n}{(1)_n(\beta'-\gamma+2)_n} \right| \sum_{m=0}^{\infty} \frac{(a_0R)^m}{\Gamma(m+1)}$$

$$\leq \left| \frac{(\beta')_n(\beta+\beta'-\gamma+1)_n}{(1)_n(\beta'-\gamma+2)_n} \right| e^{a_0 R}.$$

Since

$$\lim_{n\to\infty} \frac{\left|\frac{(\beta')_{n+1}(\beta+\beta'-\gamma+1)_{n+1}}{(1)_{n+1}(\beta'-\gamma+2)_{n+1}}\right| e^{a_0R}}{\left|\frac{(\beta')_n(\beta+\beta'-\gamma+1)_n}{(1)_n(\beta'-\gamma+2)_n}\right| e^{a_0R}} = 1,$$

the convergence of Y_0 follows immediately.

(3) Put

$$\beta_{(n,m)} = \min_{0\leq j\leq m-1} |\gamma-\beta'-n+j|.$$

We may suppose that

$$1+\frac{|\beta+\beta'-\gamma+n|}{\beta_{(n,m)}} < b_0$$

is valid for all n. Then we have, for $|x| < R$,

$$
\begin{aligned}
&\left|\frac{(\beta')_n(\beta'-\gamma+1)_n}{(1)_n}{}_1F_1(\beta,\gamma-\beta'-n,x)\right| \\
\le&\left|\frac{(\beta')_n(\beta'-\gamma+1)_n}{(1)_n}\right|\left|\sum_{m=0}^{\infty}\frac{\Gamma(\beta+m)\Gamma(\gamma-\beta'-n)}{\Gamma(\beta)\Gamma(m+1)\Gamma(\gamma-\beta'-n+m)}x^m\right| \\
\le&\left|\frac{(\beta')_n(\beta'-\gamma+1)_n}{(1)_n}\right|\left|\sum_{m=0}^{\infty}\frac{x^m}{\Gamma(m+1)}\left(1+\frac{\beta+\beta'-\gamma+n}{\gamma-\beta'-n+m-1}\right)\cdots\left(1+\frac{\beta+\beta'-\gamma+n}{\gamma-\beta'-n}\right)\right| \\
\le&\left|\frac{(\beta')_n(\beta'-\gamma+1)_n}{(1)_n}\right|\sum_{m=0}^{\infty}\frac{|x|^m}{\Gamma(m+1)}\left(1+\frac{|\beta+\beta'-\gamma+n|}{\beta_{(n,m)}}\right)^m \\
\le&\left|\frac{(\beta')_n(\beta'-\gamma+1)_n}{(1)_n}\right|\sum_{m=0}^{\infty}\frac{|x|^m b_0^m}{\Gamma(m+1)} \\
\le&\left|\frac{1}{\Gamma(\beta')\Gamma(\beta'-\gamma+1)}\right|e^{b_0R}(1+|\beta'|)^{n+1}|\Gamma(\beta'-\gamma+n+1)|.
\end{aligned}
$$

4.2 Gevrey asymptotic solutions

We construct actual solutions which have asymptotic expansions of Gevrey order 1 through the Borel-Laplace transforms of the formal solutions $\hat{Y}_-$ and $\hat{Y}_+$. Also we can find that Y_0 is equal to the Borel-Laplace transform of the solution Y_0 (Theorem 4.1).

Theorem 4.3 *We have the following explicit Borel transforms of* $\hat{Y}_-$, Y_0 *and* $\hat{Y}_+$.

$$
\begin{aligned}
(1)\hat{\mathfrak{B}}(\hat{Y}_-)(x,v) =&\frac{\Gamma(1-\beta')}{\Gamma(\gamma-\beta')}(1+v)^{\gamma-\beta'-1}v^{\beta'-1}{}_1F_1(\beta,\gamma-\beta';(1+v)x) \\
(2)\mathfrak{B}(Y_0)(x,v) =&e^{(\beta'-\beta)\pi i}\frac{\Gamma(\beta+\beta'-\gamma+1)\Gamma(1-\beta)}{\Gamma(\beta'-\gamma+2)\Gamma(\beta')} \\
&\times v^{\beta'-1}x^{\beta'-\gamma+1}{}_1F_1(\beta+\beta'-\gamma+1,2+\beta'-\gamma;(1+v)x) \\
(3)\hat{\mathfrak{B}}(\hat{Y}_+)(x,v) =&\frac{2\pi i e^{\beta'\pi i}}{\Gamma(\gamma-\beta')\Gamma(\beta')}v^{\beta'-1}(1+v)^{\gamma-\beta'-1}{}_1F_1(\beta,\gamma-\beta';(1+v)x).
\end{aligned}
$$

To compute the Borel transforms of $\hat{Y}_-$, Y_0 and $\hat{Y}_+$, we use the following propositions.

Proposition 4.1 *Let λ be a parameter. Then*

$$
\begin{aligned}
{}_1F_1(a,c;\lambda s) =&\sum_{n=0}^{\infty}\frac{(a)_n}{\Gamma(n+1)(c)_n}(\lambda-1)^n s^n{}_1F_1(a+n,c+n;s), \\
{}_1F_1(a,c;\lambda s) =&\lambda^{1-c}\sum_{n=0}^{\infty}\frac{(1-c)_n}{\Gamma(n+1)}(1-\lambda)^n{}_1F_1(a,c-n;s).
\end{aligned}
$$

The proof of Proposition 4.1 is given in Erdélyi [4].

Proposition 4.2 *We have*

$$(1-e^{(\beta')})^{-1}\int_{C(0)} u^{l-m-\beta'}(u-x)^m e^u du$$
$$= -(-1)^l e^{-\beta'\pi i}(1-\beta')_l\Gamma(1-\beta')\frac{(1)_m}{(\beta'-l)_m}L_m^{(\beta'-l-1)}(x).$$

Here the path of integration and the branch of the integrand are chosen in such a way that they have the following properties:

(1) $C(0)$ *is a loop which starts from* $u=\infty e^{-\pi i}$, *encircles* $u=0$ *in the positive sense and ends at* $u=\infty e^{\pi i}$.
(2) *The branch of* $u^{l-m-\beta'}(u-x)^m$ *is taken such that*

$$\arg u=\pi,\ \arg(u-x)=\pi$$

at the end point $u=\infty e^{\pi i}$ *of the path of integration.*

Proposition 4.2 can be found in Erdélyi [4] and Shimomura [10].

Proof of Theorem 4.3 As for (1), using Proposition 4.2, the relation between the B-function and Γ-function and the integral representation of ${}_1F_1(\beta,\gamma-\beta';(1+v)x)$, we have

$$\hat{\mathfrak{B}}(\hat{Y}_-)(x,v)$$
$$=(1-e^{(\beta')})^{-1}(1+v)^{\gamma-\beta'-1}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{(\beta)_m(\gamma-\beta-\beta')_{l-m}}{(1)_m(1)_{l-m}\Gamma(l+\gamma-\beta')}$$
$$\times(-1-v)^l\int_{C(0)}u^{l-m-\beta'}(u-x)^m e^u du$$
$$=(1-e^{(\beta')})^{-1}(1+v)^{\gamma-\beta'-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\beta)_m(\gamma-\beta-\beta')_n}{(1)_m(1)_n\Gamma(m+n+\gamma-\beta')}$$
$$\times(-1-v)^{n+m}\int_{C(0)}u^{n-\beta'}(u-x)^m e^u du$$
$$=(1-e^{(\beta')})^{-1}\frac{1}{\Gamma(\beta)\Gamma(\gamma-\beta-\beta')}(1+v)^{\gamma-\beta'-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B(\beta+m,\gamma-\beta-\beta'+n)}{(1)_m(1)_n}$$
$$\times(-1-v)^{n+m}\int_{C(0)}u^{n-\beta'}(u-x)^m e^u du$$
$$=(1-e^{(\beta')})^{-1}\frac{1}{\Gamma(\beta)\Gamma(\gamma-\beta-\beta')}(1+v)^{\gamma-\beta'-1}\int_{C(0)}\int_0^1\sum_{n=0}^{\infty}\frac{1}{m!}\{(-1-v)s(u-x)\}^m$$
$$\times\sum_{m=0}\frac{1}{n!}\{(-1-v)(1-s)u\}^n u^{-\beta'}s^{\beta-1}(1-s)^{\gamma-\beta-\beta'-1}e^u ds du$$

$$=(1-e^{(\beta')})^{-1}\frac{1}{\Gamma(\gamma-\beta')}(1+v)^{\gamma-\beta'-1}\int_{C(0)}e^{-uv}u^{-\beta'}{}_1F_1(\beta,\gamma-\beta';(1+v)x)du$$
$$=\frac{\Gamma(1-\beta')}{\Gamma(\gamma-\beta')}(1+v)^{\gamma-\beta'-1}v^{\beta'-1}{}_1F_1(\beta,\gamma-\beta';(1+v)x).$$

As for (2), using Proposition 4.1, we have

$$\begin{aligned}&\mathfrak{B}(Y_0)(x,v)\\&=e^{(\beta'-\beta)\pi i}\frac{\Gamma(\beta+\beta'-\gamma+1)\Gamma(1-\beta)}{\Gamma(\beta'-\gamma+2)}\sum_{l=0}^{\infty}\frac{(\beta')_l(\beta+\beta'-\gamma+1)_l}{(1)_l(\beta'-\gamma+2)_l\Gamma(l+\beta')}\\&\times x^{\beta'-\gamma+l+1}v^{\beta'+l-1}{}_1F_1(\beta+\beta'-\gamma+1,\beta'-\gamma+l+2;x)\\&=e^{(\beta'-\beta)\pi i}\frac{\Gamma(\beta+\beta'-\gamma+1)\Gamma(1-\beta)}{\Gamma(\beta'-\gamma+2)\Gamma(\beta')}v^{\beta'-1}x^{\beta'-\gamma+1}{}_1F_1(\beta+\beta'-\gamma+1,\beta'-\gamma+2;(1+v)x).\end{aligned}$$

As for (3), using Proposition 4.1, we have

$$\begin{aligned}&\hat{\mathfrak{B}}(\hat{Y}_+)(x,v)\\&=\frac{2\pi i e^{\beta'\pi i}}{\Gamma(\gamma-\beta')\Gamma(\beta')}\sum_{l=0}^{\infty}\frac{(\beta'-\gamma+1)_l}{(1)_l}(-v)^l v^{\beta'-1}{}_1F_1(\beta,\gamma-\beta'-l;x)\\&=\frac{2\pi i e^{\beta'\pi i}}{\Gamma(\gamma-\beta')\Gamma(\beta')}v^{\beta'-1}(1+v)^{\gamma-\beta'-1}{}_1F_1(\beta,\gamma-\beta';(1+v)x).\end{aligned}$$

By explicit formulae, the singular points of $\hat{\mathfrak{B}}(\hat{Y}_-)$ are $v=-1$ and $v=0$, and those of $\hat{\mathfrak{B}}(\hat{Y}_+)$ are $v=-1$ and $v=0$. Therefore we can consider the Laplace transforms of $\hat{\mathfrak{B}}(\hat{Y}_-)$ and $\hat{\mathfrak{B}}(\hat{Y}_+)$ except one direction. So we see that $\hat{Y}_-, Y_0, \hat{Y}_+$ are Borel summable by Laplace transforms.

Theorem 4.4 (1) $\hat{Y}_-$ *is Borel summable except for the positive direction of real axis.*
(2) Y_0 *is Borel summable in all directions.*
(3) $\hat{Y}_+$ *is Borel summable except for the negative direction of real axis.*

We construct actual solutions which have asymptotic expansions to the formal solutions uniformly for x through the Laplace transforms of $\hat{\mathfrak{B}}(\hat{Y}_-)$ and $\hat{\mathfrak{B}}(\hat{Y}_+)$.

Since $\hat{\mathfrak{B}}(\hat{Y}_-)$ is singular only at $v=-1$ and $v=0$, we can see that $\mathfrak{L}_{(-1,\theta_1)}(\hat{\mathfrak{B}}(\hat{Y}_-))$ is integrable in the direction for $\theta_1=\pi$. Here the branch of $e^{-yv}\hat{\mathfrak{B}}(\hat{Y}_-)(x,v)$ is taken such that

$$\arg v=\pi,\quad \arg(v+1)=\pi$$

at the end point $v=\infty$. Also, for $\theta_1\in(0,2\pi)$, $\mathfrak{L}_{(-1,\theta_1)}(\hat{\mathfrak{B}}(\hat{Y}_-))$ is defined in the sector

$$\frac{\pi}{2}<\arg(e^{-\pi i}yv)<\frac{3\pi}{2}$$

i.e.,

$$\frac{3\pi}{2} - \theta_1 < \arg y < \frac{5\pi}{2} - \theta_1,$$

because $\exp(e^{-\pi i}yv)$ tends to 0 as v tends to infinity. Using the fact that $e^{-yv}\hat{\mathfrak{B}}(\hat{Y}_-) \simeq_1 0$ as $y \to \infty$ and by Cauchy's integral theorem, $\mathfrak{L}_{(-1,\pi)}(\hat{\mathfrak{B}}(\hat{Y}_-))$ is analytically continuable to a function, noted $Y_-(x,y)$, in the sector

$$-\frac{\pi}{2} < \arg y < \frac{5\pi}{2},$$

where

$$Y_-(x,y) \simeq_1 \hat{Y}_-.$$

Since $\hat{\mathfrak{B}}(\hat{Y}_+)$ has singular points only at $v = -1$ and $v = 0$, we can see that $\mathfrak{L}_{(0,\theta_2)}(\hat{\mathfrak{B}}(\hat{Y}_+))$ is integrable in the direction for $\theta_2 = 0$. Here the branch of $e^{-yv}\hat{\mathfrak{B}}(\hat{Y}_+)(x,v)$ is taken such that

$$\arg v = \pi, \quad \arg(v+1) = \pi$$

at the end point $v = \infty$. Also, $\mathfrak{L}_{(0,\theta_2)}(\hat{\mathfrak{B}}(\hat{Y}_+))$ is defined in

$$\frac{\pi}{2} < \arg(e^{-\pi i}yv) < \frac{3\pi}{2}$$

i.e.,

$$\frac{3\pi}{2} - \theta_2 < \arg y < \frac{5\pi}{2} - \theta_2.$$

The function $\mathfrak{L}_{(0,0)}(\hat{\mathfrak{B}}(\hat{Y}_+))$ is analytically continuable to $Y_+(x,y)$ in the sector

$$\frac{\pi}{2} < \arg y < \frac{7\pi}{2},$$

where

$$Y_+(x,y) \simeq_1 \hat{Y}_+.$$

Theorem 4.5 *Assume that R is an arbitrarily large positive constant. Then,*

$$Y_- \simeq_1 \hat{Y}_-$$

uniformly for $|x| < R$ as $y \to \infty$ through the sector $|\arg y - \pi| < \dfrac{3\pi}{2}$, and

$$Y_+ \simeq_1 \hat{Y}_+$$

uniformly for $|x| < R$ as $y \to \infty$ through the sector $|\arg y - 2\pi| < \dfrac{3\pi}{2}$.

4.3 Stokes multipliers

Considering $\mathfrak{L}_{(0,0+2\pi)}(\hat{\mathfrak{B}}(\hat{Y}_+))(x,y)$, we can show that $\mathfrak{L}_{(0,0+2\pi)}(\hat{\mathfrak{B}}(\hat{Y}_+))(x,y)$ is defined in the sector $\frac{-\pi}{2} < \arg y < \frac{\pi}{2}$ in a similar way. Let $\tilde{Y}_+$ be analytic continuation of $\mathfrak{L}_{(0,0+2\pi)}(\hat{\mathfrak{B}}(\hat{Y}_+))(x,y)$ in the sector $\frac{-3\pi}{2} < \arg y < \frac{3\pi}{2}$.

We note the following proposition in order to compute Stokes multipliers.

Proposition 4.3 (Relation between $\hat{\mathfrak{B}}(\hat{Y}_-)$ and $\hat{\mathfrak{B}}(\hat{Y}_+)$)

$$\hat{\mathfrak{B}}(\hat{Y}_+)(x,v) = -(1-e^{(\beta')})\hat{\mathfrak{B}}(\hat{Y}_-)(x,v)$$

Proposition 4.3 immediately follows from Theorem 4.3 and the property of the Γ-function.

Theorem 4.6 *We have Stokes multipliers between the system of solutions* (Y_-, Y_0, Y_+) *in the sector* $\{y : \frac{\pi}{2} < \arg y < \frac{5\pi}{2}\}$ *and that of solutions* $(Y_-, Y_0, \tilde{Y}_+)$ *in the sector* $\{y : \frac{-\pi}{2} < \arg y < \frac{3\pi}{2}\}$*:*

$$(Y_-, Y_0, \tilde{Y}_+) = (Y_-, Y_0, Y_+)\begin{pmatrix} 1 & 0 & (1-e^{(\gamma-\beta')})(1-e^{(\beta')}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Outline of the Proof of Theorem 4.6

We would like to obtain the relation between the system of solutions $(Y_-, Y_0, \tilde{Y}_+)$ and that of (Y_-, Y_0, Y_+). We compute $\lim_{\varepsilon\to 0}\mathfrak{L}_{(0,\pi+\varepsilon)}(\hat{\mathfrak{B}}\hat{Y}_+) - \mathfrak{L}_{(0,\pi-\varepsilon)}(\hat{\mathfrak{B}}\hat{Y}_+)$. Note that the path of integration is distributed as in Figure 1.

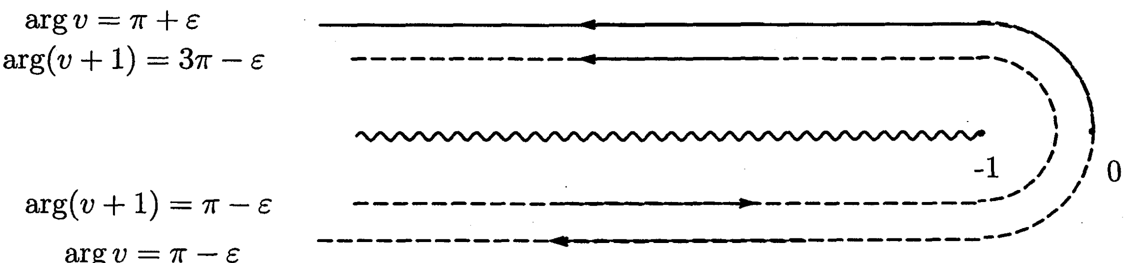


Figure 1

Then, in the sector $\{\frac{\pi}{2} < \arg y < \frac{3\pi}{2}\}$,

$$\begin{aligned}
\tilde{Y}_+ &= \int_0^{\infty(e^{(\pi+\varepsilon)i})} e^{-yv}\hat{\mathfrak{B}}(\hat{Y}_+)dv \\
&= \int_0^{\infty(e^{(\pi-\varepsilon)i})} e^{-yv}\hat{\mathfrak{B}}(\hat{Y}_+)dv \\
&+ (\int_{-1}^{\infty(e^{(3\pi-\varepsilon)i})} e^{-yv}\hat{\mathfrak{B}}(\hat{Y}_+)dv - \int_{-1}^{\infty(e^{(\pi-\varepsilon)i})} e^{-yv}\hat{\mathfrak{B}}(\hat{Y}_+)dv) \\
&= Y_+ - (1-e^{(\gamma-\beta')})\int_{-1}^{\infty(e^{(\pi-\varepsilon)i})} e^{-yv}\hat{\mathfrak{B}}(\hat{Y}_+)dv
\end{aligned}$$

is satisfied for an arbitrary small $\varepsilon > 0$. Using Proposition 4.3, we have

$$\begin{aligned}\tilde{Y}_+ &= Y_+ + (1 - e^{(\gamma-\beta')})(1 - e^{(\beta')}) \int_{-1}^{\infty(e^{(\pi-\varepsilon)i})} e^{-yv} \hat{\mathfrak{B}}(\hat{Y}_-) dv \\ &= Y_+ + (1 - e^{(\gamma-\beta')})(1 - e^{(\beta')}) Y_-\end{aligned}$$

for an arbitrary small $\varepsilon > 0$. Thus we obtain

$$(Y_-, Y_0, \tilde{Y}_+) = (Y_-, Y_0, Y_+) \begin{pmatrix} 1 & 0 & (1 - e^{(\gamma-\beta')})(1 - e^{(\beta')}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5 Solutions near $x = \infty$ (y is bounded)

5.1 Formal solutions

In the same way as in section 4, we can obtain the following results.

Theorem 5.1 *Near $x = \infty$ with bounded $|y|$, we have formal solutions written in the form*

$$\begin{aligned}\hat{X}_+ &= -\Gamma(1-\beta) e^{-\beta\pi i} x^{\beta-\gamma} e^x V(\beta, \beta', \gamma, y, 1/x), \\ X_0 &= e^{(\beta-\beta')\pi i} \frac{\Gamma(\beta+\beta'-\gamma+1)\Gamma(1-\beta')}{\Gamma(\beta-\gamma+2)} y^{\beta-\gamma+1} x^{-\beta} T(\beta, \beta', \gamma, y, 1/x), \\ \hat{X}_- &= 2\pi i e^{\beta\pi i} \frac{1}{\Gamma(\gamma-\beta)} x^{-\beta} U(\beta, \beta', \gamma, y, -1/x).\end{aligned}$$

Proposition 5.1 *We have*

$$\begin{aligned}&(1 - e^{(\beta)})^{-1} \int_{C(0)} u^{l-m-\beta} (u-y)^m e^u du \\ &\qquad = -(-1)^l e^{-\beta\pi i} (1-\beta)_l \Gamma(1-\beta) \frac{(1)_m}{(\beta-l)_m} L_m^{(\beta-l-1)}(y).\end{aligned}$$

Here the path of integration and the branch of the integrand are chosen in such a way that they have the following properties.

(1) *$C(0)$ is a loop which starts from $u = \infty e^{-\pi i}$, encircles $u = 0$ in the positive sense and ends at $u = \infty e^{\pi i}$.*
(2) *The branch of $u^{l-m-\beta}(u-y)^m$ is taken such that*

$$\arg u = \pi, \ \arg(u - y) = \pi$$

at the end point $u = \infty e^{\pi i}$ of each path of integration.

Proposition 5.1 can be found in Erdélyi [4] and Shimomura [10].

5.1.1 Gevrey asymptotic estimate

Theorem 5.2 *Assume that R is an arbitrary large positive constant. Then,*
(1) $\hat{X}_+$ is a power series of Gevrey order 1 of $1/x$ uniformly for $|y| < R$.
(2) X_0 converges uniformly for $|y| < R$ and $|x| > |y|$.
(3) $\hat{X}_-$ is a power series of Gevrey order 1 of $1/x$ uniformly for $|y| < R$.

5.2 Gevrey asymptotic solutions

We construct the actual solutions which have asymptotic expansion through the Borel-Laplace transforms of the formal solutions $\hat{X}_-$, X_0 and $\hat{X}_+$. We can find that X_0 is equal to the Borel-Laplace transform of the solution X_0.

Theorem 5.3 *We have the explicit Borel transforms of $\hat{X}_-, X_0, \hat{X}_+$ as follows :*

$$\begin{aligned}\hat{\mathfrak{B}}(\hat{X}_+)(y,u) =& \frac{\Gamma(1-\beta)}{\Gamma(\gamma-\beta)}(1+u)^{\gamma-\beta-1}u^{\beta-1}{}_1F_1(\beta',\gamma-\beta;(1+u)y);\\ \mathfrak{B}(X_0)(y,u) =& e^{(\beta-\beta')\pi i}\frac{\Gamma(\beta+\beta'-\gamma+1)\Gamma(1-\beta')}{\Gamma(\beta-\gamma+2)\Gamma(\beta)}\\ &\times u^{\beta-1}y^{\beta-\gamma+1}{}_1F_1(\beta-\beta'-\gamma+1,2+\beta-\gamma;(1+u)y);\\ \hat{\mathfrak{B}}(\hat{X}_-)(y,u) =& \frac{2\pi i e^{\beta\pi i}}{\Gamma(\gamma-\beta)\Gamma(\beta)}u^{\beta-1}(1+u)^{\gamma-\beta-1}{}_1F_1(\beta',\gamma-\beta;(1+u)y).\end{aligned}$$

In the same way as of the case $y=\infty$ (x is bounded), the singular points of $\hat{\mathfrak{B}}(\hat{X}_+)$ are $u=-1$ and $u=0$, and those of $\hat{\mathfrak{B}}(\hat{X}_-)$ are $u=-1$ and $u=0$. Therefore we consider the Laplace transforms of $\hat{\mathfrak{B}}(\hat{X}_+)$ and that of $\hat{\mathfrak{B}}(\hat{X}_-)$ except one direction. So we see that $\hat{X}_+, X_0, \hat{X}_-$ are Borel summable by Laplace transforms.

Theorem 5.4 (1) *$\hat{X}_-$ is Borel summable except for the negative direction of real axis.*
(2) *X_0 is Borel summable in all directions.*
(3) *$\hat{X}_+$ is Borel summable except for the positive direction of real axis.*

We construct actual solutions which have asymptotic expansions to the formal solutions uniformly for y through the Laplace transforms of $\hat{\mathfrak{B}}(\hat{X}_+)$ and $\hat{\mathfrak{B}}(\hat{X}_-)$. Since $\hat{\mathfrak{B}}(\hat{X}_+)$ is singular only at $u=-1$ and $u=0$, we can see that $\mathfrak{L}_{(-1,\theta_1)}(\hat{\mathfrak{B}}(\hat{X}_+))$ is integrable in the direction for $\theta_1=\pi$. Here the branch of $e^{-xu}\hat{\mathfrak{B}}(\hat{X}_+)(y,u)$ is taken such that

$$\arg u = \pi, \quad \arg(u+1) = \pi$$

at the end point $v=\infty$.

Since $\hat{\mathfrak{B}}(\hat{X}_-)$ has singular points only at $u=-1$ and $u=0$, we can see that $\mathfrak{L}_{(0,\theta_2)}(\hat{\mathfrak{B}}(\hat{X}_-))$ is integrable in the direction for $\theta_2=0$. Here the branch of $e^{-xu}\hat{\mathfrak{B}}(\hat{X}_-)(y,u)$ is taken such that

$$\arg u = \pi, \quad \arg(u+1) = \pi$$

at the end point $v = \infty$. Also, we can verify that X_0 is an analytic function, that $\mathfrak{L}_{(-1,\pi)}(\hat{\mathfrak{B}}(\hat{X}_+))(x,y)$ can be defined in $\{x : \frac{\pi}{2} < \arg x < \frac{3\pi}{2}\}$, and that $\mathfrak{L}_{(0,0)}(\hat{\mathfrak{B}}(\hat{X}_-))(x,y)$ can be defined in $\{x : \frac{3\pi}{2} < \arg x < \frac{5\pi}{2}\}$, employing an argument similar to that of the case $y = \infty$ (x is bounded). Let $X_+(x,y)$ be the analytic continuation of $\mathfrak{L}_{(-1,\pi)}(\hat{\mathfrak{B}}(\hat{X}_+))(x,y)$ in $\{x : -\frac{\pi}{2} < \arg x < \frac{5\pi}{2}\}$ and $X_-(x,y)$ the analytic continuation of $\mathfrak{L}_{(0,0)}(\hat{\mathfrak{B}}(\hat{X}_-))(x,y)$ in $\{x : \frac{\pi}{2} < \arg x < \frac{7\pi}{2}\}$.

Consequently the following theorem can be proved.

Theorem 5.5 *Assume that R is an arbitrarily large positive constant. Then,*

$$X_+ \simeq_1 \hat{X}_+$$

uniformly for $|y| < R$ as $y \to \infty$ through the sector $|\arg x - \pi| < \dfrac{3\pi}{2}$*;*

$$X_- \simeq_1 \hat{X}_-$$

uniformly for $|y| < R$ as $y \to \infty$ through the sector $|\arg x - 2\pi| < \dfrac{3\pi}{2}$.

5.3 Stokes Multipliers

Let $\tilde{X}_-(x,y)$ be the analytic continuation of $\mathfrak{L}_{(0,\pi+2\pi)}(\hat{\mathfrak{B}}(\hat{X}_-))(x,y)$ in $\{x : -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}\}$, and $\tilde{X}_+(x,y)$ the analytic continuation of $\mathfrak{L}_{(-1,0+2\pi)}(\hat{\mathfrak{B}}(\hat{X}_+))(x,y)$ in $\{x : -\frac{5\pi}{2} < \arg x < \frac{\pi}{2}\}$.

Theorem 5.6 (1) *Between the system of solutions (X_-, X_0, X_+) in the sector $\{x : \frac{\pi}{2} < \arg x < \frac{5\pi}{2}\}$ and that of solutions $(\tilde{X}_-, X_0, X_+)$ in the sector $\{x : -\frac{\pi}{2} < \arg x < \frac{3\pi}{2}\}$, we have*

$$(\tilde{X}_-, X_0, X_+) = (X_-, X_0, X_+)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (1 - e^{(\gamma-\beta)})(1 - e^{(\beta)}) & 0 & 1 \end{pmatrix}.$$

(2) *Between the system of solutions $(\tilde{X}_-, X_0, X_+)$ in the sector $\{x : -\frac{\pi}{2} < \arg x < \frac{3\pi}{2}\}$ and that of solutions $(\tilde{X}_-, X_0, \tilde{X}_+)$ in the sector $\{x : -\frac{3\pi}{2} < \arg x < \frac{\pi}{2}\}$, we have*

$$(X_-, X_0, \tilde{X}_+) = (X_-, X_0, X_+)\begin{pmatrix} 1 & 0 & e^{(-\beta)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 5.6 is proved in a similar way to that of the proof of Theorem 4.6.

6 Relation to the solutions given by Shimomura

Shimomura [10] gave the system of solutions of (1.1), as follows.

We begin with the case where (x, y) satisfies (4.1), namely

$$0 < \arg x < \pi < \arg y < 2\pi, \ \ \pi < \arg(y - x) < 2\pi.$$

Denote the integrand of (1.2) by

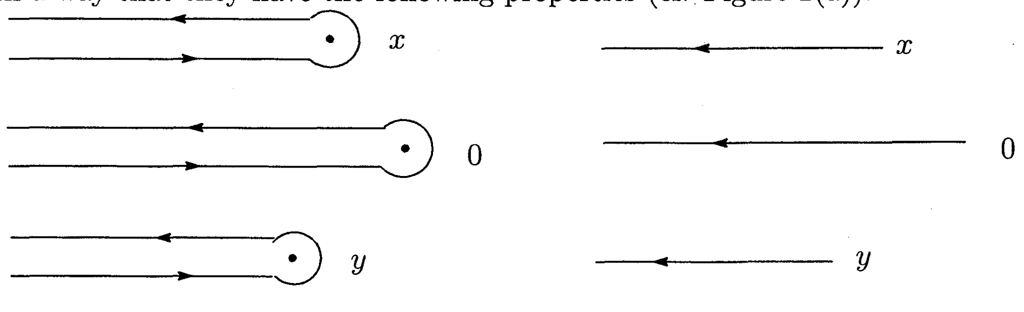
$$f(x, y, t) = t^{\beta+\beta'-\gamma}(t-x)^{-\beta}(t-y)^{-\beta'}e^t,$$

which possesses the branch points at $t = 0, x, y$.

Consider the integrals

$$\begin{aligned} z_- &= (1 - e^{(\beta')})^{-1} \int_{C(y)} f(x, y, t)dt, \\ z_0 &= (1 - e^{(\gamma-\beta-\beta')})^{-1} \int_{C(0)} f(x, y, t)dt, \\ z_+ &= (1 - e^{(\beta)})^{-1} \int_{C(x)} f(x, y, t)dt. \end{aligned}$$

Here the paths of integration and the branch of the integrand are chosen in such a way that they have the following properties (cf. Figure 2(a)):



(a) Figure 2 (b)

(1) $C(a)$ $(a = 0, x, y)$ is a loop which starts from $t = \infty e^{-\pi i}$, encircles $t = a$ in the positive sense and ends at $t = \infty e^{\pi i}$.
(2) $C(x)$ lies over $C(0)$ and $C(y)$ lies under $C(0)$ in the t-plane.
(3) The branch of f is taken such that

$$\arg t = \pi, \ \arg(t - x) = \pi, \ \arg(t - y) = \pi \tag{6.1}$$

at the end point $t = \infty e^{\pi i}$ of each path of integration.

Hereafter we treat the integrals near $x = \infty$ obtained by analytic continuation along a curve which starts from a point satisfying (4.1) and is included in the domain $|x| > |y|$. Furthermore, under the condition

$$\Re(\beta + \beta' - \gamma) > -1, \ \Re(-\beta) > -1, \ \Re(-\beta') > -1$$

we can write these integrals in the form

$$z_+ = \int_x^{-\infty} f dt, \; z_0 = \int_0^{-\infty} f dt, \; z_- = \int_y^{-\infty} f dt$$

where the branch of f is taken such that (6.1) is satisfied at $-\infty$, and the paths are located as shown in Figure 1(b) when (4.1) is satisfied.

Theorem 6.1 *Near* $y = \infty$,

$$\begin{aligned} Y_- &= (1 - e^{(\beta')})^{-1} z_-, \\ Y_0 &= z_0 - z_+, \\ Y_+ &= (1 - e^{(\beta)}) z_0 + e^{(\beta)}(1 - e^{(\gamma-\beta'-\beta')}) z_+. \end{aligned}$$

Proof of Theorem 6.1 We can verify Theorem 6.1 using Proposition 4.2 and the following propositions.

Proposition 6.1 *Assume that none of* a, c *and* $c - a$ *is an integer. Then, we have*

$$\int_{(-s,-0,+s,+0)} t^{a-1}(s-t)^{c-a-1} e^t dt = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}(1 - e^{(-a)})(1 - e^{(a-c)}) s^{c-1} {}_1F_1(a, c, s),$$

where $(-s, -0, +s, +0)$ *denotes Pochhammer's double loop which starts from* $\frac{s}{2}$, *encircles* $t = s$, $t = 0$, $t = s$, $t = 0$ *in the sense of each sign attached before and returns to* $\frac{s}{2}$, *and the branch of the integrand is taken in such a way that* $\arg t = \arg(s - t) = 2\pi$ *at* $t = \frac{s}{2}$ *in the case of* $\arg s = 2\pi$.

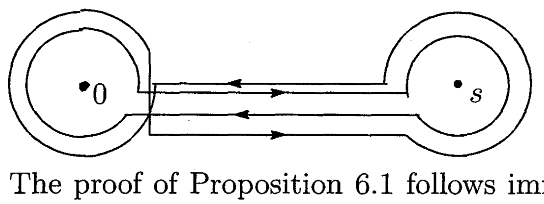


Figure 3

The proof of Proposition 6.1 follows immediately from the integral representation of ${}_1F_1(a, c, s)$.

Proposition 6.2 *Assume that none of* a, c *and* $c - a$ *is an integer. Then we have*

$$\int_{C(s,0)} t^{a-1}(s-t)^{c-a-1} e^t dt = e^{a\pi i}(1 - e^{(-c)})\Gamma(c-1) {}_1F_1(a - c + 1, 2 - c, s),$$

where $C(s, 0)$ *denotes a contour which starts from* $t = \infty e^{-\pi i}$, *encircles* $t = s$ *and* $t = 0$ *in the positive sense and ends at* $t = \infty e^{\pi i}$, *and the branch of the integrand is taken in such a way that* $\arg t = \pi, \arg(s - t) = 0$ *at the end point* $t = \infty e^{\pi i}$.

The proof of Proposition 6.2 can be found in Shimomura [10].

The following theorem is similar to that of Theorem 6.1.

Theorem 6.2 *For the solutions given in Section 5,*

$$\begin{aligned} X_+ &= z_+, \\ X_0 &= e^{(\beta-\beta')}(z_0 - z_-), \\ X_- &= e^{(-\beta)}\{e^{(\gamma-\beta-\beta')}(1-e^{(\beta')})z_- + (1-e^{(\gamma-\beta'-\beta')})z_0\}. \end{aligned}$$

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