ANOTHER INTERPRETATION OF MEASURABLE NORMS AND GAUSS CYLINDRICAL MEASURES

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Abstract. In this paper, we treat several conditions concerning the measurability of norms. Some of this note continue with the result obtained in [13]. We solve a few open questions in [13]. Moreover we reinvestigate the Gauss cylindrical measures by correlating it with Kuo's conjecture.

1. Introduction

The notion of measurable norms was introduced by Gross [4]. This gives a condition that the Gauss cylindrical measure is extensible to a measure. It plays an important role in the measure theory on infinite dimensional Hilbert spaces. Afterwards, another notion of measurable norms was introduced by Dudley-Feldman-LeCam [2]. These two notions are not equivalent for a general cylindrical measure. There are several conditions around these two notions of measurable norms, and we research them in [13]. In Section 4, we solve two open questions in [13].

In the next section, we present a new definition about Gauss cylindrical measures which was introduced by Baxendale. There we research them and compare them with the Gauss cylindrical measure defined in original sense.

Finally in Sections 6 and 7, we present Kuo's conjecture and study some facts around it.

2. Preliminaries

Let X be a locally convex Hausdorff space over the real field $\mathbb{R}$, X' its topological dual, $(\cdot,\cdot)$ the natural pairing between X and X' and $\mathcal{B}(X)$ the Borel σ-algebra of X. Let $\{\xi_1,\dots,\xi_n\}$ be a finite system of elememts of X'. Then by Ξ we denote the operator from X into $\mathbb{R}^n$ mapping x onto the vector $((x,\xi_1),\dots,(x,\xi_n))$. A set $Z\subset X$ is said to be a cylindrical set if there are $\xi_1,\dots,\xi_n\in X'$ and $B\in\mathcal{B}(\mathbb{R}^n)$ such that $Z=\Xi^{-1}(B)$. Let $\mathcal{C}_X$ denote the collection of all cylindrical sets of X.

A map μ from $\mathcal{C}_X$ into $[0,1]$ is called a cylindrical measure if it satisfies the following conditions:

(1) $\mu(X)=1$;

(2) Restrict μ to the σ-algebra of cylindrical sets which are generated by a fixed finite system of functionals. Then each such restriction is countably additive.

By putting $\mu_{\xi_1,\dots,\xi_n}(B)=\mu(\Xi^{-1}(B))$ each cylindrical measure μ defines a family of Borel probability measures.

Next we introduce two kinds of measurable norms defined on a Hilbert space. Let H be a real separable Hilbert space with norm $|\cdot|=\sqrt{\langle\cdot,\cdot\rangle}$. Let $\mathcal{F}$ be the partially ordered set of finite dimensional orthogonal projections of H and $FD(H)$ the family of all finite dimensional subspaces of H. $P>Q$ means $PH\supset QH$ for $P,Q\in\mathcal{F}$. Also a subset E of H of the form $E=\{x\in H; Px\in F\}$ is a cylindrical set, where $P\in\mathcal{F}$ and F is a Borel subset of PH.

First, we define the canonical Gauss cylindrical measure, and define two measurable norms.

Definition 2.1. The canonical Gauss cylindrical measure is the cylindrical measure γ from $\mathcal{C}_H$ into $[0,1]$ defined as follows:

If $E = \{x \in H; Px \in F\}$, then

$$\gamma(E) = \Big(\frac{1}{\sqrt{2\pi}}\Big)^n \int_F e^{-\frac{|x|^2}{2}}\, dx,$$

where n =dimPH and dx is the Lebesgue measure on PH.

Remark 2.2. If H is an infinite-dimensional space, then γ is finitely additive, but is not σ-additive. In general, we denote by $\gamma^t(Z) = (\frac{1}{\sqrt{2\pi t}})^n \int_F e^{-\frac{|x|^2}{2t}}\, dx$ the Gauss cylindrical measure with parameter $t\,(0 < t < \infty)$. γ^1 is the caonical Gauss cylindrical measure. In this paper, we denote the canonical Gauss cynlindrical measure by γ.

Definition 2.3. A semi-norm $\|\cdot\|$ in H is said to be (G)measurable or γ-measurable if for every $\varepsilon > 0$, there exists $P_0 \in \mathcal{F}$ such that $\gamma(\{x \in H; \|Px\| > \varepsilon\}) < \varepsilon$ for $\forall P \perp P_0$ and $P \in \mathcal{F}$.

This concept was introduced by Gross in 1962 [5]. It was the starting point of the successive research concerning the abstract Wiener space. In Definition 2.3 , we can replace γ with μ which is any cylindrical measure defined on H. Such a case we say that $\|\cdot\|$ is μ-(G)measurable.

We can redefine the above concept as follows:

Definition 2.4. A semi-norm $\|\cdot\|$ is said to be μ-(G)measurable if for every $\varepsilon > 0$, there exists $G \in FD(H)$ such that $\mu(N_\varepsilon \cap F + F^\perp) \geq 1 - \varepsilon$ whenever $F \in FD(H)$ and $F \perp G$, where $N_\varepsilon = \{x \in H; \|x\| \leq \varepsilon\}$ and $F^\perp$ is the orthogonal complement of F.

The following measurability was introduced by Dudley-Feldman-LeCam in 1971 [2].

Definition 2.5. A semi-norm $\|\cdot\|$ is said to be μ-(D)measurable if for every $\varepsilon > 0$ there exists $G \in FD(H)$ such that $\mu(P_F(N_\varepsilon) + F^\perp) \geq 1 - \varepsilon$ whenever $F \in FD(H)$ and $F \perp G$, where P_F is the orthogonal projection of H onto F.

Let E be the completion of H with respect to the norm $\|\cdot\|$ and i the inclusion map of H into E. If $\|\cdot\|$ is γ-(G)measurable, then the triple (i, H, E) is called an abstract Wiener space. The norm $\|\cdot\|$ is continuous and μ-(D)measurable if and only if $i(\mu)$, where $i(\mu)$ is the image of μ under the map i, is countably additive.

It is easy to see that (G)measurability implies (D)measurability. But the converse is false generally ([10], this is the 1984-Example).

We define the chraracteristic functions and the continuity of cylindrical measures.

Definition 2.6. The characteristic function ϕ of a cylindrical measure μ on H is defined by

$$\phi(\xi) = \int_H e^{i\langle \xi, x\rangle}\, \mu(dx)$$

where $\xi \in H$.

Definition 2.7. Let μ be a cylindrical measure on H. μ is said to be continuous if the characteristic function of μ is continuous.

Proposition 2.8. ([9, *Proposition* 4.4.1]) *Let μ be a cylindrical measure on H. Then the followings are equivqlent.*
(i) μ is continuous.
(ii) If the generalized sequence $\{\xi_\alpha\} \subset H$ tends to zero, then we have

$$\lim_\alpha \mu_{\xi_\alpha}([-\delta, \delta]) = 1$$

for some (each) $\delta > 0$.
(iii) Suppose $\lim_\alpha \xi_\alpha = 0$*. Then it follows that* μ_{ξ_α} *weakly converges to* δ_0.

Next we inrtoduce the definitions of Banach spaces of type 2 and of cotype 2.

Definition 2.9. Let B be a Banach space, $\{\xi_n\}$ be a sequence of independent random variables, each of which is distributed by the standard Gaussian law, and $\{x_n\}$ be a sequence in B.

(*i*) B is of type 2 if $\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} < \infty$ implies the almost sure (a.s.) convergence of the series $\sum_{n=1}^{\infty} x_n \xi_n$.

(*ii*) B is of cotype 2 if the a.s. convergence of the series $\sum_{n=1}^{\infty} x_n \xi_n$ implies $\left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} < \infty$.

3. Seven conditions

In this section we recall seven conditions that are similar to measurable norms.

Theorem 3.1. *Let* H *be a real separable Hilbert space with norm* $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$, μ *be a cylindrical measure on* H, $\|\cdot\|$ *be a continuous norm defined on* H, B *be the completion of* H *with respect to* $\|\cdot\|$ *and* i *be the inclusion map from* H *into* B*. Moreover, let* Y *be the bidual* B'' *of* B *with weak*-topology* $\sigma(B'', B')$ *and* j *be the inclusion map from* H *into* Y*. Then the following seven conditions satisfy the relations: (i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(iv), (i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(v)* $\Leftrightarrow$ *(vi)* $\Rightarrow$ *(vii)*

If μ *is continuous, then the following condition satisfy the relations: (iii)* $\Rightarrow$ *(vi) and (iv)* $\Rightarrow$ *(vii)*

(i) For any $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$*, where* $\mathbb{N}$ *is the set of all natural numbers, such that* $n > m \geq N$ *implies*

$$\mu(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) < \varepsilon$$

for every sequence $\{P_n\} \subset \mathcal{F}$ *such that* $P_n \subset P_{n+1}$ $(\forall n \in \mathbb{N})$ *and* P_n *converges strongly to the identity map* I *(we write* $P_n \nearrow I$*).*
(ii) $\|\cdot\|$ *is a* μ*-(G)measurable norm.*
(iii) There exists a sequence $\{P_n\} \subset \mathcal{F}$ *with* $P_n \nearrow I$*, which has the property that for any* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $n > m \geq N$ *implies*

$$\mu(\{x \in H; \|P_n x - P_m x\| > \varepsilon\}) < \varepsilon.$$

(iv) There exists a sequence $\{P_n\} \subset \mathcal{F}$ *with* $P_n \nearrow I$*, which has the property that for any* $\varepsilon > 0$ *there exist* $N_\varepsilon \in \mathbb{N}$ *and* $n_\varepsilon \in \mathbb{N}$ *such that* $N \geq N_\varepsilon$ *and* $n \geq n_\varepsilon$ *implies*

$$\mu(\{x \in H; \sup_{1 \leq k \leq n} \|P_k x\| > N\}) < \varepsilon.$$

(v) $\|\cdot\|$ *is a* μ*-(D)measurable norm.*
(vi) $i(\mu)$ *(i.e.* $\mu \circ i^{-1}$*) is extensible to a measure.*
(vii) $j(\mu)$ *is extensible to a measure.*

Proof. (*i*) $\Rightarrow$ (*ii*) (see Baxendale [1, Theorem 2.4])
(*ii*) $\Rightarrow$ (*iii*) (see Baxendale [1, Theorem 2.4])
(*iii*) $\Rightarrow$ (*iv*) (see Harai [8])
(*v*) $\Leftrightarrow$ (*vi*) (see Dudley-Feldman-LeCam [2, Theorem 2])
(*iii*) $\Rightarrow$ (*iv*) (see Yan [15, Theorem 3.1])
(*iv*) $\Rightarrow$ (*vii*) (see Yan [15, Theorem 3.3] and Gong [3, Theorem 1.1])

□

Remark 3.2. ([2]). The above conditions (*i*), (*ii*), (*iii*), (*v*) and (*vi*) are equivalent for γ.

4. $\|\cdot\|_1$ AND $\|\cdot\|_4$ ARE $\mu_{\mathbf{a}}$-(D)MEASURABLE

We constructed four norms ($\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_3$ and $\|\cdot\|_4$) and two cylindrical measures ($\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$), and we examined the relation with seven conditions in Theorem 3.1 in [13]. Here we study two open questions in [13]. So we recall the definitions of $\|\cdot\|_1$, $\|\cdot\|_4$, $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$.
Let $e_n = (0, \dots, 0, 1, 0, \dots)$, where 1 appears in the n-th place. It is clear that $\{e_n\}_{n=1,2,\dots}$ is a complete orthonormal system (CONS) on ℓ^2.

We define two norms $\|\cdot\|_1$ and $\|\cdot\|_4$ as follows:

Let $\{\alpha_n\}_{n=1,2,\dots}$ be the sequence of non-negative real numbers such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots$ and $\alpha_n \to \infty$ as $n \to \infty$. Denote by Γ_1 the convex hull of the set $\{\pm\alpha_n(e_1 + e_2 + \dots + e_n); n = 1, 2, \dots\}$, B_1 the open unit ball of ℓ^2 and $U_1 = \Gamma_1 + B_1$. It is obvious that U_1 is open, convex, absorbing and circled. we denote by $\|x\|_1$ the gauge of U_1 at $x \in \ell^2$. Then $\|\cdot\|_1$ is a continuous norm defined on ℓ^2.

We define $\|x\|_4 = \sqrt{\sum_{n=1}^{\infty} (\frac{x_n}{n})^2}$, where $x = \sum_{n=1}^{\infty} x_n e_n$. Then $\|\cdot\|_4$ is a continuous norm defined on ℓ^2.

We define two cylindrical measures $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$ as follows:

Let $(\ell^2)^*$ be the algebraic dual of ℓ^2, equipped with its weak topology $\sigma((\ell^2)^*, \ell^2)$, and $(\cdot, \cdot)$ be the natural pairing $(\ell^2)^* \times \ell^2 \to \mathbb{R}$. Then a cylindrical set in $(\ell^2)^*$ and in ℓ^2 can be described as

$$Z = \{x \in (\ell^2)^*; ((x, \xi_1), \dots, (x, \xi_n)) \in D\}$$

and

$$\tilde{Z} = \{x \in \ell^2; (\langle x, \xi_1\rangle, \dots, \langle x, \xi_n\rangle) \in D\}$$

where $\xi_1, \dots, \xi_n \in \ell^2$ and $D \in \mathcal{B}(\mathbb{R}^n)$, respectively.

We choose an algebraic basis $\mathcal{J}$ of ℓ^2 containing $\{e_n\}_{n=1,2,\dots}$. Define $\mathbf{a}$ and $\mathbf{b} \in (\ell^2)^*$ as follows:

$$(\mathbf{a}, e_n) = 1 \text{ for } n = 1, 2, \dots,$$
$$(\mathbf{a}, e_\alpha) = 0 \text{ for } e_\alpha \in \mathcal{J}\backslash\{e_n\}_{n=1,2,\dots}$$

$$(\mathbf{b}, e_n) = n \text{ for } n = 1, 2, \dots$$
$$(\mathbf{b}, e_\alpha) = 0 \text{ for } e_\alpha \in \mathcal{J}\backslash\{e_n\}_{n=1,2,\dots}$$

Let $\delta_{\mathbf{a}}$ and $\delta_{\mathbf{b}}$ denote the Dirac measures at the fixed points $\mathbf{a}$ and $\mathbf{b}$ in $(\ell^2)^*$ respectively. Then the induced measures $\mu_{\mathbf{a}}$ and $\mu_{\mathbf{b}}$ on ℓ^2 are defined by

$$\mu_{\mathbf{a}}(\tilde{Z}) = \delta_{\mathbf{a}}(Z),$$
$$\mu_{\mathbf{b}}(\tilde{Z}) = \delta_{\mathbf{b}}(Z), \text{ respectively.}$$

We know that $\|\cdot\|_1$ and $\|\cdot\|_4$ satisfy the condition (iii) for $\mu_{\mathbf{a}}$ in [13, Theorem 4]. So if $\mu_{\mathbf{a}}$ was continuous, they would fulfill (v) and (vi) for $\mu_{\mathbf{a}}$. But $\mu_{\mathbf{a}}$ is not continuous. So we have to prove that $\|\cdot\|_1$ and $\|\cdot\|_4$ are $\mu_{\mathbf{a}}$-(D)measurable directly. Though $\mu_{\mathbf{b}}$ has no relevance to open problems, we will prove that $\mu_{\mathbf{b}}$ is not also continuous.

Proposition 4.1. *$\mu_{\mathbf{a}}$ is not continuous.*

Proof. Let $\mu_{\mathbf{a}}$ be restricted to the linear span of the $\frac{1}{n^{\frac{3}{4}}}(e_1 + \dots + e_n)$. Let $\|\cdot\|_{\ell^2}$ be the ℓ^2-norm. Then we have

$$\left\|\frac{e_1 + \dots + e_n}{n^{\frac{3}{4}}}\right\|_{\ell^2} = \sqrt{\frac{n}{n^{\frac{3}{2}}}} = \sqrt{\frac{1}{\sqrt{n}}} \to 0$$

when $n \to \infty$. On the other hand,

$$(\mu_{\mathbf{a}})_{\frac{1}{n^{\frac{3}{4}}}(e_1+\dots+e_n)} = \delta_{(\mathbf{a}, \frac{1}{n^{\frac{3}{4}}}(e_1+\dots e_n))} = \delta_{\frac{n}{n^{\frac{3}{4}}}} = \delta_{n^{\frac{1}{4}}},$$

so that it follows from Proposition 2.8 that $\mu_{\mathbf{a}}$ is not continuous. □

Proposition 4.2. *$\mu_{\mathbf{b}}$ is not continuous.*

Proof. We consider the restriction of $\mu_{\mathbf{b}}$ to the linear span of the $\frac{1}{n}e_n$. Then we have

$$\|\frac{1}{n}e_n\|_{\ell^2} = \sqrt{\frac{1}{n^2}} = \frac{1}{n} \to 0$$

when $n \to \infty$. On the other hand,

$$(\mu_{\mathbf{b}})_{\frac{1}{n}e_n} = \delta_{(\mathbf{b},\frac{1}{n}e_n)} = \delta_1,$$

so that $\mu_{\mathbf{b}}$ is not continuous. □

Theorem 4.3. *$\|\cdot\|_1$ is $\mu_{\mathbf{a}}$-(D)measurable.*

Proof. Let E be the completion of ℓ^2 with respect to the norm $\|\cdot\|_1$, and j be the inclusion map of ℓ^2 into E and denote by j' the dual operator of j. Let $(\cdot,\cdot)_E$ be the natural pairing $E' \times E \to \mathbb{R}$.

$$E' \xrightarrow{j'} (\ell^2)' \simeq \ell^2 \xrightarrow{j} E$$

To prove that the norm $\|\cdot\|_1$ is $\mu_{\mathbf{a}}$-(D)measurable, it suffices to show that $j(\mu_{\mathbf{a}})$, the image of $\mu_{\mathbf{a}}$ under the map j, is σ-additive on $(E, \mathcal{C}_E)$.
Claim. $\mathbf{a}$ vanishes on $j'(E')$.
Proof of Claim. Suppose $y \in E'$ is given. We have to show that $(\mathbf{a}, j'(y)) = 0$. Since $(\mathbf{a}, e_\alpha) = 0$ for all $e_\alpha \in J \setminus \{e_n\}_{n=1,2,\dots}$, we may assume $j'(y)$ is of the form $\sum_{n=1}^N A_n e_n$, where $A_1, A_2, \dots, A_N \in \mathbb{R}$.
Now define a sequence $\{x^m\}_{m=1,2,\dots}$ in ℓ^2 by

$$\begin{aligned} x^1 &= e_1, \\ x^2 &= e_1 + e_2, \\ &\vdots \\ x^m &= e_1 + \dots + e_m, \\ &\vdots \end{aligned}$$

Then $\langle e_k, x^m \rangle = 1$ for $m \geq k$, so $\langle j'(y), x^m \rangle = \sum_{n=1}^N A_n$ for all $m \geq N$. Moreover, since $\langle j'(y), x^m \rangle = (y, j(x^m))_E$, we obtain $(y, j(x^m))_E = \sum_{n=1}^N A_n$ for all $m \geq N$. Therefore,

$$\lim_{m\to\infty} (y, j(x^m))_E = \sum_{n=1}^N A_n. \tag{1}$$

But by constructions of $\{\alpha_m\}$ and U_1, we know that $\alpha_m x^m \in U_1$, so that $\|x^m\|_1 \leq 1/\alpha_m$. The assumption $\alpha_m \to \infty$ as $m \to \infty$ implies $\|x^m\|_1 \to 0$ as $m \to \infty$. Therefore,

$$\lim_{m\to\infty} j(x^m) = 0 \quad \text{in } E. \tag{2}$$

Thus by (1) and (2) we deduce that $\sum_{n=1}^N A_n = 0$, and hence $(\mathbf{a}, j'(y)) = \sum_{n=1}^N A_n = 0$. This proves our claim.

Let i be the canonical map of $(\ell^2)^*$ into $(E')^*$. Then our claim implies $i(\mathbf{a}) = 0$, so that $i(\delta_{\mathbf{a}})$ is the Dirac measure δ_0 on $(E')^*$. Therefore, $j(\mu_{\mathbf{a}})$ is extendable to δ_0 on E, so it is σ-additive on $(E, \mathcal{C}_E)$. □

Theorem 4.4. *$\|\cdot\|_4$ is $\mu_{\mathbf{a}}$-(D)measurable.*

Proof. Let E be the completion of ℓ^2 with respect to the norm $\|\cdot\|_4$, and j be the inclusion map of ℓ^2 into E. Denote by j' the dual operator of j. Let $(\cdot,\cdot)_E$ be the natural pairing $E' \times E \to \mathbb{R}$.

$$E' \xrightarrow{j'} (\ell^2)' \simeq \ell^2 \xrightarrow{j} E$$

Now define the sequence $\{x^m\}_{m=1,2,\dots}$ in ℓ^2 by

$$\begin{aligned} x^1 &= e_1, \\ x^2 &= e_1 + e_2, \\ &\vdots \\ x^m &= e_1 + \cdots + e_m, \\ &\vdots \end{aligned}$$

For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|j(x^n) - j(x^m)\|_4 = \|j(e_{m+1} + \cdots + e_n)\|_4 = \sqrt{\textstyle\sum_{k=m+1}^{n} \left(\frac{1}{k}\right)^2} \le \varepsilon \quad \text{for every} \quad n > m \ge N.$$

Since $\{j(x^m)\}$ is a Cauchy sequence, there exists $x \in E$ such that

$$\lim_{m\to\infty} \|x - j(x^m)\|_4 = 0.$$

Suppose that $y \in E'$ is given. We assume $j'(y)$ is of the form $\sum_{n=1}^{N} A_n e_n$, where $A_1, A_2, \dots, A_N \in \mathbb{R}$. Then $\langle e_k, x^m \rangle = 1$ for $m \ge k$, so $\langle j'(y), x^m \rangle = \sum_{n=1}^{N} A_n$ for all $m \ge N$. Moreover, since $\langle j'(y), x^m \rangle = (y, j(x^m))_E$, we obtain $(y, j(x^m))_E = \sum_{n=1}^{N} A_n$ for all $m \ge N$. Therefore,

$$(y, x)_E = \sum_{n=1}^{N} A_n. \tag{3}$$

Let i be the canonical map of $(\ell^2)^*$ into $(E')^*$. Since $(\mathbf{a}, e_n) = 1$, we have $(\mathbf{a}, j'(y)) = \sum_{n=1}^{N} A_n$. Therefore,

$$(i(\mathbf{a}), y) = \sum_{n=1}^{N} A_n. \tag{4}$$

Thus by (3) and (4), we have

$$(i(\mathbf{a}), y) = (y, x)_E. \tag{5}$$

The set $\{y \in E'; j'(y) = \sum_{n=1}^{N} A_n e_n\,,\ N \in \mathbb{N},\ A_n \in \mathbb{R}\}$ is dense in E'. Then (5) and the definition of $\mathbf{a}$ induce that $i(\mathbf{a})$ is continuous at zero in E'. So we have $(i(\mathbf{a}), y) = (y, x)_E$ for all $y \in E'$. Therefore, $j(\mu_{\mathbf{a}})$ is extendable to δ_x on E, so it is σ-additive on $(E, \mathcal{C}_E)$. □

5. Another Gauss cylindrical measure

The comparison between the Gauss cylindrical measures by Baxendale and the original one was researched in [14]. Here we concretely construct a cylindrical measure on ℓ^2 using the same $\mathbf{a}$ as we introduced in Section 4, and we denote it by $\gamma_{\mathbf{a}}$. We mention the definition of $\gamma_{\mathbf{a}}$ later. It is the Gauss cylindrical measure in the sense of Baxendale but it is not the original Gauss cylindrical measure. In this section, we study the seven conditions in Theorem 3.1 by comparing γ and $\gamma_{\mathbf{a}}$.

First we recall the Gauss cylindrical measures by Baxendale.

Definition 5.1. ([1]) Let E be a separable Banach space.

(*a*) A Borel probability measure λ on $\mathbb{R}$ is Gaussian if either
(*i*) $\lambda = \delta_0$
or
(*ii*) There exists $t > 0$ such that for $B \in \mathcal{B}(\mathbb{R})$
$$\lambda(B) = \frac{1}{\sqrt{2\pi t}} \int_B e^{-\frac{|x|^2}{2t}}\, dx.$$
(*b*) A cylindrical measure μ is Gaussian or μ is a Gauss cylindrical measure if every one dimensional distribution μ_ξ (i.e. the image of μ under $\xi : E \to \mathbb{R}$) of μ is Gaussian on $\mathbb{R}$.
(*c*) A Gauss measure on E is a Borel probability measure on E which restricts to a Gauss cylindrical measure.

Remark 5.2. γ^t and γ^1 are also the Gauss cylindrical measures in the sense of Baxendale.

Definition 5.3. Let E be a real separable Banach space and μ be a Gaussian cylindrical measure on E. We say μ has variance A if there is a self adjoint $A \in L(E, E')$ such that $\phi(\mu, \xi) = \exp\{-(\xi, A\xi)/2\}$ for all $\xi \in E'$. Here A is called self adjoint if $(\xi, A\eta) = (\eta, A\xi)$ for any $\xi, \eta \in E'$. Let $L(E, E')$ be the family of bounded linear operator from E into E' and $\phi(\mu, \xi)$ is the characteristic function of μ.

Let $\mathbf{a}$ be the element of $(\ell^2)^*$ such that
$$(\mathbf{a}, e_n) = 1 \text{ for } n = 1, 2, \ldots,$$
$$(\mathbf{a}, e_\alpha) = 0 \text{ for } e_\alpha \in \mathcal{J} \backslash \{e_n\}_{n=1,2,\ldots}.$$

Let $\gamma_{\mathbb{R}}$ be the canonical Gauss measure on $\mathbb{R}$, $\gamma_{\mathbf{a}}^*$ be the measure induced by a function $f : \mathbb{R} \to (\ell^2)^*$ $(t \to t\mathbf{a})$. i.e. $\gamma_{\mathbf{a}}^*$ is the image measure of $\gamma_{\mathbb{R}}$ under f. Define a cylindrical measure $\gamma_{\mathbf{a}}$ on ℓ^2 by $\gamma_{\mathbf{a}}(\tilde{Z}) = \gamma_{\mathbf{a}}^*(Z)$, where $Z = \{x \in (\ell^2)^*; ((x, \xi_1), \ldots, (x, \xi_n)) \in D\}$, $\tilde{Z} = \{x \in \ell^2; (\langle x, \xi_1\rangle, \ldots, \langle x, \xi_n\rangle) \in D\}$, $\xi_1, \ldots, \xi_n \in \ell^2$ and $D \in \mathcal{B}(\mathbb{R}^n)$. Then $\gamma_{\mathbf{a}}$ is a Gauss cylindrical measure in the sense of Baxendale, but is not a canonical Gauss cylindrical measure. Further, it does not coincide with γ^t and does not have a variance [14].

The following results are shown in the similar way as Theorems 4.3 and 4.4.

Theorem 5.4. *$\|\cdot\|_1$ is $\gamma_{\mathbf{a}}$-(D)measurable.*

Theorem 5.5. *$\|\cdot\|_4$ is $\gamma_{\mathbf{a}}$-(D)measurable.*

Next we consider a norm $\|\cdot\|_2$ again and study the relation of $\gamma_{\mathbf{a}}$ and $\|\cdot\|_2$.
Maeda showed that $\|\cdot\|_2$ is $\mu_{\mathbf{a}}$-(D)measurable and not $\mu_{\mathbf{a}}$-(G)measurable in 1984 [10]. This example gave the fact that the measurability of Dudley-Feldman-LeCam differs from that of Gross.

Let $\{\beta_n\}_{n=1,2,\ldots}$ be a sequence of non-negative real numbers such that $\beta_{2m} = 0$ for $m = 1, 2, \ldots$ and $\{\beta_{2m-1}\}_{m=1,2,\ldots}$ be an increasing sequence such that $\beta_{2m-1} \to \infty$ as $m \to \infty$. Denote by Γ_2 the convex hull of the set $\{\pm\beta_n(e_1 + e_2 + \cdots + e_n); n = 1, 2, \ldots\}$, B_1 the open unit ball of ℓ^2 and $U_2 = \Gamma_2 + B_1$. It is obvious that U_2 is open, convex, absorbing and circled. We denote by $\|x\|_2$ the gauge of U_2 at $x \in \ell^2$.

Theorem 5.6. *$\|\cdot\|_2$ is $\gamma_{\mathbf{a}}$-(D)measurable.*

Proof. It can be shown by the similar method of Theorems 4.3 and 5.3. □

Theorem 5.7. *$\|\cdot\|_2$ is not $\gamma_{\mathbf{a}}$-(G)measurable.*

Proof. It suffices to show that there exists a positive number $\varepsilon_0 > 0$ such that for every $G \in FD(\ell^2)$ there exists $F \in FD(\ell^2)$ satisfying $F \perp G$ and $\gamma_{\mathbf{a}}(\varepsilon_0 U_2 \cap F + F^\perp) < 1 - \varepsilon_0$.

Let G be an arbitrary finite dimensional subspace of ℓ^2, and $\{\xi^j\}_{j=1,2,\ldots,n}$ be a CONS in G. Then each ξ^j is of the form $\xi^j = \sum_{i=1}^\infty \alpha_i^j e_i$ where $\alpha_i^j \in \mathbb{R}$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots$. Then we have the following matrix A:
$$A = \begin{pmatrix} \alpha_1^1 & \cdots & \alpha_n^1 & \cdots & \alpha_{n+m}^1 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_1^n & \cdots & \alpha_n^n & \cdots & \alpha_{n+m}^n \end{pmatrix},$$

where m is chosen such that rank $A = n$. Suppose $N > n+m$. Then the next equation has its solution in $\mathbb{R}^{n+m}$.

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} -\alpha^1_{2N+1} \\ \vdots \\ -\alpha^n_{2N+1} \end{pmatrix}. \tag{6}$$

By construction we know that $\alpha^j_i \to 0$ as $i \to \infty$ for $j = 1, 2, \dots, n$. Therefore, for every $\delta > 0$, we may choose a positive integer $N(> n+m)$, N sufficiently large, such that the equation (6) has the solution $x_1 = \eta_1, \dots, x_{n+m} = \eta_{n+m}$ satisfying

$$\max_{1 \le l \le n+m} |\eta_l| < \delta. \tag{7}$$

Now choose a number $\delta > 0$ in (7) such that

$$\frac{\eta_1 + \eta_2 + \cdots + \eta_{n+m} + 1}{(\eta_1)^2 + (\eta_2)^2 + \cdots + (\eta_{n+m})^2 + 1} > \frac{1}{2}. \tag{8}$$

Let $\tau = \eta_1 e_1 + \cdots + \eta_{n+m} e_{n+m} + e_{2N+1}$ and F be the one dimensional subspace of ℓ^2 generated by τ. $\langle \tau, \xi^j \rangle = 0$ for $j = 1, 2, \dots, n$, so that $F \perp G$.

Put $\phi = \frac{\tau}{|\tau|}$, where $|\cdot|$ is the Hilbert norm of ℓ^2, then

$$(\mathbf{a}, \phi) = \frac{(\mathbf{a}, \tau)}{|\tau|} = \frac{\eta_1 + \eta_2 + \cdots + \eta_{n+m} + 1}{({\eta_1}^2 + \cdots + {\eta_{n+m}}^2 + 1)^{\frac{1}{2}}}.$$

Let $\varepsilon_1 = \int_{(\mathbf{a},\phi)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$ then $0 < \varepsilon_1 < \frac{1}{2}$. Put $\varepsilon_0 = \min(\varepsilon_1, \frac{1}{6})$.

Now we show that $(\mathbf{a}, \phi)\phi \notin \varepsilon_0 U_2$. Suppose this is not the case. Then $(\mathbf{a}, \phi)\phi \in \varepsilon_0 U_2$ implies that $(\mathbf{a}, \phi)\phi$ may be written $(\mathbf{a}, \phi)\phi = X + Y$, where $X \in \varepsilon_0 \Gamma_2$ and $Y \in \varepsilon_0 B_1$. Since $X, Y \in \ell^2$, X and Y are of the form $X = \sum_{i=1}^{\infty} X_i e_i$ and $Y = \sum_{i=1}^{\infty} Y_i e_i$, where $X_i, Y_i \in \mathbb{R}$ for $i = 1, 2, \dots$. Then $(\mathbf{a}, \phi)\phi = \sum_{i=1}^{\infty} (X_i + Y_i) e_i$ and by (8) we have

$$X_{2N} + Y_{2N} = 0$$

and

$$X_{2N+1} + Y_{2N+1} = \frac{\eta_1 + \eta_2 + \cdots + \eta_{n+m} + 1}{{\eta_1}^2 + \cdots + {\eta_{n+m}}^2 + 1} > \frac{1}{2}.$$

The fact $X \in \varepsilon_0 \Gamma_2$ implies that $X_{2N} = X_{2N+1}$. Therefore,

$$\left| \frac{\eta_1 + \eta_2 + \cdots + \eta_{n+m} + 1}{{\eta_1}^2 + \cdots + {\eta_{n+m}}^2 + 1} - Y_{2N+1} \right| = |X_{2N+1}| = |Y_{2N}| < \varepsilon_0 \le \frac{1}{6}.$$

On the other hand by (8),

$$\left| \frac{\eta_1 + \eta_2 + \cdots + \eta_{n+m} + 1}{{\eta_1}^2 + \cdots + {\eta_{n+m}}^2 + 1} - Y_{2N+1} \right| > \frac{1}{2} - |Y_{2N+1}| \ge \frac{1}{3},$$

and since N is sufficiently large, we reach a contradiction. Therefore we have $(\mathbf{a}, \phi)\phi \notin \varepsilon_0 U_2$.

By $(\mathbf{a}, \phi)\phi \notin \varepsilon_0 U_2$, we obtain $q\phi \notin \varepsilon_0 U_2$ for any $q \ge (\mathbf{a}, \phi)$. Since $(\gamma_{\mathbf{a}})_\phi = \gamma_{\mathbf{a}} \circ \phi^{-1}$,

$$\gamma_{\mathbf{a}}(\varepsilon_0 U_2 \cap F + F^{\perp}) = (\gamma_{\mathbf{a}})_\phi(\varepsilon_0 U_2 \cap F) \le 1 - 2\int_{(\mathbf{a},\phi)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = 1 - 2\varepsilon_1 < 1 - \varepsilon_1 \le 1 - \varepsilon_0,$$

and the proof is complete. □

We conclude that Gauss cylindrical measures of Baxendale contain not only the canonical Gauss cylindrical measure and γ^t but also another cylindrical measure. And they do not have a particular property that the conditions (i), (ii), (iii), (v) and (vi) in Theorem 3.1 are equivalent. This property characterizes the Gauss cylindrical measures in the original sense.

6. Kuo's conjecture and its periphery

In 1973, Kuo offered the following question:

"Let $\|\cdot\|$ be a γ-measurable norm in H, does there exist an orthnormal basis $\{e_n\}$ of H such that $\sum_{n=1}^{\infty} \|e_n\|^2 < +\infty$?"

In 1980 Kwapien, Szymanski and Tarieladze solve this problem affirmatively [5].

In this section, we give three examples. In the examples, we show that there is an orthonormal basis $\{e_n\}$ such that $\sum_{n=1}^{\infty} \|e_n\|^2 = +\infty$ for γ-measurable norms. The first and second examples are well-known. The third example contain a result of Goodman when $\alpha = 1/2$. We could not get the paper of Goodman, so we give the proof and generalize it.

Example 1. Let A be a Hilbert-Schmidt operator. Put $\|x\| = |Ax|$ for $x \in H$, then $\|\cdot\|$ is γ-measurable. Moreover, it is clear that $\sum_{n=1}^{\infty} \|e_n\|^2 < \infty$ and its value is constant for all orthonormal basis $\{e_n\}$.

Example 2. Let $C[0,1]$ denote the set of $\mathbb{R}$-valued functions $x(t)$ in the unit interval $[0,1]$ with $x(0) = 0$. Let C' consist of all $\mathbb{R}$-valued functions f such that f is absolutely continuous, $f(0) = 0$ and $f' \in L^2[0,1]$. Define an inner product $\langle \cdot, \cdot \rangle$ in C' by

$$\langle f, g \rangle = \int_0^1 f'(t) g'(t)\, dt.$$

Then C' is a Hilbert space and $C' \subset C[0,1]$, and $C[0,1]$ is a Banach space with the supreum norm $\|x\| = \sup_{0 \le t \le 1} |x(t)|$. Hence $(i, C', C[0,1])$ is an abstract Wiener space. $\|\cdot\|$ is a measurable norm for any Gauss cylindrical measures on C'. Let $\{e_n(t)\}$ be an orthonormal basis in C' defined as follows:

$$e_n(t) = \sqrt{2}\{1 - \frac{1}{(n-\frac{1}{2})\pi} \cos(n - \frac{1}{2})\pi t\}.$$

Since $\|e_n\| \ge \sqrt{2}$, it follows that $\sum_{n=1}^{\infty} \|e_n\|^2 = +\infty$.

Example 3. For $(x_1, x_2, \dots, x_n, \dots) \in \ell^2$, put

$$\|(x_1, x_2, \dots, x_n, \dots)\| = \sup n^{-\alpha} |x_n|.$$

Then if $\alpha \ge \frac{1}{2}$, we have that $\|\cdot\|$ is γ-measurable. Moreover, if $\alpha > \frac{1}{2}$, there exists an orthonormal basis $\{e_n\}$ such that $\sum \|e_n\|^2 < \infty$, and if $\alpha = \frac{1}{2}$, there exists an orthonormal basis $\{e_n\}$ such that $\sum \|e_n\|^2 = \infty$.

Proof. If $\|\cdot\|$ is γ-measurable when $\alpha = \frac{1}{2}$, then $\|\cdot\|$ is γ-measurable when $\alpha > \frac{1}{2}$. Moreover, the existence of $\{e_n\}$ such that $\sum_{n=1}^{\infty} \|e_n\|^2 < \infty$ or $\sum_{n=1}^{\infty} \|e_n\|^2 = \infty$ is shown by taking an orthonormal basis $\{e_n\}_{n=1,2,\dots}$, where $e_n = (0, \dots, 0, 1, 0, \dots)$, 1 appears in the n-th place. Therefore here we show that $\|\cdot\|$ is γ-measurable when $\alpha = \frac{1}{2}$.

Note that the following inequality holds:

$$\int_{-k}^{k} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx > \sqrt{1-e^{-k^2/2}}.$$

In fact, put $x = r\cos\theta, y = r\sin\theta$. Then we have

$$\left(\int_0^k e^{-x^2/2}dx\right)^2 = \int_0^k \int_0^k e^{-(x^2+y^2)/2}dxdy > \int_0^{\pi/2} \int_0^k e^{-r^2/2} r dr d\theta = \frac{\pi}{2}\left(1-e^{-k^2/2}\right),$$

and this proves the above inequality.

Fix $\varepsilon > 0$ and put

$$a_N = \prod_{k=1}^{N} \int_{-\varepsilon\sqrt{k}}^{\varepsilon\sqrt{k}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad (N = 1, 2, \ldots).$$

Since the series $\sum_{k=1}^{\infty} e^{-\varepsilon^2 k/2}$ converges, the infinite product $\prod_{k=1}^{\infty} \sqrt{1-e^{-\varepsilon^2 k/2}}$ also converges. The sequence $\{a_N\}_{N=1,2,\ldots}$ is decreasing and it follows from the above inequality that

$$a_N \geq \prod_{k=1}^{\infty} \sqrt{1-e^{-\varepsilon^2 k/2}} > 0$$

for all $N = 1, 2, \ldots$. Thus, the infinite product $\prod_{k=1}^{\infty} \int_{-\varepsilon\sqrt{k}}^{\varepsilon\sqrt{k}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$ converges, so that there exsits $N \in \mathbb{N}$ such that $n > m > N$ implies

$$\left| \prod_{k=m+1}^{n} \int_{-\varepsilon\sqrt{k}}^{\varepsilon\sqrt{k}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - 1 \right| < \varepsilon.$$

Let P_n be the projection from ℓ^2 onto the linear span of $\{e_1, \ldots, e_n\}$. Then, for any $n > m > N$

$$\begin{aligned} \gamma(\{x \in \ell^2 : \|P_n x - P_m x\| \leq \varepsilon\}) &\geq \gamma(\{x \in \ell^2 : |x_k| \leq \varepsilon\sqrt{k}, k = m+1, \ldots, n\}) \\ &= \prod_{k=m+1}^{n} \int_{-\varepsilon\sqrt{k}}^{\varepsilon\sqrt{k}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &> 1 - \varepsilon, \end{aligned}$$

and this implies the γ-measurability of $\|\cdot\|$. □

7. The extention to a Banach space of type 2 or of cotype 2

In the preceding section, we see that the γ-measurability of a norm $\|\cdot\|$ implies the existence of orthonormal basis $\{e_n\}$ in H such that $\sum_{n=1}^{\infty} \|e_n\|^2 < \infty$. When Banach spaces completed with respect to $\|\cdot\|$ is of type 2 or of cotype 2, we have the following theorems.

Let $\{\xi_n\}$ be a sequence of independent random variables, each of which is distributed by the standard Gaussian law.

Theorem 7.1. *Let H be a real separable Hilbert space, $\|\cdot\|$ be a γ-measurable norm defined on H, B be the completion of H with respect to $\|\cdot\|$ and i be the inclusion map from H to B. If B is of cotype 2, we have $\sum \|e_n\|^2 < \infty$ for every orthonormal basis $\{e_n\}$ in H.*

Proof. Since $\|\cdot\|$ is γ-measurable, $i(\gamma)$ is extendable to a measure on B. For any orthonormal basis $\{e_n\}$ in H, the distribution of $\sum_{n=1}^{\infty} i(e_n)\xi_n$ is $i(\gamma)$ and $\sum_{n=1}^{\infty} i(e_n)\xi_n$ almost surely converges. Since B is of cotype 2, we have $\sum_{n=1}^{\infty} \|e_n\|^2 < \infty$. □

Theorem 7.2. *Let H be a real separable Hilbert spce, $\|\cdot\|$ be a norm defined on H, B be the completion of H with respect to $\|\cdot\|$, i be the inclusion map from H into B. If B is a Banach spce of type 2, then the following statements are equivarnt:*

(a) $\|\cdot\|$ *is* γ*-measurable.*

(b) *There exists an orthonormal basis* $\{e_n\}$ *in* H *such that* $\sum_{n=1}^{\infty}\|e_n\|^2 < +\infty$.

Proof. It follows from [5] that (a) implies (b). We prove that (b) implies (a). By assumption, there exists $\{e_n\}$ such that $\sum_{n=1}^{\infty}\|i(e_n)\|^2 < \infty$. Since B is of type 2, it follows that $\sum_{n=1}^{\infty} i(e_n)\xi_n$ almost surely converges. Since the distribution of $\sum_{n=1}^{\infty} i(e_n)\xi_n$ is $i(\gamma)$, $\|\cdot\|$ is γ-measurable. $\square$

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